CS 4820, Spring 2014	**Homework 6, Problem 3**
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(3) *(5 points) In this problem — as in the lectures this semester, but contrary to the textbook — a flow network may have edges entering the source and/or leaving the sink.*

A flow network is called *Eulerian* if the combined capacity of the incoming edges at any vertex is equal to the combined capacity of the outgoing edges at that vertex. If a and b are any two distinct vertices of an Eulerian flow network, prove that the maximum flow from a to b is equal to the maximum flow from b to a.

Solution

Lemma 1. *In an Eulerian graph, for any cut A, B, we have C(A,B) = C(B,A) where C(A,B) is the capacity of all edges going from A to B.*

Proof. Let $C(x, A)$ where x is any vertex in the graph G and A is any subset of vertices in G denote the sum of capacities of all edges going from x to any vertex in A.

Let $C(A, x)$ be the sum of capacities of all edges going from any vertex in A to vertex x.

Since we know that capacity going out from each node is equal to the capacity going in, we have $C(x, V) = C(V, x)$ where V is the vertex set in G.

Now for any cut A,B we have $V = A + B$.

This means that C(x, A) + C(x,B) = C(A,x) + C(B,x) for any vertex x

Now let all the vertices in A be $a_1, a_2...a_{|A|}$. Then we have

$C(a_1, A) + C(a_1, B) = C(A, a_1) + C(B, a_1)$
$C(a_2, A) + C(a_2, B) = C(A, a_2) + C(B, a_2)$
..
..
$C(a_{|A|}, A) + C(a_{|A|}, B) = C(A, a_{|A|}) + C(B, a_{A|})$

Adding all these equations

$C((a_1 + a_2.. + a_{|A|}), A) + C((a_1 + a_2.. + a_{|A|}), B) = C(A, (a_1 + a_2.. + a_{|A|})) + C(B, (a_1 + a_2.. + a_{|A|}))$

Since $a_1 + a_2.. + a_{|A|} = A$, we have

$C(A, A) + C(A, B) = C(B, A) + C(A, A)$

$C(A, B) = C(B, A)$ □

Proof

Now let the maximum flow from a to b has a minimum cut A_1, B_1. From max flow min cut theorem we know that this is also the minimum cut among all cuts from a to b. Since $C(A_1, B_1)$ is also equal to

$C(B_1, A_1)$ from lemma 1, this means that $C(B_1, A_1)$ is the minimum cut for all flows from b to a.

Now let the maximum flow from b to a has a minimum cut B_2, A_2. From max flow min cut theorem we know that this it min cut among all cuts from b to a. But we also know that $C(B_1, A_1)$ is the minimum cut from b to a. This means that

$C(B_2, A_2) = C(B_1, A_1)$
$C(B_2, A_2) = C(A_1, B_1)$

Also max flow from a to b $= C(A_1, B_1)$
max flow from b to a $= C(B_2, A_2)$

This means that max flow from a to b is equal to max flow from b to a.