GRAVITATION

* Central force

$$\vec{F}(r) = f(r)\,(\pm\hat{r}) = \pm\frac{f(r)}{r}\cdot\vec{r}.$$

→ The force act on a particle in such a way directed towards a fixed point or away from a fixed point. & its magnitude only depends magnitude of dist. from fixed point

* Properties of central force.

* when a particle moving under influence of central force, than Torque acting on it is zero.

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times\frac{f(r)}{r}\cdot\vec{r} = \frac{f(r)}{r}\left[\vec{r}\times\vec{r}\right) = 0$$

* also angular momentum $(\vec{L})$ is conserved

* Central force motion is plane (2D) motion is $\hat{r}$ is always $\perp^r$ to $\vec{L}$.

→ $\vec{L} = \vec{r}\times\vec{p}$

$\vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times\vec{p})$

$\vec{r}\cdot\vec{L} = (\vec{r}\times\vec{r})\cdot P = 0$

$\vec{r}\cdot\vec{L} = 0$ mean vector is $\perp$

* Central force is a conservative force.

$$\boxed{F = -\frac{dU}{dr}}$$

$F = \pm \frac{K}{r^n}$ n^{th} power law of force

→ $\frac{+K}{r^n} = \frac{+dU}{dr}$

$\int du = \int K r^{-n} . dr$

$U = K \frac{r^{-n+1}}{-n+1} + C$

$U = K \frac{r^{-(n-1)}}{-(n-1)} + C$

$$\boxed{U = \frac{-K r^{-(n-1)}}{(n-1)} + C} \quad \boxed{U = \frac{-K}{(n-1) r^{(n-1)}}}$$

→ if $n = -1$

$$\boxed{F = + K r}$$

$U = \frac{-K r^2}{-2} = \frac{1}{2} K r^2$

* Gravitational Field Intensity

→ The intensity of gravitation field at a point is defined as the force experienced by unit mass, placed at that point

$\vec{F} = -G \frac{M m'}{r^2}$ $\vec{E} = \frac{\vec{F}}{m} = \frac{-GM}{r^2}$

$$\boxed{\vec{E} = \frac{-GM}{r^2}}$$

* Gravitational potential

$V = \int_{\infty}^{r} \frac{GM}{r^2} . dr$.

$V = GM \left[\frac{-1}{r} \right]_{\infty}^{r}$

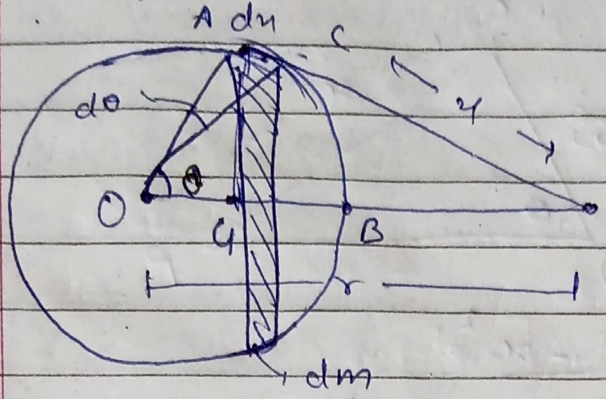
$V = GM \left[\frac{-1}{r} \right] = -\frac{GM}{r}$

$$\boxed{V = \frac{-GM}{r}}$$

→ Gravitational potential energy:

$U = \frac{-G m_1 m_2}{r_{12}}$

Gravitational Potential & field due to thin spherical·



$$dV = -\frac{G\,(\text{mass of the Ring})}{\text{distance}}$$

⇒ Surface mass density is const. of the sphere.

mass of the ring $= \sigma \cdot$ (area of the ring)

in Δ OCG $\quad \boxed{GC = R\sin\theta}$

AC

$\boxed{AC = R\,d\theta}$

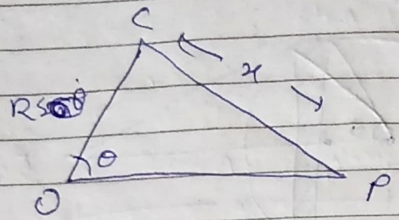
area of Ring $= 2\pi\,(R\sin\theta)\cdot R\,d\theta$
$= 2\pi R^2 \sin\theta \cdot d\theta$

mass of the ring $= \sigma \cdot \left[2\pi R^2 \sin\theta \cdot d\theta\right]$

$$dV = \frac{-G\,\sigma\,(2\pi R^2 \sin\theta \cdot d\theta)}{x} \quad \text{— (i)}$$

* Case I^{st} $r > R$.

in $\triangle OCP$



$$\cos\theta = \frac{R^2 + r^2 - x^2}{2Rr}$$

$$2Rr\cos\theta = R^2 + r^2 - x^2$$

$$\boxed{x^2 = R^2 + r^2 - 2Rr\cos\theta}$$

$$2x\,dx = 2Rr\sin\theta\cdot d\theta$$

$$\boxed{\sin\theta\cdot d\theta = \frac{x\,dx}{R\cdot r}}$$

now in eq (ii)

$$dv = -G\sigma\left(2\pi R^2 \frac{x\,dx}{R\cdot r}\right)\cdot\frac{1}{x}$$

$$dv = -\frac{G\sigma(2\pi R\,dx)}{r} \quad \text{— (iii)}$$

Case I $r > R$.

$$V = \int dv = -\frac{G\sigma 2\pi R}{r}\int_{r-R}^{r+R} dx.$$

$$V = -\frac{G\sigma 2\pi R}{r}\left[r + R - r + R\right]$$

$$V = -\frac{G\sigma 2\pi\cdot 2R^2}{r}$$

$$V = -\frac{G\sigma 4\pi R^2}{r}$$

$\sigma\cdot 4\pi R^2 =$ mass of the thin spherical shell

$$\boxed{V = -\frac{GM}{r}}$$ gravitational potential.

* gravitational field $= -\text{grad}\,V = -\left\{\frac{d}{dr}\frac{+GM}{r}\right\} = -\frac{GM}{r^2}$

$$\boxed{E = -\frac{GM}{r^2}}$$

Case 2^{nd} $r = R$.

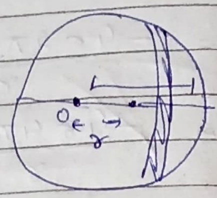
$$V = \int dV = \int \frac{-G\sigma 2\pi R}{\cancel{r}R}\,dx.$$

$$V = \int_0^{2R} -G\sigma 2\pi\,dx = -G\sigma 2\pi\left[2R\right] = -\frac{G\sigma 4\pi R^2}{R}$$

$$\boxed{V_B = -\frac{GM}{R}}$$

$$\boxed{E_B = -\frac{GM}{R^2}}$$

Case III $\boxed{r<R}$ inside.



$$V = \int dV = \int \frac{G\sigma 2\pi R\, dx}{r}$$

$$= \int_{R-r}^{R+r} \frac{G\sigma 2\pi R}{r} dx$$

$$-G\frac{\sigma 2\pi R}{r} \cdot [2r]$$

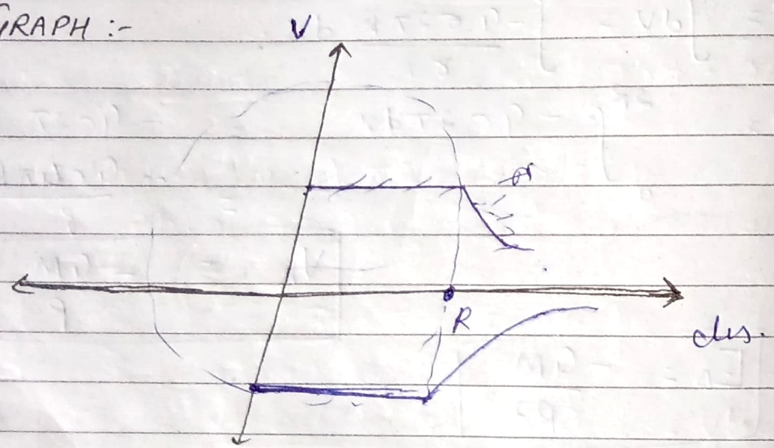
$$V = -\frac{G\sigma 4\pi R^2}{R}$$

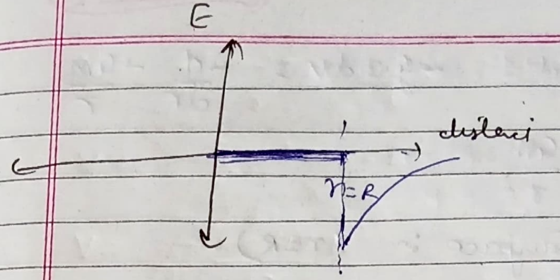
$$\boxed{V = -\frac{GM}{R}} \longrightarrow \text{Constant potential.}$$

E Gravitational field $= -\text{grad } V = +\frac{d}{dr}\frac{GM}{R} = 0.$

$$\boxed{E \text{ inside} = 0}$$

* GRAPH :-

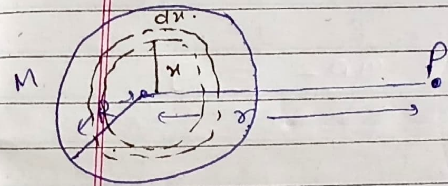




* Gravitational, potential & Field due to solid sphere.

Mass - M Volume = V.

volume mass density $\longrightarrow \boxed{\rho = \frac{M}{V}}$



1. Let us consider a thin spherical shell

$$dV = -\frac{G\,dm}{r}$$

$$dm = \rho \cdot V_{(\text{Thin shell})} = \rho dV = 4\pi x^2 . dx . \rho$$

$$dm = \rho . 4\pi x^2 . dx$$

$$dV = -G\frac{\rho 4\pi x^2 . dx}{r}$$

$$\therefore V = \int dV = \int_0^R \frac{-G\rho 4\pi x^2 dx}{r} = -\frac{G\rho 4\pi}{r}[$$

$$= -\frac{G\rho 4\pi R}{3\, r}$$

$$\boxed{V = -\frac{GM}{r}}$$

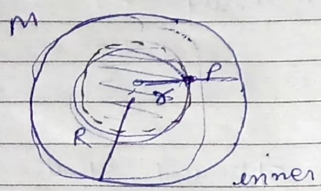
Gravitational field = $-\text{grad}\, V = \frac{-d}{dr}\,\frac{-GM}{r}$

$$\vec{E} = -\frac{GM}{r^2}$$

(ii) at the Surface :- $(r=R)$.

$V = -\frac{GM}{R}$ & $\vec{E} = \frac{-GM}{R^2}$

(iii) Potential at internal part of solid sphere ;



inner outer

net potential $= V_1 + V_2$

$V_1 = -\frac{G\,dm_1}{r}$

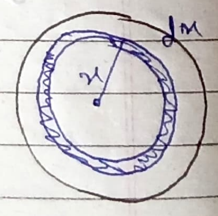
$dm_1 = \rho \cdot dV_1 = \rho \cdot \left(\frac{4\pi r^3}{3}\right)$

$V = -G\rho\left(\frac{4}{3}\pi r^2\right)$

$V_1 = -\frac{4}{3}\pi r^2 G\rho$

for V_2

$dm_2 = 4\pi x^2 \cdot dx$



$dV_2 = \frac{-G\cdot dm_2}{r} = \frac{-G\cdot 4\pi x^2 \cdot dx\,\rho}{x}$

$dV_2 = \int_r^R -G\pi x\,dx\,\rho$

$V_2 = \int_r^R -G\pi\rho\,x\,dx = -G4\pi\rho\left[\frac{x^2}{2}\right]_r^R$

$V_2 = -G4\pi\rho\left[\frac{R^2-r^2}{2}\right]$

$V_2 = -\frac{G4\pi\rho}{3}\left[\frac{3R^2}{2} - \frac{3r^2}{2}\right]$

$V = V_1 + V_2 = -\left[\frac{4}{3}\pi r^2 G\rho + \frac{G4\pi\rho}{3}\left(\frac{3R^2}{2} - \frac{3r^2}{2}\right)\right]$

$V = -\left[\frac{4}{3}G\rho\pi\left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right]\right]$

$V = -\left(\frac{4}{3}\pi\rho G\left[\frac{3R^2 - r^2}{2}\right]\right)$

$$\boxed{V_{insi} = -\frac{4}{3}\pi\rho G\left(\frac{3R^2 - r^2}{2}\right)}$$

$V_{ins} = \left(-\frac{4}{3}\pi R^3\rho\right)G\left[\frac{3R^2 - r^2}{2R^3}\right]$

$$\boxed{V_{ins} = GM\left(-\left(\frac{3R^2 - r^2}{2R^3}\right)\right)} \rightarrow -GM\left(\frac{3R^2 -}{2R}\right.$$

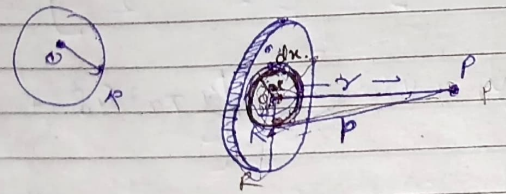
Gravitational field $= -\frac{dV}{dr}$

$$= -\frac{d}{dr}\left[-GM\left[\frac{3R^2-r^3}{2R^3}\right]\right]$$

$$= GM\left[\frac{-r}{R^3}\right]$$

$$= \boxed{\frac{-GMr}{R^3}}$$

* Gravitational potential & field at a point on the axis of disk.



Gravitational potential at point P due to elementary ring $dV = \frac{-G(\text{mass of Ring})}{\text{Distance}} = \frac{-G\,dm}{p}$

Surface mass density $\sigma = \frac{M}{A}$

mass of element ring $= \sigma\,(2\pi x).dx$

area of ds

$p = \sqrt{x^2+r^2}$

$$dV = \frac{-G\,(\sigma\,2\pi x\,dx)}{\sqrt{x^2+R^2}}$$

$$V = \int dV = -\int_0^R G\sigma 2\pi \frac{x\,dx}{\sqrt{x^2+R^2}}$$

$$V = -G\sigma 2\pi \int_0^R \frac{x\,dx}{\sqrt{x^2+R^2}}$$

$x^2 + R^2 = t$

$2x = \frac{dt}{dx}$

$dx = \frac{dt}{2x}$

$x\,dx = \frac{dt}{2}$

$$V = -G\sigma 2\pi \frac{1}{2}\int_0^R \frac{dt}{\sqrt{t}}$$

$$V = -G\sigma\pi\left[2\sqrt{t}\right]_0^R$$

$$V = -G\sigma\pi\left[\sqrt{x^2+R^2}\right]_0^R$$

$\int \frac{1}{\sqrt{t}} = 2\sqrt{t}$

~~$V = -G\sigma\pi\left[\sqrt{2R^2}\right.$~~

~~$V = -G\sigma\pi\left[R\sqrt{2} - R\right]$~~

~~$V = -G\pi\sigma$~~

$E = -\text{grad}\,V = \frac{d}{dr}V$

$$V = -G\sigma\pi 2\left(\sqrt{R^2+r^2} - \sqrt{r^2}\right)$$

$$V = -G\sigma\pi\left[\sqrt{R^2+r^2} - r\right] \rightarrow \text{gravitational potential}$$

$$E = -\frac{dV}{dr} = G\sigma\pi\frac{d}{dr}\left(\sqrt{R^2+r^2} - r\right)$$

$$= G\sigma 2\pi\left[\frac{2r}{2\sqrt{R^2+r^2}} - 1\right]$$

$$E = G\sigma\pi\left[\frac{r}{\sqrt{R^2+r^2}} - 1\right]$$

$$E = -G\sigma 2\pi\left[1 - \frac{r}{\sqrt{R^2+r^2}}\right]$$

GRAVITATIONAL SELF ENERGY

A body may considered to be formed infinitesimaly small element, infinite dist. from each other

The G. self energy is defined as the work done on assembling the body from infinitesimaly small elements which are distributed initially at ∞.

The potential energy of body itself called, Gravitational self energy.

Consider a body is formed from 'n' atoms

$m_1 - a_1, a_2 \; m_2 \; \text{---} \; a_n - m_n$

& let r_{ij} is separation btw a_i^{th} & a_j^{th} atom.

GPE btw a_1 & a_2 $= -G \frac{m_1 m_2}{r_{12}}$

" " a_2 & a_3 $= -\frac{G m_2 m_3}{r_{23}}$

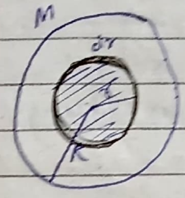
GPE btw a_i & a_j $= \frac{-G m_i m_j}{r_{ij}}$

GP self Energy of the body

$$= -G\frac{m_1 m_2}{r_{12}} - \frac{G m_2 m_3}{r_{23}} + \cdots - \frac{G m_i m_j}{r_{ij}}$$

$$\boxed{U = -\sum_{i>j=1}^{n} G \frac{m_i m_j}{r_{ij}}}$$

"GRAVITATIONAL SELF ENERGY" of "UNIFORM SOLID SPHERE"



G.P.E = (Grav. Potential) × mass

$dU_s =$

• Consider a spherical core of radius 'r' of mass $= \frac{4}{3}\pi r^3 . (\rho)$ volume mass density

→ let dr → Thickness of layer

Mass of layer $= \rho . (4\pi r^2) . dr$ $\quad = -\frac{G m_1 m_2}{r}$

Gravitation Potential Energy = Potential × mass

= potential of inner core × mass layer

$$dU = \frac{-G\,(\text{mass})}{r} . \rho(4\pi r^2)\,dr$$

$$= -G . \frac{\frac{4}{3}\pi r^3 \rho . \rho(4\pi r^2) . dr}{r}$$

$$du \Rightarrow -G \frac{16}{3}\pi^2 \rho^2 r^4 . dr$$

$$U = \int du = \int_0^R \frac{16}{3}\pi^2 \rho^2 r^4 . dr$$

$$= -G\frac{16}{3}\pi^2 \rho^2 \frac{R^5}{5} \times \frac{3R}{3R} \quad \left(\frac{4\pi r^3 \rho}{3}\right)$$

$$U = -\frac{G M^2 3}{5R} = \boxed{-\frac{3}{5}\frac{G M^2}{R}}$$

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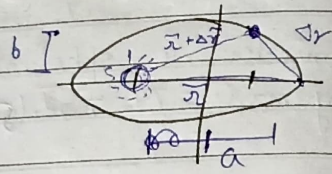
Kepler's Law of planetary motion :-

(I) Law of elliptical orbit.

- all the planets moves around the sun / star in an elliptical ~~par~~ orbit & the sun is being at one of the focii.



(II) Law of area :-

The radious vector of any planet relative to sun sweeps out equal area in equal interval of time [Areal velocity remains constant]

$\boxed{\frac{\Delta A}{\Delta t} = \frac{L}{2m}}$ $\left[\frac{\Delta \vec{A}}{\Delta T} = \text{areal velocity}\right]$

(III) Harmonic Law :-

The square of period of revolution of any planet around the sun is proportional to cube of the semi major axis of the elliptical orbit.

$\boxed{T^2 \propto a^3}$

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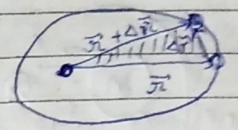
Proofs of Kepler's 2nd Law [Law of area]



area of $\Delta = \frac{1}{2}|\vec{A}\times\vec{B}|$

area of the $\Delta\vec{A} = \frac{1}{2}\vec{r}\times\Delta\vec{r}$
triangle

dividing both side by Δt

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r}\times\frac{\Delta\vec{r}}{\Delta t} \qquad \left[\frac{\Delta\vec{r}}{\Delta t} = \vec{v}\right]$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r}\times\vec{v}$$

$\Delta\vec{A}$ $m \to$ multiply & divide

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\frac{\vec{r}\times m\vec{v}}{m} = \frac{1}{2}\frac{\vec{r}\times\vec{p}}{m}$$

$\vec{L} = \vec{r}\times\vec{p}$
↳ (angular momentum)

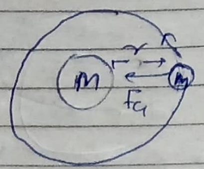
$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m}$$

$\Delta t \to 0 \qquad \frac{\Delta\vec{A}}{\Delta t} \to \frac{d\vec{A}}{dt}$

$$\frac{dA}{dt} = \frac{\vec{L}}{2m}$$ → const. here $\vec{L}$ & m are constant

→ con

3rd Law proof [Harmonic Law] [Not exact proof] because we assume as circular orbit



also in a circular motion object experience a centripetal force. which is

$$F_c = \frac{mv^2}{r}$$

& this force is equal to Gravitational force

$$F_G = F_c$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v^2 = \frac{GM}{r}$$

we know that $v = r\omega$

$$r^2\omega^2 = \frac{GM}{r}$$

$$r^3 \cdot \left(\frac{2\pi}{T}\right)^2 = GM \qquad \because \omega = \frac{2\pi}{T}$$

$$\rightarrow \quad r^3 \cdot \frac{4\pi^2}{T^2} = GM$$

$$r^3 = T^2 \cdot \left(\frac{GM}{4\pi^2}\right)$$

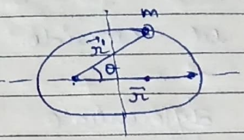
Here $\frac{GM}{4\pi^2}$ is constant

$\boxed{r^3 \propto T^2}$ Hence prove

Proof of Kepler's 1st Law 8n.

→ Proof with the help of Newtons Laws.

$$\vec{F_r} = -\frac{GMm}{r^2} \cdot \hat{r} \quad \text{— (i)}$$



A/c to Newton's 2nd Law in radial direction.

$$\vec{F_r} = m\vec{a_r} = m a_r \hat{r}$$

$$\vec{a_r} = \text{radial acceleration} = \ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$\vec{F_r} = \left[m \cdot \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\right] \cdot \hat{r} \quad \text{— (ii)}$$

Equate eq (i) & eq. (ii)

$$= \frac{-GMm}{r^2} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

$$= \frac{-GM}{r^2} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

Both Side Multiply by $[r^3]$:-

$$-GMr = r^3\left(\frac{d^2r}{dt^2}\right) - r^4\left(\frac{d\theta}{dt}\right)^2$$

put $r^4\left(\frac{d\theta}{dt}\right)^2 = h^2$

$\boxed{h = \omega r^2}$ $\quad \boxed{\frac{d\theta}{dt} = \omega}$

$$= r^3 \frac{d^2r}{dt^2} - h^2 = -GMr \quad \text{—— (iii)}$$

$$h = \omega r^2 \quad \text{—— (iv)} \qquad \boxed{\omega = \frac{h}{r^2}} \quad \text{—— (v)}$$

let consider $\boxed{r = \frac{1}{u}}$

differentiate w.r.t t.

$$\frac{dr}{dt} = \frac{-1}{u^2} \cdot \frac{du}{dt}$$

→ multiply by $\frac{d\theta}{d\theta}$ in R.H.S.

$$\frac{dr}{dt} = \frac{-1}{u^2} \cdot \frac{du}{dt} \cdot \frac{d\theta}{d\theta} = \frac{-1}{u^2} \left(\frac{du}{d\theta}\right) \cdot \left(\frac{d\theta}{dt}\right)$$

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \cdot (\omega) = \frac{-1}{u^2} \frac{du}{d\theta} \cdot \frac{h}{r^2}$$

from eq (v)

& also $r = \frac{1}{u} \Rightarrow u^2 = \frac{1}{r^2}$

$$\frac{dr}{dt} = -\frac{1}{\cancel{u^2}} \cdot \frac{du}{d\theta} \cdot h \cancel{u^2} = -h \cdot \frac{du}{d\theta}$$

$$\frac{dr}{dt} = -h \frac{du}{d\theta}$$

Second derivative ::

multiply by $\frac{d\theta}{d\theta}$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left[-h \frac{du}{d\theta}\right] \times \left(\frac{d\theta}{d\theta}\right)$$

$$= -h \frac{d^2u}{d\theta^2} \cdot \left(\frac{d\theta}{dt}\right)$$

$$\omega = \frac{h}{r^2} = h \cdot \frac{1}{r^2} = \boxed{h \cdot u^2}$$

$$\frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2}\, \omega = -h \frac{d^2u}{d\theta^2} \cdot (hu^2)$$

$$\frac{d^2r}{dt^2} = -h^2u^2 \frac{d^2u}{d\theta^2} \quad \text{—— eq. (vi)}$$

Put eq vi in equation (iii)

$$r^3 \frac{d^2r}{dt^2} - h^2 = -GMr$$

$$\Rightarrow r^3 \cdot \left[-h^2u^2 \frac{d^2u}{d\theta^2}\right] - h^2 = -GMr.$$

$$\Rightarrow \frac{1}{u^3}\left[-h^2u^2 \frac{d^2u}{d\theta^2}\right] - h^2 = -GMr$$

$$= \frac{-h^2}{u} \frac{d^2u}{d\theta^2} - h^2 = -GMr$$

$$= -\frac{h^2}{u}\left[\frac{d^2u}{d\theta^2} + u\right] = -\frac{GM}{u}$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}} \quad \text{differential eq.}$$

Const.

$\frac{GM}{h^2}$ is constant Term. so we can write

$$\frac{d^2}{d\theta^2}\left[u - \frac{GM}{h^2}\right] + \left(u - \frac{GM}{h^2}\right) = 0$$

$$\frac{d^2}{d\theta^2}\left[u - \frac{GM}{h^2}\right] + u - \frac{GM}{h^2} = 0$$

let $u - \frac{GM}{h^2} = x.$

$$\frac{d^2 x}{d\theta^2} + x = 0$$

Solution of Homogeneous differential equation

$x = A\cos\theta$

$$u - \frac{GM}{h^2} = A\cos\theta$$

$$u = A\cos\theta + \frac{GM}{h^2}.$$

$$\frac{1}{r} = A\cos\theta + \frac{GM}{h^2}$$

multiply bot side by $\frac{h^2}{GM}$.

$$\frac{h^2}{GM.r} = 1 + \frac{Ah^2\cos\theta}{GM.} \longrightarrow \text{eq. (vii)}.$$

Compare eq vii with genral eq. of conic section

$$\left(\frac{l}{r} = 1 + e\cos\theta.\right)$$

* $e = \frac{Ah^2}{GM}$ = eccentricity   for ellips $0<e<1$

$$\boxed{l = \frac{h^2}{GM}}$$

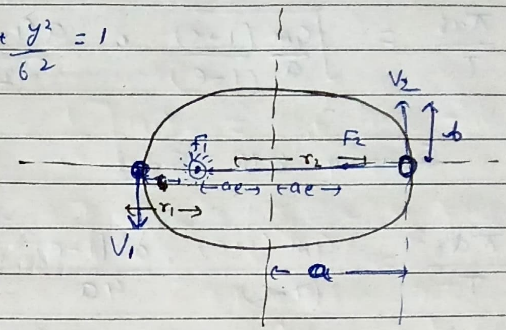
Proof of kepler's IIIrd law :-

→ $T^2 \propto a^3$

Ellips :- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

e- eccentricity

$0<e<1$



** $b^2 = a^2[1-e^2]$

r_1 → nearest / smallest dist. of planet & sun.

$\boxed{r_1 = a - ae.}$

r_2 → fenthest $\boxed{r_2 = a + ae}$

* V_1 is more than V_2 because of ~~Conse~~ the angular momentum is same / constant.

$$V_1 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)} \qquad V_2 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

→ using kepler's IInd law [areal velocity same / constant]

$$\text{area velocity} = \frac{\text{area}}{\text{Time}} = \frac{L}{2m} = \frac{mvr}{2m}$$

area ⟶ area of ellips = πab

$$\frac{\pi ab}{T} = \frac{m V_1 r_1}{2m}$$ ⟶ at ~~po~~ nearest point (becau... angular mome... is const. where so w... Take a poi...

$$\frac{\pi a b}{T} = \frac{v_1 r_1}{2}$$

$$\frac{\pi a b}{T} = \sqrt{\frac{GM}{a}\frac{(1+e)}{(1-e)}} \cdot \frac{a(1-e)}{2}$$

Squaring both Sides

$$\frac{\pi^2 a^2 b^2}{T^2} = \frac{GM(1+e)}{\cancel{(1-e)}} \cdot \frac{a^2(1-e)^2}{4a}$$

using Relation $b^2 = a^2[1-e^2]$

$$\frac{\pi^2 b^2}{T^2} = \frac{GM(1+e)(1-e)}{4a}$$

$$\frac{\pi^2[a^2\cancel{(1-e^2)}]}{T^2} = \frac{GM\cancel{(1-e^2)}}{4a}$$

$$\frac{\pi^2(4a^3)}{T^2} = GM$$

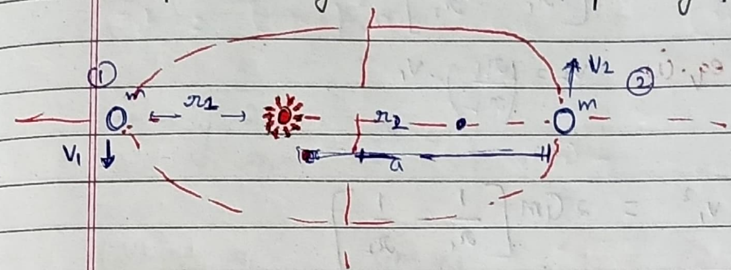
$$a^3 = \left[\frac{GM}{4\pi^2}\right] \cdot T^2$$

$\rightarrow$ Constant

$$T^2 = a^3 \cdot \left[\frac{4\pi^2}{GM}\right]$$

$\boxed{T^2 \propto a^3}$ Hence proved

→ Expression of min & max speed of planet;



applying conservation of angular momentum at point (1) & (2)

$$(L)_1 = (L)_2$$

$$m V_1 r_1 = m V_2 r_2$$

$$\boxed{V_2 = \frac{V_1 r_1}{r_2}} \longrightarrow \text{eq (1)}$$

applying conservation of energy at 1 & 2 about Sun.

$$(T.E)_1 = (T.E)_2$$

$$[K.E_1 + U_1] = [K.E_2 + U_2]$$

$$\Rightarrow \frac{1}{2} m V_1^2 - \frac{GMm}{r_1^2} = \frac{1}{2} m V_2^2 - \frac{GMm}{r_2^2}$$

$$\frac{V_1^2 - V_2^2}{2} = \frac{GM}{r_1^2} - \frac{GM}{r_2^2}$$

$$V_1^2 - V_2^2 = 2GM\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right]$$

$$v_1^2 - v_2^2 = 2GM\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Now from eq. (i) $v_2 = \left(\frac{r_1}{r_2}\right) \cdot v_1$

$$v_1^2 - \frac{r_1^2}{r_2^2} v_1^2 = 2GM\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$v_1^2\left(\frac{r_2^2 - r_1^2}{r_2^2}\right) = 2GM\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

$$\frac{v_1^2 (r_2 - r_1)(r_2 + r_1)}{r_2} = 2GM(r_2 - r_1)$$

$\because a^2 - b^2 = (a+b)(a-b)$

$$\frac{v_1^2 (r_1 + r_2)}{r_2} = \frac{2GM}{r_1}$$

$$v_1^2 = \frac{2GM r_2}{r_1 + r_2}$$

$$v_1 = \sqrt{\frac{2GM}{r_1 + r_2} \cdot \frac{r_2}{r_1}}$$

$r_1 = a(1-e)$ $\quad r_2 = a(1+e)$

$$v_1 = \sqrt{\frac{2GM}{2a} \frac{1+e}{1-e}} = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

$$\boxed{v_2 = \frac{r_1}{r_2} v_1}$$ from eq (i)

$$v_2^2 = \frac{r_1^2}{r_2^2} \frac{GM}{a}\left[\frac{1+e}{1-e}\right]$$

$$\cancel{v_2^2 = \frac{r_1^2}{r_2^2} \frac{2GM}{(r_1 + r_2)} \frac{r_2}{r_1}}$$

$$v_1 = \sqrt{\frac{2GM}{r_1 + r_2} \cdot \frac{r_2}{r_1}}$$

$$v_2^2 = \frac{r_1^2}{r_2^2} \cdot \frac{2GM r_2}{(r_1 + r_2) r_1}$$

$$v_2 = \sqrt{\frac{2GM}{(r_1 + r_2)} \cdot \frac{r_1}{r_2}}$$

$$v_2 = \sqrt{\frac{2GM}{2a} \frac{a(1-e)}{a(1+e)}} = \sqrt{\frac{GM}{a} \frac{(1-e)}{(1+e)}}$$

v_2 is minimum & v_1 is maximum.

⇒ Total Energy of a planet &r.

T.E = K.E + P.E

$$E = \frac{1}{2} m v_1^2 - \frac{GMm}{r_1}$$

Since total energy is conse[rved] it is equal at every poi[nt]

* at nearest point.

$$v_1 = \sqrt{\frac{2GM}{(r_1 + r_2)} \frac{r_2}{r_1}}$$

$$E = \frac{1}{2} m \frac{2GM r_2}{(r_1 + r_2) r_1} - \frac{GMm}{r_1}$$

$$E = \frac{GMm}{r_1}\left[\frac{r_2}{r_1 + r_2} - 1\right] = \frac{GMm}{r_1}\left[\frac{-r_1}{r_1 + r_2}\right]$$

$$\boxed{T.E = \frac{-GMm}{r_1 + r_2}}$$

$r_1 = a(1-e)$

$r_2 = a(1+e)$

$r_1 + r_2 = 2a.$

$r_1 \cdot r_2 = a^2(1-e^2) = b^2$

$\boxed{r_1 \cdot r_2 = b^2}$

$$\boxed{\text{Total energy} = \frac{-GMm}{2a}}$$

* Velocity of planet at any instant

$$T.E = K.E + P.E$$

$$\frac{-GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{v^2}{2} = GM\left[\frac{1}{r} - \frac{1}{2a}\right]$$

$$v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{2a}\right)} = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

* Angular Momentum :-

* it is conserved, it's value remains same at all points.

$$|L| = mvr \sin 90^\circ = mvr.$$

$$|L| = mv_1 r_1 = m\sqrt{\frac{2GM}{(r_1+r_2)} \cdot \frac{r_2}{r_1}} \cdot r_1 =$$

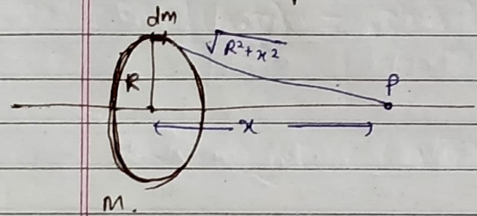
$$|L| = \sqrt{\frac{2GM r_1 \cdot r_2}{(r_1+r_2)}}$$

$$|L| = \sqrt{\frac{2GMb^2}{2a}} = m\sqrt{\frac{GMb^2}{a}}$$

$$|L| = \text{angular momentum} = m\sqrt{\frac{GMb^2}{a}}$$

Gravitational potential & field due to Thin Ring.

* At Axial position.



gravitational potential due to small length element of mass dm at a point 'p'.

$$dV = \frac{-Gm}{r} = \frac{-Gdm}{\sqrt{R^2+x^2}}$$

$$V = \int dV = -\int \frac{G\,dm}{\sqrt{R^2+x^2}} = -\frac{G}{\sqrt{R^2+x^2}} \int dm = \frac{-GM}{\sqrt{R^2+x^2}}$$

$$\boxed{V = \frac{-GM}{\sqrt{R^2+x^2}}}$$

∵ Gravitational field due to Ring

$$E = -\frac{dV}{dr}, = \frac{-dV}{dx} = +\frac{d}{dx}\frac{GM}{\sqrt{R^2+x^2}}$$

$$E = GM \cdot \left(-\frac{1}{2}\right) \frac{1}{(x^2+R^2)^{3/2}} \cdot 2x$$

$$E = -\frac{GM}{\cancel{2}}\left(R^2+x^2\right)^{-3/2} \cdot \cancel{2}x$$

$$\boxed{E = \frac{-GMx}{\left(R^2+x^2\right)^{3/2}}} \rightarrow \text{Gravulatenal Field}$$

→ At center $x = 0$.

$$V_{cen} = -\frac{GM}{\sqrt{R^2+x^2}} = -\frac{GM}{R}$$

& $E_{centr} = 0$