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VECTOR ALGEBRA

Surface integral ⟷ Volume integral

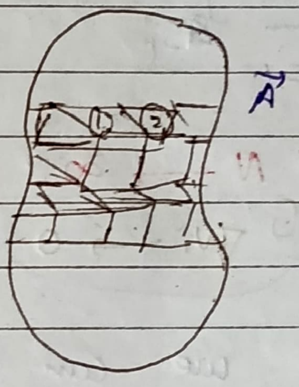
→ Gauss Divergence Theorem.

* This Theorem states that the surface integral of the normal component of a vector $\vec{A}$, taken over a closed surface S is equals to volume integral of the divergence of $\vec{A}$ taken over the volume V enclosed by surface 'S'.

$$\iint_{S} \vec{A}\cdot d\vec{s} = \iiint_{V} div\,\vec{A}\cdot dV.$$

→ Proof :-

Consider a closed surface (S) which enclosed in a vector field A.



* let us devide the given volume V into a large no. of infinitismly small volume element each having volume $\Delta V_1, \Delta V_2 \ldots \Delta V_n$ enclosed by surface $\Delta S_1, \Delta S_2 \ldots \Delta S_n$

\# The flux of a vector field $\vec{A}$ through volume element ΔV_1

$$\oint_{S_1} \vec{A}\cdot d\vec{s} = (\vec{\nabla}\cdot A)\cdot \Delta V_1 \quad —(1)$$

∴ divergence of a any vector is flux per unit Volume.

$\nabla\cdot A = \frac{flux}{Volume} \Rightarrow flux = \vec{\nabla}\cdot\vec{A} \times Volume$

$\phi_i = \oint A \cdot ds$

$\Delta V_1 \to$ Small Volume element

$$\oint_{\Delta S_1} \vec{A} \cdot \vec{ds} = (\vec{\nabla} \cdot \vec{A}) \cdot \Delta V_1 \quad \text{--- (i)}$$

Similarly in $\oint_{\Delta S_2} \vec{A} \cdot \vec{ds} = (\vec{\nabla} \cdot \vec{A}) \Delta V_2 \quad \text{--- (ii)}$

$$\oint_{\Delta S_n} \vec{A} \cdot \vec{ds} = (\vec{\nabla} \cdot \vec{A}) \Delta V_n \quad \text{--- (iii)}$$

add eqnation (i) & (ii) (iii)

$$\oint_{\Delta S_1} \vec{A} \cdot \vec{ds} + \oint_{\Delta S_2} \vec{A} \cdot ds + \dots + \oint_{\Delta S_n} \vec{A} \cdot ds_n = (\vec{\nabla} \cdot \vec{A}) [\nabla V_1 + \nabla V_2 + \nabla V_3 \dots \nabla V_n]$$

$$\sum_{i=1}^{N} \oint_{\Delta S_i} \vec{A} \cdot \vec{ds} = \sum_{i=1}^{N} (\vec{\nabla} \cdot \vec{A}) \cdot (\nabla V_i)$$

if $N \longrightarrow \infty$ (more no. of volume element)

& $\nabla V_i \longrightarrow 0$

So decreases the volume.

So we can ~~Right~~ write

$$\oint_S A \cdot \vec{ds} = \iiint_V (\vec{\nabla} \cdot \vec{A})\, dV \quad \text{or}$$

$$\boxed{\iint A \cdot \vec{ds} = \iiint_V (\vec{\nabla} \cdot \vec{A})\, dv.}$$

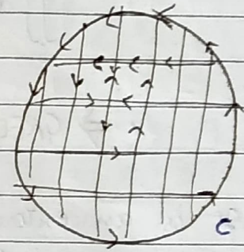
line intregal $\longleftrightarrow$ Surface integral

⇒ Stoke's Theorem

Stoke's The. states that, the line integral of vector $\vec{A}$ taken around a closed curve C, which bounds a surface S, is equal to the surface integral of the curl of $\vec{A}$ taken over S.

$$\oint_C \vec{A} \cdot \vec{dr} = \iint_S \text{curl} \vec{A} \cdot ds = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \vec{ds}$$

→ Proof in.



* Suppose an area is enclosed by the closed path C in a vector field $\vec{A}$.

* This area can be considerd to by made up of infinitesmly closed path

Consider such an element "ΔS_i".

[Note:* Curl of any vector gives us the "line integral of that vector in per unit area"

$$\text{curl} \vec{A} = \vec{\nabla} \times \vec{A} = \text{line integral of } \vec{A} / \text{unit area.}$$

So, ($\vec{\nabla}$ line integral of $\vec{A}$ = $(\vec{\nabla} \times \vec{A}) \times$ unit area

$$\boxed{\oint_{C_i} A \cdot \vec{dr} = (\vec{\nabla} \times \vec{A}) \cdot \Delta S_i}$$

if we add all such equations

$$\sum_{i=1}^{N} (\vec{\nabla}\times\vec{A})\nabla S_i = \sum_{i=1}^{N} \oint_{C_i} \vec{A}\cdot\vec{dr}$$

if $N \to \infty$ then $S \to 0$

[No. of small area tends to ∞ than area tends to 0]

So. then:

$$\iint_S (\vec{\nabla}\times\vec{A}).d\vec{s} = \oint_C \vec{A}\cdot\vec{dr}$$

⇒ GREEN'S THEOREM.

It is an extended form of gauss div theorm and stokes theorem:

- From Gauss's div Theorem :- First & second form of Green's theorem.

$$\iiint \text{div } \vec{A}\cdot dv = \iint_S \vec{A}\cdot d\vec{s} \quad \#$$

$$\iiint_V (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1) dV = \iint_S \left[\phi_1\frac{\partial\phi_2}{\partial n} - \phi_2\frac{d\phi_1}{dn}\right]$$

From stoke's theorem :- Green's theorem in plane.

$$\iint_S \text{curl } \vec{A}\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{l} \quad \# \quad \oint_C M dx + \oint_C N dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

→ First & second form of green's theorem or Greens Identity

From Gauss' div theorem.

$$\oiint \vec{A}\cdot d\vec{s} = \iiint (\vec{\nabla}\cdot\vec{A}) dv.$$

let $A = \phi_1\vec{\nabla}\phi_2$ & $B = \phi_2\vec{\nabla}\phi_1$

here ϕ_1, ϕ_2 are scalar functions.

$$\Rightarrow \oiint \vec{A}\cdot d\vec{s} = \iiint (\vec{\nabla}\cdot\vec{A}) dv.$$

$$\oiint (\phi_1\vec{\nabla}\phi_2) d\vec{s} = \iiint \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\phi_1\vec{\nabla}\phi_2) \cdot dv$$

$$= \iiint \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\phi_1\left(\hat{i}\frac{\partial}{\partial x}\phi_2 + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial}{\partial z}\phi_2\right)\right]\cdot dv$$

$$= \iiint \frac{\partial}{\partial x}\left[\phi_1\frac{\partial}{\partial x}\phi_2\right] + \frac{\partial}{\partial y}\left[\phi_1\frac{\partial}{\partial y}\phi_2\right] + \frac{\partial}{\partial z}\left[\phi_1\frac{\partial}{\partial z}\phi_2\right]$$

= $\iiint$ using uv rule :- $\frac{d}{dx}(x_1\cdot x_2) = \frac{dx_1}{dx}\cdot x_2 + \frac{dx_2}{dx}\cdot x_1$

$$= \iiint \left[\frac{\partial\phi_1}{\partial x}\cdot\frac{\partial}{\partial x}\phi_2 + \phi_1\frac{\partial^2}{\partial x^2}\phi_2 + \frac{\partial}{\partial y}\phi_1\cdot\frac{\partial\phi_2}{\partial y} + \phi_1\frac{\partial^2\phi_2}{\partial y^2} + \frac{\partial}{\partial z}\phi_1\frac{\partial}{\partial z}\phi_2 + \phi_1\frac{\partial^2}{\partial z^2}\phi_2\right]$$

$$= \iiint \phi_1\left[\frac{\partial^2}{\partial x^2}\phi_2 + \frac{\partial^2}{\partial y^2}\phi_2 + \frac{\partial^2}{\partial z^2}\phi_2\right] + \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x} + \frac{\partial}{\partial y}\phi_1\frac{\partial}{\partial y}\phi_2 + \frac{\partial}{\partial z}\phi_1\frac{\partial}{\partial z}\phi_2$$

$$\Rightarrow \iint (\vec{A}\cdot d\vec{s}) = \iiint \phi_1 \left[\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\phi_2$$

$$+ \frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial x}\phi_2 + \frac{\partial}{\partial y}\phi_1 \frac{\partial}{\partial y}\phi_2 + \frac{\partial}{\partial z}\phi_1 \frac{\partial}{\partial z}\phi_2$$

we know $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\iint_S (\vec{A}\cdot d\vec{s}) = \iiint_V (\phi_1 \nabla^2 \phi_2) + \left[\frac{\partial \phi_1}{\partial x}\cdot\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_1}{\partial z}\cdot\frac{\partial \phi_2}{\partial z}\right] dv \rightarrow \text{eq (i)}$$

Similarly $\iint \vec{B}\cdot d\vec{s} = \iint (\phi_2 \vec{\nabla}\vec{\phi_1})\, d\vec{s}$

$$= \iiint_V (\phi_2 \nabla^2 \phi_1) + \left[\frac{\partial \phi_1}{\partial x}\cdot\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial y}\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial z}\cdot\frac{\partial \phi_1}{\partial z}\right] dv \rightarrow \text{(ii)}$$

Substract eq (ii) from (i) $\rightarrow$ [(ii) - i]

$$\Rightarrow \iint \vec{A}\cdot d\vec{s} - \vec{B}\cdot d\vec{s} = \iiint_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1)\, dv$$

$$\iint [\phi_1 \vec{\nabla}\phi_2 - \phi_2 \vec{\nabla}\phi_1] = \iint \left(\phi_1 \frac{\partial}{\partial n}\phi_2 - \phi_2 \frac{\partial}{\partial n}\phi_1\right)$$

$$= \iiint_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_2)\cdot dv.$$

$\Rightarrow$ by Stokes Theorem or Proof of Green's Theorem.

If $M(x,y)$ and $N(x,y)$ and $\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial y}$ be continuous function over the region R bounded by simple closed curve 'C' in x-y plane is.

$$\oint M dx + \oint N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

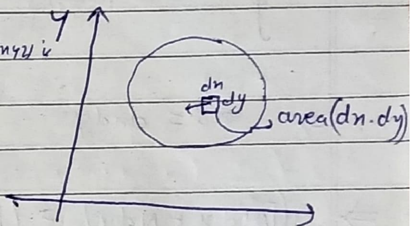
$$\boxed{\text{Stokes Theo.}: \oint \vec{A}\cdot d\vec{l} = \iint (\vec{\nabla}\times\vec{A})\cdot d\vec{s}} \rightarrow \text{(i)}$$

let $\vec{A} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

$\vec{A} = M\hat{i} + N\hat{j} + P\hat{k}$

because curve is in the x-y plane

$d\vec{l} = dx\hat{i} + dy\hat{j}$



$d\vec{s} = ds\hat{k} = dx\,dy\,\hat{k}$

↳ area vector which is $\perp^{er}$ to the plane. iz

$$\vec{A}\cdot d\vec{l} = (M\hat{i} + N\hat{j} + P\hat{k})\cdot(dx\hat{i} + dy\hat{j}) = \boxed{M dx + N dy}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M\hat{i} & N\hat{j} & P\hat{k} \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}P\hat{k} - \frac{\partial}{\partial z}N\hat{j}\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}P\hat{k} - \frac{\partial}{\partial z}M\hat{i}\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}N\hat{j} - \frac{\partial}{\partial y}M\hat{i}\right)$$

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$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)-\hat{j}\left(\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z}\right)+\hat{k}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

$$\boxed{(\vec{\nabla}\times\vec{A})\cdot d\vec{S} = (\vec{\nabla}\times\vec{A})\cdot ds\,\hat{k} = \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)ds}$$

Put value of $(A\cdot dl)$ and $(\vec{\nabla}\times\vec{A})\cdot d\vec{S}$ in equation (i)

$$\oint \vec{A}\cdot d\vec{l} = \iint (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

$$= \oint M\,dx + N\,dy = \iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\cdot ds$$

$\Rightarrow$ $ds = dx\,dy$

$$\Rightarrow \oint_l [M\,dx + N\,dy] = \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy.$$

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Numericals :-

$V(x,y,z) = C[x^2+y^2+z^2]$ Show that the force is radial ie along the radius vector.

$$\vec{F} = -\text{grad}\,V = -\nabla V.$$

$$= -\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)V$$

$$= -\left[\hat{i}\frac{\partial V}{\partial x}+\hat{j}\frac{\partial V}{\partial y}+\hat{k}\frac{\partial V}{\partial z}\right]$$

$$\vec{F} = \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}[C(x^2+y^2+z^2) = 2Cx$$

$$\frac{\partial V}{\partial y} = 2Cy, \quad \frac{\partial V}{\partial z} = 2Cz.$$

$$\vec{F} = -\left[\hat{i}\,2Cx+\hat{j}\,2Cy+\hat{k}\,2Cz\right]$$

$$\vec{F} = -2C[x\hat{i}+y\hat{j}+z\hat{k}]$$

we know radius vector $= x\hat{i}+y\hat{j}+z\hat{k}$

$\boxed{\vec{F} = -2C\vec{r}}$ hence proof

$\boxed{\vec{F}\propto -\vec{r}}$

Prove that grad r^n or $\vec{\nabla} r^n = n r^{n-2} \vec{r}$, where $\vec{r}$ is any point.

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$|\vec{r}| = r = (x^2+y^2+z^2)^{1/2}$.

LHS $\vec{\nabla} \cdot r^n$.

$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^2+y^2+z^2)^{n/2}$.

$\Rightarrow \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \hat{j}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}$

$+ \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$.

$\Rightarrow$ $\cancel{\frac{\partial}{\partial x}}$

$\Rightarrow \hat{i}\left(\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot 2x\right) + \hat{j}\left(\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot 2y\right)$

$+ \hat{k}\left(\frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot \cancel{2}y\right)$

$= \hat{i}\left(nx(x^2+y^2+z^2)^{\frac{n}{2}-1}\right) + \hat{j}\left(ny(x^2+y^2+z^2)^{\frac{n}{2}-1}\right)$

$+ \hat{k}\left(nz(x^2+y^2+z^2)^{\frac{n}{2}-1}\right)$

$= n \cdot (x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$

we know $\frac{a^m}{a^n} = a^{m-n}$

$= n \cdot \frac{(x^2+y^2+z^2)^{n/2}}{x^2+y^2+z^2}(x\hat{i}+y\hat{j}+z\hat{k})$ —— eq (1)

$\therefore r = (x^2+y^2+z^2)^{1/2}$ & $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$\boxed{r^2 = x^2+y^2+z^2}$

Put these value in eq (1)

$\vec{\nabla} \cdot r^n = n \cdot \frac{\left[(x^2+y^2+z^2)^{1/2}\right]^n}{r^2} \cdot \vec{r}$

$\vec{\nabla} \cdot r^n = n \cdot \frac{r^n}{r^2} \cdot \vec{r} = n r^{n-2} \vec{r}$

$\vec{\nabla} \cdot r^n = n r^{n-2} \cdot \vec{r}$ Hence prove

Show that div $\left(\frac{\vec{r}}{r^3}\right) = 0$.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$|\vec{r}| = r = (x^2+y^2+z^2)^{1/2}$.

div $\left(\frac{\vec{r}}{r^3}\right) = \nabla \cdot (r^{-3} \cdot \vec{r}) = \vec{\nabla} \cdot r^{-3}[x\hat{i}+y\hat{j}+z\hat{k}]$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[r^{-3}x\hat{i} + r^{-3}y\hat{j} + r\right.$

$= \frac{\partial}{\partial x} r^{-3} \cdot x + \frac{\partial}{\partial y} r^{-3} y + \frac{\partial}{\partial z} z.$

$$\frac{\partial}{\partial x}[r^{-3}x] = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-3/2}\cdot x\right]$$

$$= (x^2+y^2+z^2)^{-3/2} + x\cdot\frac{-3}{2}(x^2+y^2+z^2)^{-5/2}\cdot 2x$$

$$= (x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}$$

Similarly $\frac{\partial}{\partial y}[r^{-3}y] = (x^2+y^2+z^2)^{-3/2} - 3y^2(x^2+y^2+z^2)^{-5/2}$

$$\frac{\partial}{\partial z}(r^{-3}z) = (x^2+y^2+z^2)^{-3/2} - 3z^2(x^2+y^2+z^2)^{-5/2}.$$

$$\Rightarrow \frac{\partial}{\partial x}[r^{-3}x] + \frac{\partial}{\partial y}[r^{-3}y] + \frac{\partial}{\partial z}(r^{-3}z)$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2\right]$$

$3\cancel{r^2}$ $\quad a^m\cdot a^n = a^{m+n} \quad\quad -5/2+1 = -3/2$

$$3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$= 0$ Hence proved

Show that div $(r^n\vec{r}) = (3+n)\, r^n$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = (x^2+y^2+z^2)^{1/2}$

$$\text{div}(r^n\cdot\vec{r}) = \vec{\nabla}\cdot(r^n\cdot\vec{r})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(x^2+y^2+z^2)^{n/2}\cdot[x\hat{i}+y\hat{j}+z\hat{k}]\right]$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}\cdot x + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}\cdot y + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\cdot z$$

$$\Rightarrow \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}\cdot x = (x^2+y^2+z^2)^{n/2} + x\cdot\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2x$$

$$= (x^2+y^2+z^2)^{n/2} + x^2 n\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1} \quad \text{— (i)}$$

Similarly $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}\cdot y = (x^2+y^2+z^2)^{n/2} + y^2 n(x^2+y^2+z^2)^{\frac{n}{2}-1}$ — (ii)

$$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\cdot z = (x^2+y^2+z^2)^{n/2} + z^2 n(x^2+y^2+z^2)^{\frac{n}{2}-1} \quad \text{(iii)}$$

so add (i) (ii) & (iii)

$$\nabla(r^n\vec{r}) = 3(x^2+y^2+z^2)^{n/2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x^2+y^2+z^2\right]$$

$$= 3(x^2+y^2+z^2)^{n/2} + n(x^2+y^2+z^2)^{n/2}$$

$$\nabla(r^n\cdot\vec{r}) = (3+n)\left[(x^2+y^2+z^2)^{1/2}\right]^n = (3+n)\, r^n$$

$$\boxed{\nabla(r^n\cdot\vec{r}) = (3+n)\, r^n}$$