Covariance Steering

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Introduction and Background

Problem Formulation

- Stochastic System Definition
- Augmented System Definition
- Controller Definition
- Decoupled Controller and Dynamics
- Updated Terminal Condition
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- Numerical Example
- Result

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- Chance Constraints

Optimal Control in Stochastic environment

- LQR is an optimal regulator for linear state feedback control but it designed for deterministic environment.
- In a real world scenario both system dynamics and state measurements are corrupted by noise.
- Optimal techniques used under such stochastic environment can be:

LQG - Linear Quadratic Gaussian is a combination of LQR controller and Linear Quadratic Estimator (Kalman filter). LQR and LQE shows duality and thus have similar formulation.

CS - Covariance Steering is a stochastic optimal control technique to compute control commands that start from an initial probability distribution and converges to a target distribution.

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Stochastic System Definition

We assume a linear discrete time stochastic system:

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k \tag{1}$$

where, $k = [0, 1, 2, \dots, N-1]$ is the time index,
$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $w_k \in \mathbb{R}^r$ are the state, control and noise at time k respectively
Also, A_k, B_k and D_k vary with time

- Boundary conditions: Initial state (x_0) and Final state (x_f):

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0) \quad \text{and} \quad x_f \sim \mathcal{N}(\mu_f, \Sigma_f) \tag{2}$$

- The form of cost function taken as:

$$J = \mathbb{E}\left[\sum_{k=0}^{N-1} \left(x_k^T Q_k x_k + u_k^T R_k u_k\right)\right] \tag{3}$$

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Augmented System Definition

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W \tag{4}$$

where, we have following definitions:

- Augmented state: $X \in \mathbb{R}^{n(N+1)\times 1}$ $\quad X = \begin{bmatrix} x_0 & x_1 & \dots & x_N \end{bmatrix}^T$
- Augmented $\mathcal{A}$: $\mathcal{A} \in \mathbb{R}^{n(N+1)\times 1}$ $\quad \mathcal{A} = \begin{bmatrix} I_{n\times n} & \hat{A}_1 & \hat{A}_2 & \dots & \hat{A}_N \end{bmatrix}^T$
- Augmented $\mathcal{B}$: $\mathcal{B} \in \mathbb{R}^{nN\times mN}$ and $\mathcal{D}$: $\mathcal{D} \in \mathbb{R}^{nN\times rN}$

$$\mathcal{B} = \begin{bmatrix} 0_{n\times m} & 0_{n\times m} & \dots & 0_{n\times m} \\ \hat{B}_0 & 0_{n\times m} & \dots & 0_{n\times m} \\ \hat{B}_{1,0} & \hat{B}_0 & \dots & 0_{n\times m} \\ \vdots & & & \\ \hat{B}_{N-1,0} & \hat{B}_{N-1,1} & \dots & \hat{B}_{N-1} \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \begin{bmatrix} 0_{n\times r} & 0_{n\times r} & \dots & 0_{n\times r} \\ \hat{D}_0 & 0_{n\times r} & \dots & 0_{n\times r} \\ \hat{D}_{1,0} & \hat{D}_0 & \dots & 0_{n\times r} \\ \vdots & & & \\ \hat{D}_{N-1,0} & \hat{D}_{N-1,1} & \dots & \hat{D}_{N-1} \end{bmatrix}$$

Augmented System Definition

- Augmented control: $U \in \mathbb{R}^{m(N+1)\times 1}$ $\quad U = \begin{bmatrix} u_0 & u_1 & \dots & u_N \end{bmatrix}^T$
- Augmented noise: $W \in \mathbb{R}^{r(N+1)\times 1}$ $\quad W = \begin{bmatrix} w_0 & w_1 & \dots & w_N \end{bmatrix}^T$
- Other defn.

$$\hat{A}_k = A_{k-1,0} = A_{k-1}.A_{k-2}\dots A_2 A_1 A_0$$

$$\hat{B}_k = \begin{bmatrix} B_{k-1,0} & B_{k-1,1} & \dots & B_{k-1} \end{bmatrix}$$

$$B_{k1,k0} = A_{k1,k0+1} B_{k0}$$

$$A_{k1,k0} = A_{k1}.A_{k1-1}\dots A_{k0}$$

$$\hat{D}_k = \begin{bmatrix} D_{k-1,0} & D_{k-1,1} & \dots & D_{k-1} \end{bmatrix}$$

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Augmented System Definition

Cost Function in terms of augmented state and augmented control is defined as:

$$J = \mathbb{E}[X^T Q X + U^T R U] \tag{5}$$

where $Q = blkdiagonal(Q_0, Q_1, \ldots, Q_{N-1}, 0)$ and
$R = blkdiagonal(R_0, R_1, \ldots, R_{N-1})$
Augmented Boundary Conditions: Initial and Final conditions are defined as:

$$\mu_0 = E_0 \mathbb{E}[X] \quad \text{and} \quad \Sigma_0 = E_0 \Sigma_{xx} E_0^T \tag{6}$$

$$\mu_f = E_N \mathbb{E}[X] \quad \text{and} \quad \Sigma_f = E_N \Sigma_{xx} E_N^T \tag{7}$$

where, $\Sigma_{xx} = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T]$

Controller Definition

The controller is defined as a sum of close loop and open loop controller. Two forms of controller possible:

- Form 1: Controller utilizes full state history: Optimal and computational intensive

$$u_k = \sum_{j=0}^{k} K_{kj} \tilde{x}_k + v_k \tag{8}$$

- Form 2: Controller utilizes only the current state: Sub-optimal but computational advantage

$$u_k = K_k \tilde{x}_k + v_k \tag{9}$$

In this study, The form 2 controller is considered for its computational benefits and optimization ease.

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Augmented Controller

The Controller is defined as:

$$U = K\tilde{X} + V \tag{10}$$

where

$$K = diag(k_0, k_1, \ldots, k_{N-1})$$

Note: In case of Form 1 controller, K matrix would be a lower diagonal matrix rather than a diagonal matrix. $\tilde{X}$ is the augmented State deviation, whereas K and V are our design variables.

Decoupled Controller and Dynamics

- Mean Controller:

$$\bar{U} = V \tag{11}$$

- Deviation Controller:

$$\tilde{U} = L(\mathbb{1} + \mathcal{B}L)^{-1}\tilde{X} \tag{12}$$

- Mean Dynamics:

$$\bar{X} = \mathcal{A}\mu_0 + \mathcal{B}V \tag{13}$$

- Deviation Dynamics:

$$\tilde{X} = (\mathbb{1} + \mathcal{B}L)\mathcal{A}x_0 + (\mathbb{1} + \mathcal{B}L)\mathcal{D}W \tag{14}$$

Note that our design variables are now L and V

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Updated Terminal Condition

- Terminal Mean:

$$\mu_f = E_N(\mathcal{A}\mu_0 + \mathcal{B}V) \tag{15}$$

- Terminal Covariance:

$$\Sigma_f \leq E_N(\mathbb{1} + \mathcal{B}L)\, S\, (\mathbb{1} + \mathcal{B}L)^T E_N^T \tag{16}$$

Note: L and V are design variables. Terminal Mean is a linear function of V whereas Σ_f is a Linear Matrix Inequality (LMI) in L

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Resulting Optimization Problem

$$\min_{L,V} \quad J = \underbrace{(\bar{X}^T Q \bar{X} + \bar{U}^T R \bar{U})}_{\text{Mean Cost}(J_\mu)} + \underbrace{(tr(Q\Sigma_{xx}) + tr(R\Sigma_{uu}))}_{\text{Covariance Cost}(J_\Sigma)}$$

$$\begin{aligned} \text{s.t.} \quad & \mu_f = E_N(\mathcal{A}\mu_0 + \mathcal{B}V) \\ & \Sigma_f \leq E_N(\mathbb{1} + \mathcal{B}L)\, S\, (\mathbb{1} + \mathcal{B}L)^T E_N^T \end{aligned}$$

where

$$\bar{X} = \mathcal{A}\mu_0 + \mathcal{B}V \qquad \bar{U} = V \qquad S = (\mathcal{A}\Sigma_0\mathcal{A}^T + \mathcal{D}\mathcal{D}^T)$$

$$\Sigma_{xx} = (\mathbb{1} + \mathcal{B}L)\, S\, (\mathbb{1} + \mathcal{B}L)^T \qquad \Sigma_{uu} = LSL^T$$

$$Q = blkdiag(Q_0, Q_1, \ldots, Q_{N-1}, 0) \qquad R = blkdiag(R_0, R_1, \ldots, R_{N-1})$$

Modified Mathematical Formulation

Mean Steering problem can also be derived in a close form (apart from optimization problem formulation as before).

$$\bar{U} = \mathcal{R}^{-1}(\mathcal{B}^T Q \mathcal{A} \mu_0 + \bar{B}_N^T (\bar{B}_N \mathcal{R}^{-1} \bar{B}_N^T)^{-1} (\mu_N - \bar{A}_N \mu_0 - \bar{B}_N \mathcal{R}^{-1} \mathcal{B}^T Q \mathcal{A} \mu_0)) \tag{17}$$

where $\mathcal{R} = \mathcal{B}^T Q \mathcal{B} + R$

Modified Mathematical Formulation

$$\min_{L} \quad J_{cov}(L) = \|\bar{\bar{Q}}(\mathbb{1} + BL)\bar{\bar{\Sigma}}\|_F^2 + \|\bar{\bar{R}}L\bar{\bar{\Sigma}}\|$$

$$\text{subject to} \quad E_0(\mathbb{1} + \mathcal{B}L)\Sigma_{xx}(\mathbb{1} + \mathcal{B}L)^T E_0^T = \Sigma_0$$

$$1 - \|\bar{\bar{\Sigma}}(I + BL)^T E_N^T \Sigma_f^{1/2}\| \geq 0$$

where

$$\Sigma_{xx} = V_{xx}\Lambda_{xx}V_{xx}^T \qquad \bar{\bar{\Sigma}} = V_{xx}\Lambda_{xx}^{1/2}$$

$$Q = V_Q\Lambda_Q V_Q^T \qquad \bar{\bar{Q}}^T = V_Q\Lambda_Q^{1/2}$$

$$R = V_R\Lambda_R V_R^T \qquad \bar{\bar{R}}^T = V_R\Lambda_R^{1/2}$$

Numerical Example

The Initial mean and covariance is taken as:

$$\mu_0 = \begin{bmatrix} -10 & 1 & 0 & 0 \end{bmatrix}^T \qquad \Sigma_0 = \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix} \tag{18}$$

The Final mean and covariance is taken as:

$$\mu_f = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad \Sigma_f = 0.2 \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix} \tag{19}$$

Numerical Example

The transition matrix for state, input and disturbance are assumed as:

$$A_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B_k = \begin{bmatrix} 0.5\Delta t^2 & 0 \\ 0 & 0.5\Delta t^2 \\ \Delta t^2 & 0 \\ 0 & \Delta t^2 \end{bmatrix} \qquad D_k = \eta I_4 \tag{20}$$

$$Q_k = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.05 \end{bmatrix} \qquad R_k = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \tag{21}$$

Result

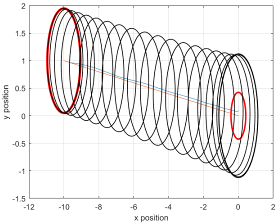


Figure: Mean steering (without co-variance steering) for N=20 time steps.

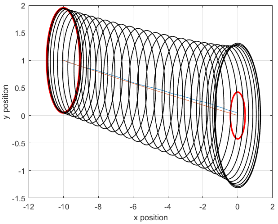


Figure: Mean steering (without co-variance steering) for N=30 time steps.

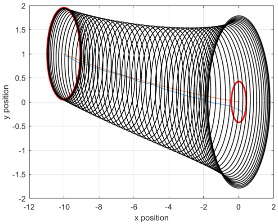


Figure: Mean steering (without co-variance steering) for N=50 time steps.

Result

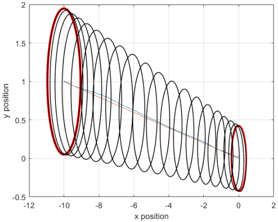
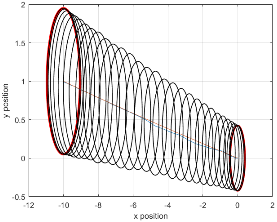


Figure: Mean and co-variance steering for N=20 time steps.



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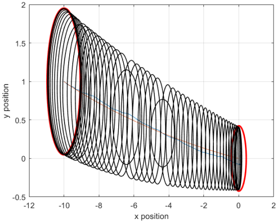

Figure: Mean and co-variance steering for N=50 time steps.

Covariance Steering with Chance Constraints

In real life, systems would face additional constraints like:

- Limitations on Controller: For instance, Control $\in [U_{max}, U_{min}]$
- Path Constraints: Vehicle should not deviate away from a region along the entire path.
- Sensor Viewing Constraints: Sensor's field of view may induce path constraints which could be polytopes or cones.
  In chance constraints we deal with polytopes.

Mathematical interpretation of polytopes and cone:

- Half space interpretation of Polytopes:

$$\mathcal{X}^P = \bigcap_{j=1}^{Nx} \{x : \alpha_j^T x \leq \beta_j\}$$

- Cones:

$$\mathcal{X}^C = \{x : \|Ax + b\|_2 \leq c^T x + d\}$$

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Chance Constraints

Chance Constraints for system state lie within the polytope region ($x_k \in \mathcal{X}^P$) will be:

$$\mathbb{P}[x_k \in \mathcal{X}^P] \geq 1 - \Delta$$

where Δ is very small (close to zero e.g. 0.01) .
The Joint chance constraint (chance constraint satisfied for all time) will be:

$$\mathbb{P}[\forall_{k=1}^{N} x_k \in \mathcal{X}^P] \geq 1 - \Delta$$

This leads to additional constraints (along with those derived earlier in Mean and Covariance steering):

- Additional constraint 1:

$$\alpha_j^T E_k(\mathcal{A}\mu_0 + \mathcal{B}V) + \|S^{1/2}(\mathbb{1} + \mathbb{B}L)^T E_k^T \alpha_j\|_2^{-1}(1 - \delta_{jk} \leq \beta_j)$$

- Additional Constraint 2:

$$\sum_{k=1}^{N} \sum_{j=1}^{Nx} \delta_{jk} \leq \Delta$$

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