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Lecture R – Review

Section R.1 – Basic Algebra Review

R.1–1 Adding and Subtracting Polynomials

Properties of Real numbers

For all real numbers a, b, and c:

$a+b=b+a$ $ab=ba$	***Commutative properties***
$(a+b)+c=a+(b+c)$ $(ab)c=a(bc)$	***Associative properties***
$a(b+c)=ab+ac$	***Distributive properties***

***Example* 1**

Perform the indicated operation:

$$\begin{aligned}\left(8x^3-4x^2+6x\right)+\left(3x^3+5x^2-9x+8\right)&=8x^3-4x^2+6x+3x^3+5x^2-9x+8\\&=\left(8x^3+3x^3\right)+\left(-4x^2+5x^2\right)+(6x-9x)+8\\&=11x^3+x^2-3x+8\end{aligned}$$

R.1–2 Multiplication Polynomials

***Example* 2**

Perform the indicated operation:

$$\begin{aligned}8x(6x-4)&=8x(6x)-8x(4)\\&=48x^2-32x\end{aligned}$$

Example 3

Perform the indicated operation:

$$(3p-2)\left(p^2+5p-1\right)=3p^3+15p^2-3p-2p^2-10p+2$$

$$=3p^3+13p^2-13p+2$$

Example 4

Perform the indicated operations:

$$2\left(3x^2+4x+2\right)-3\left(-x^2+4x-5\right)=6x^2+8x+4+3x^2-12x+15$$

$$=9x^2-4x+19$$

R.1–3 Factoring

Prime Factorization

A process that allows us to write a composite number as a product of two or more prime numbers.

Factoring Trinomial

Example 5

Factor $y^2+8y+15$

Solution

Product 15	***Sum*** 8
15 *x* 1	15 + 1
3 *x* 5	3 + 5

$$y^2+8y+15=(y+3)(y+5)$$

Special Factorization

$a^2 - b^2 = (a-b)(a+b)$ $\qquad$ $a^2 - 2ab + b^2 = (a-b)^2$

$a^2 + 2ab + b^2 = (a+b)^2$

$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$ $\qquad$ $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

R.1–4 Foiling

Pascal's Triangle

								1								
							1		**1**							
						1		**2**		**1**						
					1		**3**		**3**		**1**					
				1		**4**		**6**		**4**		**1**				
			1		**5**		**10**		**10**		**5**		**1**			
		1		**6**		**15**		**20**		**15**		**6**		**1**		
	1		**7**		**21**		**35**		**35**		**21**		**7**		**1**	
1		**8**		**28**		**56**		**70**		**56**		**28**		**8**		**1**

$$(a+b)^n = a^n + *a^{n-1}b + *a^{n-2}b^2 + \ldots + *a^2b^{n-2} + *ab^{n-1} + b^n$$

Example 6

Use Pascal's triangle to expand $(a+b)^7$

Solution

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

R.1–5 Exponents

Integer Exponents

Definition of exponent

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n-times}$$

a appears as a factor n times

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n} \qquad a^{m/n} = \left(a^{1/n}\right)^m$$

$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (ab)^m = a^m b^m$$

$$\left(a^m\right)^n = a^{mn} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad \left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$$

Example 7

Simplify $27^{2/3}$

Solution

$$\begin{aligned} 27^{2/3} &= \left(27^{1/3}\right)^2 \\ &= \left(\left(3^3\right)^{1/3}\right)^2 \\ &= \left(3^{3.\frac{1}{3}}\right)^2 \\ &= (3)^2 \\ &= 9 \end{aligned}$$

R.1–6 Radical

$$a^{1/n} = \sqrt[n]{a}$$

Properties

$$\left(\sqrt[n]{a}\right)^n = a \qquad \sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \qquad \sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

Example 7

Simplify $\dfrac{1}{1-\sqrt{2}}$

Solution

$$\frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}}\,\frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$= \frac{1+\sqrt{2}}{1-2}$$

$$= \frac{1+\sqrt{2}}{-1}$$

$$= -1-\sqrt{2}$$

R.1–7 Cartesian / Rectangular Coordinates

In geometry, a point is a location that defines a point. A point has no length or no size. A point is notated by a dot.

Example 8

Draw the point $(3,\ 2)$

Solution

$(3,\ 2)$

R.1–8 Lines

A ***linear*** equation, in mathematics, is an equation between two points (x_1, y_1) and (x_2, y_2) that give a straight ***line*** when plotted on a graph.

A line can be written as $y = mx + b$

m: is the slope can be found by: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

b: is y-intercept.

The general line formula can be written: $y = m(x - x_1) + y_1$

A *line* ***segment*** is a piece or part of a line having two endpoints.
Unlike a line, a line segment has a definite length.

Notes:

Two lines said to be parallel $(\ //\ )$ if the slopes are equal $(m_1 = m_2)$

Two lines said to be parallel $(\ \perp\ )$ if the slopes $m_1 \bullet m_2 = -1$

Distance

A ***distance***, d, between the two points in the rectangular coordinate system is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To complete the distance between two points. Find the square of the difference between the x-coordinate plus the square of the difference between the y-coordinates. The principal square root of this sum is the distance.

Midpoint

Consider a line segment whose endpoints are (x_1, y_1) and (x_2, y_2). The coordinates of the segment's midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

To find the midpoint, take the average of the two x-coordinates and the average of the y-coordinates

Example 9

For the given 2 points: $A(2,\ 3)$ & $B(-3,-1)$

1. Find an equation of a line that passes through the 2 points *A* & *B*. Then, plot the line
2. Find the distance between the 2 points.
3. Find the middle point.
4. Plot the segment *AB*.

Solution

a) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-1-3}{-3-2}$

$= \dfrac{-4}{-5}$

$= \dfrac{4}{5}$

$y = \dfrac{4}{5}(x-2)+3$ $\qquad$ $y = m(x - x_1) + y_1$

$= \dfrac{4}{5}x - \dfrac{8}{5} + 3$

$= \dfrac{4}{5}x + \dfrac{7}{5}$

Or $\quad 5y - 4x = 7$

b) The distance between $A(2,\ 3)$ & $B(-3,-1)$ is:

$d = \sqrt{(-2-3)^2 + (-1-3)^2}$ $\qquad$ $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-5)^2 + (-4)^2}$

$= \sqrt{25+9}$

$= \sqrt{34} \quad unit$

c) Let M be the middle point between $A(2,\ 3)$ & $B(-3,-1)$

$$M = \left(\frac{2-3}{2},\ \frac{3-1}{2}\right) \qquad M = \left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$$

$$= \left(-\frac{1}{2},\ 1\right)$$

d)

R.1–9 The Standard Form of the Equation of a *Circle*

The *standard form of the equation of a circle* with center $(h,\ k)$ and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

A circle with center $(0,\ 0)$ and radius r has equation: $x^2 + y^2 = r^2$

Example 10

Write in a standard form of a circle center at $(-3,4)$, radius 6.

Solution

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-3))^2+(y-4)^2=6^2$$

$$(x+3)^2+(y-4)^2=36$$

Exercises *Section R.1* – Basic Algebra Review

$(\mathbf{1-7})$ Expand and simplify:

1. $(4x-y)^3$

2. $\left(\sqrt{x}-\sqrt{3}\right)^4$

3. $(ax+by)^5$

4. $(x+y)^6$

5. $(2x+5y)^7$

6. $(a+b)^8$

7. $\left(x-\dfrac{1}{x^2}\right)^9$

$(\mathbf{8-11})$ Find a line equation that passes through the given points

8. $P(-4,\ 3)$ & $Q(2,\ -5)$

9. $P(8,\ 2)$ & $Q(3,\ 5)$

10. $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

11. $(-4,\ 9)$ & $(1,\ -3)$

$(\mathbf{12-15})$ Find the distance between the two given points

12. $P(-4,\ 3)$ & $Q(2,\ -5)$

13. $P(8,\ 2)$ & $Q(3,\ 5)$

14. $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

15. $(-4,\ 9)$ & $(1,\ -3)$

$(\mathbf{16-19})$ Find the midpoint of the line segment with endpoints

16. $P(-2,\ -1)$ & $Q(-8,\ 6)$

17. $P(8,\ 2)$ & $Q(3,\ 5)$

18. $(1,\ 2)$ & $(7,\ -3)$

19. $(7,\ -2)$ & $(9,\ 5)$

$(\mathbf{20-23})$ Write the standard form of the equation of the circle

20. center $\left(-\sqrt{3},-\sqrt{3}\right)$, radius $\sqrt{3}$.

21. center $(-5,-3)$ and $r=\sqrt{5}$.

22. center $(6,\ -5)$ that passes through $(1,\ 7)$

23. diameter whose endpoints are $(4,\ 4)$ and $(-2,\ 3)$.

Section R.2 – Solving Equations

The numbers of solutions to a polynomial with ***n*** degree, where ***n*** is Natural Number, are ***n*** solutions.

R.2–1 Definition of a Linear Equation

A linear equation in one variable x is an equation that can be written in the form

$$ax + b = 0$$

where a and b are real number, and $a \neq 0$

Addition and Multiplication Properties of Equalities

$$\text{If } a = b, \text{ then } a + c = b + c$$

$$\text{If } a = b, \text{ then } ac = bc$$

Example 1

Solve: $3(2x-4) = 7-(x+5)$

Solution

$$6x - 12 = 7 - x - 5$$
$$6x - 12 + x = 2 - x + x$$
$$7x - 12 = 2$$
$$7x - 12 + 12 = 2 + 12$$
$$7x = 14$$
$$\frac{7}{7}x = \frac{14}{7}$$
$$\underline{x = 2}$$

Example 2

Solve: $\frac{x}{x-2} = \frac{2}{x-2} - \frac{2}{3}$

Solution

$$3(x-2)\frac{x}{x-2} = 3(x-2)\frac{2}{x-2} - (x-2)\frac{2}{3}$$

Restriction: $\boldsymbol{x \neq 2}$

$$3x = 6 - 2(x-2)$$
$$3x = 6 - 2x + 4$$
$$3x + 2x = 10$$
$$5x = 10$$
$$x = \frac{10}{5}$$
$$= 2$$

No Solution ***or*** $\{\varnothing\}$, since the restriction $x \neq 2$

R.2–2 The *Square Root* Property

If u is an algebraic expression and d is a nonzero real number, then $u^2 = d$ has exactly two solutions:

$$\textit{If} \quad u^2 = d, \textit{ then} \quad u = \sqrt{d} \quad \textit{or} \quad u = -\sqrt{d}$$

Equivalently,

$$\textit{If} \quad u^2 = d \;\Rightarrow\; u = \pm\sqrt{d}\,.$$

Example 3

Solve $3x^2 - 21 = 0$

Solution

$$3x^2 = 21$$
$$x^2 = 7$$
$$x = \pm\sqrt{7}$$

R.2–3 Completing the Square

If $x^2 + bx$ is a binomial, then by ***adding*** $\left(\frac{b}{2}\right)^2$ which is the square of half the coefficient of x. a perfect square trinomial will result. That is.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \qquad\qquad x^2 + bx + \left(\frac{1}{2}b\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 4

Solve: $x^2 + 4x - 1 = 0$

Solution

$$x^2 + 4x = 1$$

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = 1 + \left(\frac{4}{2}\right)^2$$

$$x^2 + 4x + (2)^2 = 1 + 4$$

$$(x+2)^2 = 5$$

$$x + 2 = \pm\sqrt{5}$$

$$\underline{x = -2 \pm \sqrt{5}}$$

R.2–4 Quadratic Formula

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5

Solve $x^2 - 4x = -2$

Solution

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \qquad \Rightarrow a = 1 \quad b = -4 \quad c = 2 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 8}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2}$$

$$= \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$

$$\underline{= 2 \pm \sqrt{2}}$$

$$ax^2 + bx + c = 0$$

If **a + b + c = 0** $\Rightarrow$ $x = 1, \ \frac{c}{a}$

***Example* 6**

$$2x^2 + x - 3 = 0$$
$$2 + 1 - 3 = 0$$
$$\Rightarrow \ x = 1, \ -\frac{3}{2}$$

If **a – b + c = 0** $\Rightarrow$ $x = -1, \ -\frac{c}{a}$

***Example* 7**

$$2x^2 - x - 3 = 0$$
$$2 - (-1) - 3 = 0$$
$$\Rightarrow \ x = -1, \ \frac{3}{2}$$

R.2–5 Solving a Polynomial Equation by factoring

***Example* 8**

Solve: $4x^4 = 12x^2$

Solution

$$4x^4 - 12x^2 = 0$$

$$4x^2\left(x^2 - 3\right) = 0$$

$4x^2 = 0$	$x^2 - 3 = 0$
$x^2 = 0$	$x^2 = 3$
$\rightarrow \ x = 0, 0$	$x = \pm\sqrt{3}$

Example 9

Solve: $2x^3 + 3x^2 = 8x + 12$

Solution

$2x^3 + 3x^2 - 8x - 12 = 0$ *Factor by grouping*

$x^2(2x+3) - 4(2x+3) = 0$

$(2x+3)(x^2-4) = 0$

$2x + 3 = 0$ $\quad$ $x^2 - 4 = 0$

$2x = -3$ $\quad$ $x^2 = 4$

$x = -\frac{3}{2}$ $\quad$ $x = \pm\sqrt{4}$

$= \pm 2$

R.2–6 Equations that Are Quadratic in Form

$$ax^2 + bx + c = 0$$
$$a(x)^2 + b(x)^1 + c = 0$$
$$a(u)^2 + b(u)^1 + c = 0$$
$$a(x^n)^2 + b(x^n)^1 + c = 0$$
$$au^2 + bu + c = 0$$

Example 10

Solve: $x^4 - 5x^2 + 6 = 0$

Solution

$(x^2)^2 - 5(x^2) + 6 = 0$ $\quad$ ***or*** $(x^2-2)(x^2-3) = 0$

$(U)^2 - 5(U) + 6 = 0$ $\quad$ $x^2 - 2 = 0$ $\quad$ $x^2 - 3 = 0$

$U^2 - 5U + 6 = 0$ $\quad$ $x^2 = 2$ $\quad$ $x^2 = 3$

Solve for U $\quad$ $x = \pm\sqrt{2}$ $\quad$ $x = \pm\sqrt{3}$

$$U = \frac{5 \pm \sqrt{(-5)^2 - 4(6)(6)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25-24}}{2}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$\rightarrow \begin{cases} U = \frac{5-1}{2} = 2 \\ U = \frac{5+1}{2} = 3 \end{cases}$$

$$x^2 = U$$

$$\rightarrow \begin{cases} x^2 = 2 \rightarrow \underline{x = \pm\sqrt{2}} \\ x^2 = 3 \rightarrow \underline{x = \pm\sqrt{3}} \end{cases}$$

R.2–7 Solving a *Radical* Equation

Power Property

If P and Q are algebraic expressions, then every solution of the equation $P = Q$ is also a solution of the equation $P^n = Q^n$; for any positive integer n.

Example 11

Solve $x - \sqrt{15 - 2x} = 0$

Solution

$$x = \sqrt{15 - 2x}$$

$$x^2 = \left(\sqrt{15 - 2x}\right)^2$$

$$x^2 = 15 - 2x$$

$$x^2 + 2x - 15 = 0$$

$$(x-3)(x+5) = 0$$

$$x - 3 = 0 \qquad x + 5 = 0$$

$$x = 3 \qquad x = -5$$

Check

$x = 3$ $\qquad\qquad$ $x = -5$

$3-\sqrt{15-2(3)}=0$ $\qquad$ $-5-\sqrt{15-2(-5)}=0$

$3-\sqrt{9}=0$ $\qquad$ $-5-\sqrt{25}=0$

$3-3=0 \quad (true)$ $\qquad$ $-5-5\neq 0 \quad (false)$

$\therefore$ $x=3$ is the only solution

R.2–8 Interval / Set Notations

Type of Interval	*Set*	*Interval Notation*
Open Interval	$\{x \mid x<a\}$ *or* $x<a$	$(-\infty,\ a)$
	$\{x \mid a<x<b\}$ *or* $a<x<b$	$(a,\ b)$
	$\{x \mid x>b\}$ *or* $x>b$	$(b,\ \infty)$
Closed Interval	$\{x \mid a\le x\le b\}$ *or* $a\le x\le b$	$[a,\ b]$
Other Intervals	$\{x \mid x\le a\}$ *or* $x\le a$	$(-\infty,\ a]$
	$\{x \mid a\le x<b\}$ *or* $a\le x<b$	$[a,\ b)$
	$\{x \mid a<x\le b\}$ *or* $a<x\le b$	$(a,\ b]$
	$\{x \mid x\ge b\}$ *or* $x\ge b$	$[b,\ \infty)$
Disjoint Interval	$\{x \mid x<a \text{ or } x>b\}$	$(-\infty,\ a)\cup(b,\ \infty)$
All Real Numbers $\mathbb{R}$	$\{x \mid x\in\mathbb{R}\}$	$(-\infty,\ \infty)$

R.2–9 Solving an *Absolute Value* Equation

If c is a positive real number and X represents any algebraic expression, then $|X| = c$ is equivalent to $X = c$ or $X = -c$

$$|X| = c \quad \rightarrow \quad X = c \text{ or } X = -c$$

Properties of Absolute Value

1. For b > 0, $|a|=b$ if and only if (*iff*) $a=b$ *or* $a=-b$
2. $|a|=|b|$ *iff* $a=b$ *or* $a=-b$

For any positive number b:

3. $|a| < b \text{ iff } -b < a < b$

4. $|a| < b \text{ iff } a < -b \text{ or } a > b$

Example 12

Solve: $|2x - 1| = 5$

Solution

$2x - 1 = 5$ $\qquad$ $2x - 1 = -5$

$2x = 6$ $\qquad$ $2x = -4$

$x = 3$ $\qquad$ $x = -2$

Solutions: $x = -2,\ 3$

Properties of inequality

1. *If a < b, then a + c < b + c*
2. *If a < b and if c > 0, then ac < bc*
3. *If a < b and if c < 0, then ac > bc*

Example 13

Solve $\quad 3x + 1 > 7x - 15$

Solution

$3x - 7x > -15 - 1$

$-4x > -16$ $\qquad$ *Divide by – 4 both sides*

$x < 4$ $\qquad$ *or* $(-\infty,\ 4)$ $\qquad$ *or* $\{x \mid x < 4\}$

R.2–10 Solving an *Absolute* Value Inequality:

If X is an algebraic expression and c is a positive number,

1. The solutions of $|X| < c$ are the numbers that satisfy $-c < X < c$.
2. The solutions of $|X| > c$ are the numbers that satisfy $X < -c$ or $X > c$.

Example 14

Solve: $-3|5x-2|+20 \geq -19$

Solution

$$-3|5x-2| \geq -39$$

$$|5x-2| \leq 13$$

$$-13 \leq 5x-2 \leq 13$$

$$-11 \leq 5x \leq 15$$

$$\boxed{-\frac{11}{5} \leq x \leq 3} \quad \text{or} \quad \boxed{\left[-\frac{11}{5},\ 3\right]}$$

R.2–11 *Definition* of a Polynomial Inequality

A polynomial inequality is any inequality that can be put into one of the forms

$$f(x)<0 \qquad f(x)>0 \qquad f(x)\leq 0 \qquad f(x)\geq 0$$

✓ $ax^2+bx+c \geq 0 \;\rightarrow\; if\ a>0 \;\Rightarrow\; x \leq x_1,\quad x \geq x_2$

✓ $ax^2+bx+c \leq 0 \;\rightarrow\; if\ a>0 \;\Rightarrow\; x_1 \leq x \leq x_2$

Example 15

Solve $2x^2+5x-12 \geq 0$

Solution

$$2x^2+5x-12=0$$
$$(2x-3)(x+4)=0$$

$$x=-4,\ \frac{3}{2}$$

	-4		0		$\frac{3}{2}$	
		$+$		$-$		$+$

$$\left(-\infty,\ -4\right] \cup \left[\frac{3}{2},\ \infty\right)$$

Solution: $\boxed{x \leq -4 \quad x \geq \frac{3}{2}}$

Example 16

Solve $\dfrac{2x-1}{3x+4} < 5$

Solution

$\dfrac{2x-1}{3x+4} - 5 = 0$ *Restriction:* $3x+4 \neq 0 \Rightarrow \boxed{x \neq -\dfrac{4}{3}}$

$(3x+4)\dfrac{2x-1}{3x+4} - 5(3x+4) = 0$

$2x - 1 - 15x - 20 = 0$

$-13x - 21 = 0$

$\boxed{x = -\dfrac{21}{13}}$

	$-\frac{21}{13}$		$-\frac{4}{3}$		0
$-$		$+$		$-$	

Solution: $\boxed{x < -\dfrac{21}{13} \quad x > -\dfrac{4}{3}}$ $\boxed{\left(-\infty,\ -\dfrac{21}{13}\right) \cup \left(-\dfrac{4}{3},\ \infty\right)}$

R.2–12 The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\dfrac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\textit{possible rational zeros} = \frac{\textit{factors of the constant term } a_0}{\textit{factors of the leading coefficient } a_n} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example 17

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$
possibilities for a_n	$\pm 1,\ \pm 3$
possibilities for c/d	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

The result will show that –2 is a ***zero*** (plug in).

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3+8x^2-2x-4\right)=0$

For $3x^3+8x^2-2x-4 \quad \Rightarrow \quad \frac{c}{d}=\frac{\pm 1,\ \pm 2,\ \pm 4}{\pm 1,\ \pm 3}$

$x=-\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x+\frac{2}{3}\right)\left(3x^2+6x-6\right)=0$

By applying quadratic formula to solve: $3x^2+6x-6=0 \Rightarrow x=-1\pm\sqrt{3}$

Hence, the polynomial has two rational roots $x=-2$ *and* $-\frac{2}{3}$ and two irrational roots $x=-1\pm\sqrt{3}$.

Exercises *Section R.2* – Solving Equations

(1 – 78) Solving Equations

1. $3\left[2x-(4-x)+5\right]=7x-2$

2. $-4(2x-6)+8x=5x+24+x$

3. $-8(3x+4)+6x=4(x-8)+4x$

4. $\frac{1}{2}(4x+8)-16=-\frac{2}{3}(9x-12)$

5. $\frac{3}{4}(24-8x)-16=-\frac{2}{3}(6x-9)$

6. $\frac{x-3}{4}=\frac{5}{14}-\frac{x+5}{7}$

7. $\frac{x+1}{4}=\frac{1}{6}+\frac{2-x}{3}$

8. $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

9. $\frac{6}{x+1}-\frac{5}{x+2}=\frac{10}{x^2+3x+2}$

10. $x^2=-25$

11. $x^2=49$

12. $9x^2=100$

13. $4x^2+25=0$

14. $5x^2-45=0$

15. $(x-4)^2=12$

16. $(x+3)^2=-16$

17. $x^2+8x+15=0$

18. $x^2+5x+2=0$

19. $x^2+x-12=0$

20. $x^2-2x-15=0$

21. $x(8x+1)=3x^2-2x+2$

22. $3x^2-x-2=0$

23. $3x^2+x-2=0$

24. $2x^2+3x-5=0$

25. $2x^2-3x-5=0$

26. $3x^3+2x^2=12x+8$

27. $x^3+x^2-4x-4=0$

28. $x^3-x^2=16x-16$

29. $x^4+3x^2=10$

30. $x^4-4x^3+3x^2=0$

31. $x^4+6x^2-7=0$

32. $3x^4-x^2-2=0$

33. $x-3\sqrt{x}-4=0$

34. $\left(5x^2-6\right)^{1/4}=x$

35. $\sqrt[3]{6x-3}=3$

36. $\sqrt{2x+3}=5$

37. $\sqrt{x-3}+6=5$

38. $\sqrt{3x-2}=4$

39. $\sqrt{2x+5}+11=6$

40. $\sqrt{x+2}+\sqrt{x-1}=3$

41. $\sqrt{x+2}+\sqrt{3x+7}=1$

42. $|x|=-9$

43. $|x|=9$

44. $|x-2|=7$

45. $|x-2|=0$

46. $2|x-6|=8$

47. $3|2x-1|=21$

48. $2|3x-2|=14$

49. $|4x+1|+4=10$

50. $|3x-1|+2=16$

51. $|x+1|=|1-3x|$

52. $|3x-1|=|x+5|$

53. $|2x-4|=|x-1|$

54. $-3x+5>-7$

55. $4-3x\le 7+2x$

56. $4(x+1)+2\ge 3x+6$

57. $8x+3>3(2x+1)+x+5$

58. $5(3-x)\le 3x-1$

59. $\frac{2x-5}{-8}\le 1-x$

60. $8(x+1)\le 7(x+5)+x$

61. $|x-2|<1$

62. $|x+2|\ge 1$

63. $|2(x-1)+4|\le 8$

64. $\left|\frac{2x+6}{3}\right|>2$

65. $|12-9x|\ge -12$

66. $|6-3x|<-11$

67. $2x^2-9x\le 18$

68. $x^2-5x+4>0$

69. $x^2+7x+10<0$

70. $x^3-3x^2-9x+27<0$

71. $x^3+3x^2\le x+3$

72. $x^4-20x^2+64\le 0$

73. $x^4-10x^2+9\ge 0$

74. $\frac{x+4}{x-1}<0$

75. $\frac{x-2}{x+3}>0$

76. $\frac{x-4}{x+6}\le 1$

77. $\frac{x}{2x+7}\ge 4$

78. $\frac{x-3}{x+4}\ge\frac{x+2}{x-5}$

Section R.3 – Functions

R.3−1 Relations

A ***relation*** is any set of ordered pairs. The set of all first components of ordered pairs is called the domain of the relation and the set of second components is called the range of the relation.

R.3−2 *Definition* of a Function

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

Range

Function machine

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.

Notation for Function

$f(x)$ read "f of x" or "f at x" represents the value of the function at the number x.

Example 1

Let $f(x) = -x^2 + 5x - 3$ Find: $f(2)$

Solution

$$f(x) = -x^2 + 5x - 3$$
$$f(\text{--}) = -(\text{--})^2 + 5(\text{--}) - 3$$

$$f(2) = -(2)^2 + 5(2) - 3$$
$$\underline{= 3}$$

R.3–3 *Piecewise*-Defined Functions

Function is sometimes described by more than one expression; we call such functions ***piecewise-defined functions***.

Example 2

Graph the piecewise function

$$f(x) = \begin{cases} -2x+5 & if \quad x \le 2 \\ x+1 & if \quad x > 2 \end{cases}$$

Find: $f(2),\ f(0),\ f(4)$

Solution

$$f(2) = -2(2) + 5 = 1$$

$$f(0) = -2(0) + 5 = 5$$

$$f(4) = 4 + 1 = 5 \quad \mathbb{R} - \{3\}$$

R.3–4 The *Domain* of a Function

1. ***Rational*** function: $\dfrac{f(x)}{h(x)} \Rightarrow$ ***Domain***: $h(x) \ne 0$

Example: $f(x) = \dfrac{1}{x-3}$

Domain: $\underline{x \ne 3}$ $\qquad \{x \mid x \ne 3\}$

Or $\qquad (-\infty, 3) \cup (3, \infty) \qquad$ *Interval Notation*

Or

2. ***Irrational*** function: $\sqrt{g(x)} \qquad \Rightarrow$ ***Domain***: $g(x) \ge 0$

Example: $g(x) = \sqrt{3-x} + 5$

$$3 - x \ge 0$$

$-x \geq -3$

Domain: $x < 3$ $\quad (-\infty,\ 3]$

3. ***Otherwise***: Domain all real numbers $(-\infty,\ \infty)$

Example: $f(x) = x^3 + |x|$

Domain: All real numbers $\quad \mathbb{R}$

$(-\infty,\ \infty)$

(1) & (2) → Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$

$x > 3$

Domain: $(3,\ \infty)$

Example 3

Find the domain

a) $f(x) = x^2 + 3x - 17$

Domain: $\mathbb{R}$

b) $g(x) = \frac{5x}{x^2 - 49}$

$x^2 \neq 49$

$x \neq \pm 7$

Domain: $\begin{cases} \{x \mid x \neq \pm 7\} & \text{or} \\ (-\infty, -7) \cup (-7, 7) \cup (7, \infty) \end{cases}$

c) $h(x) = \sqrt{9x - 27}$

$9x - 27 \geq 0$

$9x \geq 27$

***Domain*:** $x \geq 3$ $\quad [3,\ \infty)$

R.3–5 The *Algebra* of Functions

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

Example 4

Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find each of the following $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned}(f+g)(1)&=f(1)+g(1)\\&=1^2+1+3(1)+5\\&=1+1+3+5\\&=10\end{aligned}$$

$$\begin{aligned}(f-g)(-3)&=f(-3)-g(-3)\\&=(-3)^2+1-(3(-3)+5)\\&=14\end{aligned}$$

$$\begin{aligned}(fg)(5)&=f(5)\cdot g(5)\\&=\left(5^2+1\right)\cdot(3(5)+5)\\&=(26)\cdot(20)\\&=520\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(0)&=\frac{f(0)}{g(0)}\\&=\frac{0^2+1}{3(0)+5}\\&=\frac{1}{5}\end{aligned}$$

R.3–6 *Composition* of Functions

The composite function $g \circ f$, the composite of f and g, is defined as

$$(g \circ f)(x) = g\big(f(x)\big)$$

Where x is in the domain of f

and $g(x)$ is in the domain of f

Example 5

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$(f \circ g)(x) = f\big(g(x)\big)$ $\qquad = f(2x^2 - x - 1)$ $\qquad$ ***Domain:*** All real numbers

$= 5(\text{---------}) + 6$

$= 5(2x^2 - x - 1) + 6$

$= 10x^2 - 5x - 5 + 6$

$= 10x^2 - 5x + 1$ $\qquad$ ***Domain:*** All real numbers

$(g \circ f)(x) = g\big(f(x)\big)$

$= g(5x + 6)$ $\qquad$ ***Domain:*** All real numbers

$= 2(\qquad)^2 - (\qquad) - 1$

$= 2(5x + 6)^2 - (5x + 6) - 1$

$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$

$= 50x^2 + 120x + 72 - 5x - 7$

$= 50x^2 + 115x + 65$ $\qquad$ ***Domain:*** All real numbers

R.3–8 Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients a_n, a_{n-1}, $\ldots, a_2$, a_1, a_0 are real numbers and the exponents are whole numbers.

Non-polynomial Functions: $\frac{1}{x} + 2x$; $\sqrt{x^2 - 3} + x$; $\frac{x-5}{x^2+2}$

R.3–9 Graphing a Quadratic Function

Definition of a Parabola

A ***parabola*** is the set of all points in a plane equidistant from a fixed–point F (the ***focus***) and a fixed line l (the ***directrix***) that lie in the plane

A function f is a ***quadratic function*** if $f(x) = ax^2 + bx + c$

Formula | ***Example* 6**

***Vertex* of a Parabola**

The ***vertex*** of the graph of $f(x)$ is

V_x or $x_v = -\frac{b}{2a}$

V_y or $y_v = f\left(-\frac{b}{2a}\right)$

Vertex Point $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

$f(x) = x^2 - 4x - 2$

$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$

$y = f\left(-\frac{b}{2a}\right) = f(2)$

$= (2)^2 - 4(2) - 2$

$= -6$

Vertex point: $(2,\ -6)$

Axis of Symmetry:

$x = V_x = -\frac{b}{2a}$

Axis of Symmetry: $x = 2$

***Minimum* or *Maximum* Point**

If $a > 0 \Rightarrow$ *f(x)* has a ***minimum*** point

If $a < 0 \Rightarrow$ *f(x)* has a ***maximum*** point

@ vertex point $(V_x,\ V_y)$

Minimum point @ $(2,\ -6)$

Range

If $a > 0 \Rightarrow [V_y,\ \infty)$

If $a < 0 \Rightarrow (-\infty,\ V_y]$

$[-6,\ \infty)$

Domain: $(-\infty,\ \infty)$

R.3–10 Graphing a Polynomial Function

Example 7

Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)(x^2-4) \\ &= (x+1)(x+2)(x-2) = 0 \end{aligned}$$

The zeros of $f(x)$ (x-intercepts) are: $-2,\ -1,\ and\ 2$

Interval	$-\infty$ -2	-1 **0**	2 ∞	
Sign of $f(x)$	–	+	–	+
Position	**Below *x*-axis**	**Above *x*-axis**	**Below *x*-axis**	**Above *x*-axis**

We can conclude from the chart and the graph that:

$$f(x) > 0 \quad if\ x\ is\ in\ (-2,\ -1) \cup (2,\ \infty)$$

$$f(x) < 0 \quad if\ x\ is\ in\ (-\infty,\ -2) \cup (-1,\ 2)$$

R.3–11 Graphing Rational Functions

R.3–12 Vertical Asymptote (*VA*) - *Think Domain*

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

R.3–13 Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator ($n < m$) $\Rightarrow$ y = 0

$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ($n > m$)$\Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$

R.3–14 *Slant* or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote, and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2-1}{x+2}$$

$$\begin{array}{r} 3x-6 \\ x+2\overline{\big)\,3x^2+0x-1} \\ \underline{3x^2+6x} \\ -6x-1 \\ \underline{-6x-12} \\ R=11 \end{array}$$

$$y=\frac{3x^2-1}{x+2}$$
$$=3x-6+\frac{11}{x+2}$$

∴ The ***oblique asymptote*** is the line **$y = 3x - 6$**

Example 8

Find all the asymptotes of $f(x)=\dfrac{2x^2-3x-1}{x-2}$

Solution

$$\begin{array}{r} 2x+1 \\ x-2\overline{\big)\,2x^2-3x-1} \\ \underline{-2x^2+4x} \\ x-1 \\ \underline{-x+2} \\ 1 \end{array}$$

$$f(x)=\frac{2x^2-3x-1}{x-2}$$
$$=(2x+1)+\frac{1}{x-2}$$

The ***oblique asymptote*** is the line $y=2x+1$

VA: $x=2$

R.3–15 Graph That Has a *Hole*

Example 9

Sketch the graph of g if $g(x)=\dfrac{3x^2+x-4}{2x^2-7x+5}$

Solution

$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$

$$=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1\rightarrow f(1)=-\frac{7}{3}$

x	y
$-\frac{4}{3}$	0
0	$-\frac{4}{5}$

Exercises *Section R.3* – Functions

1. $f(x)=\begin{cases} 2+x & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 2 \\ 3x & \text{if } x>2 \end{cases}$ **Find**: $f(-5),\ f(-1),\ f(0),\ \text{and } f(3)$

2. $f(x)=\begin{cases} -2x & \text{if } x<-3 \\ 3x-1 & \text{if } -3\le x\le 2 \\ -4x & \text{if } x>2 \end{cases}$ **Find**: $f(-5),\ f(-1),\ f(0),\ \text{and } f(3)$

3. $f(x)=\begin{cases} 3x+5 & \text{if } x<0 \\ 4x+7 & \text{if } x\ge 0 \end{cases}$ **Find**

a) $f(0)$ *c)* $f(1)$ *e)* Graph $f(x)$

b) $f(-2)$ *d)* $f(3)+f(-3)$

$(\mathbf{4-37})$ Find the Domain.

4. $f(x)=7x+4$

5. $f(x)=|3x-2|$

6. $f(x)=3x+\pi$

7. $f(x)=-2x^2+3x-5$

8. $f(x)=x^3-2x^2+x-3$

9. $f(x)=4-\frac{2}{x}$

10. $f(x)=\frac{1}{x^4}$

11. $y=\frac{2}{x-3}$

12. $f(x)=\frac{3x}{x+2}$

13. $f(x)=x-\frac{2}{x-3}$

14. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

15. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

16. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

17. $f(x)=\frac{x}{x^2+3x+2}$

18. $f(x)=\frac{x^2}{x^2-5x+4}$

19. $g(x)=\frac{2}{x^2+x-12}$

20. $h(x)=\frac{5}{\frac{4}{x}-1}$

21. $y=\sqrt{x}$

22. $f(x)=\sqrt{3-2x}$

23. $f(x)=\sqrt{3+2x}$

24. $f(x)=\sqrt{6-3x}$

25. $f(x)=\sqrt{2x+7}$

26. $f(x)=\sqrt{x^2-16}$

27. $f(x)=\sqrt{16-x^2}$

28. $f(x)=\sqrt{x^2-5x+4}$

29. $f(x)=\sqrt{x^2+5x+4}$

30. $f(x)=\frac{\sqrt{x+1}}{x}$

31. $g(x)=\frac{\sqrt{x-3}}{x-6}$

32. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

33. $f(x)=\frac{\sqrt{5-x}}{x}$

34. $f(x)=\frac{x}{\sqrt{5-x}}$

35. $f(x)=\frac{1}{(x-3)\sqrt{x+3}}$

36. $f(x)=\frac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

37. $f(x)=\frac{\sqrt{2x+3}}{x^2-6x+5}$

38. Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

39. Let $f(x)=x^2-2x-3$ and $g(x)=x^2+3x-2$. Find each of the following and give the domain.

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

40. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

(41 – 55) For the given function, find:

a) Find $(f\circ g)(x)$ and the ***domain*** of $f\circ g$

b) Find $(g\circ f)(x)$ and the ***domain*** of $g\circ f$

41. $f(x)=x-3$ *and* $g(x)=x+3$

42. $f(x)=\frac{2}{3}x$ *and* $g(x)=\frac{3}{2}x$

43. $f(x)=x-1$ *and* $g(x)=3x^2-2x-1$

44. $f(x)=x^2-2$ *and* $g(x)=4x-3$

45. $f(x)=\sqrt{x}$ *and* $g(x)=x+3$

46. $f(x)=\sqrt{x}$ *and* $g(x)=2-3x$

47. $f(x)=x^4$ *and* $g(x)=\sqrt[4]{x}$

48. $f(x)=x^n$ *and* $g(x)=\sqrt[n]{x}$

49. $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

50. $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

51. $f(x)=x^5-2$ *and* $g(x)=\sqrt[5]{x+2}$

52. $f(x)=1-x^2$ *and* $g(x)=\sqrt{x^2-25}$

53. $f(x)=\frac{1}{x-2}$ *and* $g(x)=\frac{x+2}{x}$

54. $f(x)=\frac{3x+5}{2}$ *and* $g(x)=\frac{2x-5}{3}$

55. $f(x)=\frac{2x+3}{x+4}$ *and* $g(x)=\frac{-4x+3}{x-2}$

(56 – 80) Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f

56. $f(x)=x^2+6x+3$

57. $f(x)=x^2+6x+5$

58. $f(x)=-x^2-6x-5$

59. $f(x)=x^2-4x+2$

60. $f(x)=-2x^2+16x-26$

61. $f(x)=x^2+4x+1$

62. $f(x)=x^2+6x-1$

72. $f(x)=2x^3+11x^2-7x-6$

63. $f(x)=x^2+6x+3$

64. $f(x)=x^2-10x+3$

65. $f(x)=x^2-3x+4$

66. $f(x)=x^2-4x-5$

67. $f(x)=2x^2-3x+1$

68. $f(x)=-x^2-4x+5$

69. $f(x)=x^3+2x^2-4x-8$

70. $f(x)=x^3+2x^2-5x-6$

71. $f(x)=x^3-3x^2-9x+27$

73. $f(x)=x^3+8x^2+11x-20$

74. $f(x)=-x^4+12x^2-27$

75. $f(x)=x^4+x^2-2$

76. $f(x)=x^4-x^3-6x^2+4x+8$

77. $f(x)=2x^4-x^3-5x^2+2x+2$

78. $f(x)=6x^5+19x^4+x^3-6x^2$

79. $f(x)=x^5-x^4-7x^3+7x^2+12x-12$

80. $f(x)=x^5-7x^4+19x^3-37x^2+60x-36$

(81 – 98) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

81. $f(x)=\dfrac{-3x}{x+2}$

82. $f(x)=\dfrac{x+1}{x^2+2x-3}$

83. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

84. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

85. $f(x)=\dfrac{x^2-x-6}{x+1}$

86. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

87. $f(x)=\dfrac{x-1}{1-x^2}$

88. $f(x)=\dfrac{x^2+x-2}{x+2}$

89. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

90. $f(x)=\dfrac{2x^2-3x-1}{x-2}$

91. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

92. $f(x)=\dfrac{x^2-1}{x^2+x-6}$

93. $f(x)=\dfrac{1}{x-3}$

94. $f(x)=\dfrac{-2}{x+3}$

95. $f(x)=\dfrac{x}{x+2}$

96. $f(x)=\dfrac{x-5}{x+4}$

97. $f(x)=\dfrac{2x^2-2}{x^2-9}$

98. $f(x)=\dfrac{x^2-3}{x^2+4}$

99. A cone has an altitude of 15 *cm* and a radius of 3 *cm*.
A right circular cylinder of radius r and height h is inscribed in the cone.
Use similar triangles to write h as a function of r.

100. Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

a) Write the radius *r* of the surface of the water as a function of its depth *h*.
b) Write the volume *V* of the water as a function of its depth *h*.

101. A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius *r* (in *feet*) of the surface of the water is given by $r = 1.5t$, where *t* is the time (in *minutes*) that the water has been running.

a) The area *A* of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ *minutes*.
b) The volume *V* of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ *minutes*

102. The surface area *S* of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following:

a) A function $S(r)$ for the surface area as a function of *r*.
b) A function $S(h)$ for the surface area as a function of *h*.

103. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

104. A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.
b) Express the volume *V* of the cylinder as a function of *r*.
c) Express the volume *V* of the cylinder as a function of *h*.

105. A child 4 *feet* tall is standing near a streetlamp that is 12 *feet* high. The light from the lamp casts a shadow.

a) Find the length *l* of the shadow as a function of the distance *d* of the child from the lamppost.

b) What is the domain of the function?
c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

106. An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.

a) Express the volume V of the box as a function of x.
b) Determine the domain of V.

107. A rectangle is bounded by the x- and y-*axis* of $y = -\frac{1}{2}x + 4$

a) Find the area of the rectangle as a function of x.
b) What is the domain of this function?

Section R.4 – Exponential & Logarithm

R.4–1 *Definition*: Exponential

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x) = 2^x \qquad f(x) = \left(\frac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$$

Example 1

If $f(x) = 2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\frac{5}{2}\right)$

Solution

a) $f(-1) = 2^{-1}$

$= \frac{1}{2}$

b) $f(3) = 2^3$

$= 8$

c) $f\left(\frac{5}{2}\right) = 2^{\frac{5}{2}}$

$= 4\sqrt{2}$

R.4–2 Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$y = 0 \pm \text{d}$

The exponential function always equals to 0

$x \to \infty$ *or* $x \to -\infty \Rightarrow f(x) \to 0$

Example 2

$f(x) = 3^x$

Asymptote: $y = 0$

2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	$f(x)$
$x-1$	
x	
$x+1$	
$x+2$	

x	$f(x)$
-1	1/3
0	1
1	3
2	9

Domain: $(-\infty, \infty)$

Range: (d, ∞)

Example 3

Graph $f(x) = 2^x + 3$

Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$

R.4–3 Natural Base *e*

The irrational number e is called natural base.

$f(x) = e^x$ is called natural exponential function.

$e^0 = 1$ $\quad$ $e \approx 2.7183$ $\quad$ $e^2 \approx 7.3891$ $\quad$ $e^{-1} \approx 0.3679$

Example 4

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x *years* after 1978. Project the gray population in the recovery area in 2012.

Solution

x = 2012 - 1978 = 34

$$f(x = 34) = 1066e^{0.042(34)}$$
$$= 4445.6$$
$$\approx 4,446$$

***Example* 5**

Graph $f(x) = e^x$

Solution

Asymptote: $y = 0$

x	$f(x)$
–1	.4
0	1
1	2.7

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

R.4–4 Logarithmic Function (*Definition*)

For $\boldsymbol{x} > \mathbf{0}$ and $b > 0$, $b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \quad \Leftrightarrow \quad x = b^y$$

Base

The function $f(x) = \log_b x$ is the lo unction with base b.

$\log_b x$: ***read*** log base b of x

$\log x$ ***means*** $\log_{10} x$

$\ln x$ ***means*** $\log_e x$

The logarithmic function with base $\boldsymbol{e}$ is called natural logarithmic function.

$\ln x$ *read* "el en of x"

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

R.4–5 Logarithm *Domain*

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers.
(*Inside* the log has to be > 0)

Range: $(-\infty, \infty)$

Example 6

Find the ***domain*** of

a) $f(x) = \log_4(x-5)$

$$x - 5 > 0 \;\Rightarrow\; x > 5$$

Domain: $(5, \infty)$

b) $f(x) = \ln(4-x)$

$$4 - x > 0$$
$$-x > -4$$
$$x < 4$$

Domain: $(-\infty, 4)$

c) $h(x) = \ln\left(x^2\right)$

$x^2 > 0 \;\Rightarrow$ all real numbers except 0.

Domain: $\{x \mid x \neq 0\}$

or $(-\infty, 0) \cup (0, \infty)$ ***or*** $\mathbb{R} - \{0\}$

R.4–6 Graphs of *Logarithmic* Functions

Example 7

Graph $g(x) = \log x$

Solution

Asymptote: $x = 0$
(*Force inside log to be equal to zero, then solve for* x)

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$g(x)$
0	
0.5	–.3
1	0
2	.3
3	.5

R.4–7 Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N$$

For M > 0 and N > 0

Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

Power Rule

$$\log_b M^p = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \tfrac{1}{3} \ln x$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$
$$= 5 - \ln 11$$

Example 8

Express each of the following in terms of sums and differences of logarithm: $\log_a\left(\frac{mnq}{p^2r^4}\right)$

Solution

$$\log_a\left(\frac{mnq}{p^2r^4}\right) = \log_a(mnq) - \log_a\left(p^2r^4\right) \qquad \textit{Quotient Rule}$$
$$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right) \qquad \textit{Product Rule}$$
$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$
$$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r \qquad \textit{Power Rule}$$

Example 9

Express each of the following in terms of sums and differences of logarithm: $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right) \qquad \textit{Quotient Rule}$$
$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right] \qquad \textit{Product Rule}$$
$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right) \qquad \log_5\left(5^2\right) = 2$$
$$= \frac{1}{2}\log_5 x - 2 - 3\log_5 y$$

Example 10

Write as a single logarithmic $2\ln x + \frac{1}{3}\ln(x+5)$

Solution

$$2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{1/3} \qquad \textit{Power Rule}$$

$$= \ln x^2 \ (x+5)^{1/3}$$ *Product Rule*

$$= \ln\left(x^2 \ \sqrt[3]{x+5}\right)$$

Example 11

Write as a single logarithmic $2\log(x-3) - \log x$

Solution

$$2\log(x-3) - \log x = \log(x-3)^2 - \log x$$ *Power Rule*

$$= \log\frac{(x-3)^2}{x}$$ *Quotient Rule*

R.4–8 Solving Exponential Equations

1. Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example 12

Solve $5^{3x-6} = 125$

Solution

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$3x = 9$$

$$x = 3$$

2. Using *Natural Logarithms*

a. Isolate the exponential expression

b. Take the natural logarithm on both sides of the equation

c. Simplify using one of the following properties: $\ln b^x = x\ln b$ *or* $\ln e^x = x$

d. Solve for the variable.

Note: This method will work on any exponential equation.

Example 13

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$	*Isolate the exponential expression*
$7e^{2x} = 63$	*Divide by 7 both sides*
$e^{2x} = 9$	*Natural logarithm on both sides*
$\ln e^{2x} = \ln 9$	*Use inverse Property*
$2x = \ln 9$	
$x = \frac{\ln 9}{2}$ ≈ 1.0986	

R.4–9 Solving Logarithm Equations

1. Logarithmic Equations

a. Express the equation in the form $\log_b M = c$

b. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

c. Solve for the variable

d. Check proposed solution in the original equation. Include only the set for $M > 0$

Example 14

Solve: $\log(x) + \log(x-3) = 1$

Solution

$\log(x(x-3)) = 1$	*Product Rule*
$x(x-3) = 10^1$	*Convert to exponential form*
$x^2 - 3x = 10$	
$x^2 - 3x - 10 = 0$	*Solve for x*
$x = -2, 5$	

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

$\therefore$ ***Solution***: $x = 5$

2. Property of Logarithmic Equality

For any $M > 0,\ N > 0,\ \&\ b > 0,\ \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example 15

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right) \qquad \textit{Quotient Rule}$$

$$x - 3 = \frac{7x-23}{x+1}$$

$$(x-3)(x+1) = 7x - 23$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0$$

$$\underline{x = 4,\ 5}$$

Check: $x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

$\therefore$ ***Solution***: $\underline{x = 4,\ 5}$

R.4–10 Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time t)

A_0: At time zero (initial value).

t: Time

k: Exponential rate.

- If $\boldsymbol{k > 0}$, the function models of a growing entity
- If $\boldsymbol{k < 0}$, the function models of a decay entity.

Example 16

In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 *million*, or two *billion*?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln\frac{813}{643} = \ln e^{10k}$$

$$\ln\frac{813}{643} = 10k$$

$$\frac{1}{10}\ln\frac{813}{643} = k$$

$$k \approx 0.023$$

$$\Rightarrow A(t) = 643e^{0.023t}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$$t \approx 49$$

$$Year: \ 2039$$

R.4–11 *Definition*

The quantity described by the function $y(t) = y_0 e^{kt}$ *for* $k > 0$, has a constant doubling time of $T_2 = \frac{\ln 2}{k}$, with the same units as *t*.

Formula To find either *k* or *T*:

$$A = A_0 e^{kt} \quad \Rightarrow \quad kT = \ln \frac{A}{A_0}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$

Example 17

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a) The half-life of Strontium-90 is 28 *years*, meaning that all after 28 *years* a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.

b) Suppose the nuclear accident occurs and releases 60 *grams* of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 *grams*?

Solution

a) $k = \frac{1}{28} \ln \frac{1}{2}$ $\qquad kT = \ln \frac{A}{A_0}$

$$\approx -0.0248$$

$$A(t) = A_0 e^{-0.0248t}$$

b) $A = A_0 e^{-0.0248t}$

$$t = \frac{\ln \frac{1}{6}}{-0.0248}$$

$$\approx 72.25 \ yrs$$

$$kT = \ln \frac{A}{A_0}$$

R.4−12 Exponential Decay Function

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$.

Rate constant: $k > 0$.
Initial value: y_0

Half-life is $T_{1/2} = \frac{\ln 2}{k}$

Example 18

Researchers determine that a fossilized bone has 30% of the *C*-14 of a live bone. Estimate the age of the bone. Assume a half-life for *C*-14 of ~5730 yrs.

Solution

$$k = \frac{\ln 2}{T_{1/2}}$$

$$= \frac{\ln 2}{5730}$$

$$\approx 0.000121$$

$$T = \frac{\ln \frac{y}{y_0}}{k}$$

$$= \frac{\ln 0.3}{-0.000121}$$

$$\approx 9950 \ yrs$$

R.4−13 Financial Model

The balance in the account increases exponentially at a rate that can be determined from the advertised ***annual percentage yield*** (or **APY**) of the account.

R.4−14 Effective Rate

The ***effective rate*** corresponding to a started rate of interest r compounded m times per year is

$$r_e = \left(1+\frac{r}{m}\right)^m - 1$$

APY is also referred to as ***effective rate*** or true interest rate.

Example 19

The APY of a savings account is the percentage increase in the balance over the course of a year. Suppose you deposit $500 in a savings account that has an APY of 6.18% per year. Assume that the interest rate remains constant and that no additional deposits or withdrawals are made. How long will it take the balance to reach $2500?

Solution

In one year the balance: $y(1) = (1+.0618)y_0 = 1.0618y_0$

$$k = \frac{1}{T}\ln\left(\frac{y(1)}{y_0}\right)$$
$$= \ln 1.0618$$
$$\approx 0.05997$$

$$y(t) = 500e^{0.05997t}$$

$$T = \frac{1}{k}\ln\left(\frac{y}{y_0}\right)$$
$$= \frac{1}{0.05997}\ln\left(\frac{2500}{500}\right)$$
$$\approx 26.8 \ \ yrs$$

Exercises *Section R.4* – Exponential & Logarithm

$(\mathbf{1-20})$ Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph.

1. $f(x)=2^x+3$

2. $f(x)=2^{3-x}$

3. $f(x)=-\left(\frac{1}{2}\right)^x+4$

4. $f(x)=4^x$

5. $f(x)=2-4^x$

6. $f(x)=-3+4^{x-1}$

7. $f(x)=1+\left(\frac{1}{4}\right)^{x+1}$

8. $f(x)=e^{x-2}$

9. $f(x)=3-e^{x-2}$

10. $f(x)=e^{x+4}$

11. $f(x)=2+e^{x-1}$

12. $f(x)=\log_4(x-2)$

13. $f(x)=\log_4|x|$

14. $f(x)=\left(\log_4 x\right)-2$

15. $f(x)=\log(3-x)$

16. $f(x)=2-\log(x+2)$

17. $f(x)=\ln(x-2)$

18. $f(x)=\ln(3-x)$

19. $f(x)=2+\ln(x+1)$

20. $f(x)=1-\ln(x-2)$

$(\mathbf{21-32})$ Write the equation in its equivalent logarithmic form:

21. $2^6=64$

22. $5^4=625$

23. $5^{-3}=\frac{1}{125}$

24. $\sqrt[3]{64}=4$

25. $b^3=343$

26. $8^y=300$

27. $\sqrt[n]{x}=y$

28. $\left(\frac{2}{3}\right)^{-3}=\frac{27}{8}$

29. $\left(\frac{1}{2}\right)^{-5}=32$

30. $e^{x-2}=2y$

31. $e=3x$

32. $\sqrt[3]{e^{2x}}=y$

$(\mathbf{33-41})$ Write the equation in its equivalent exponential form:

33. $\log_5 125=y$

34. $\log_4 16=x$

35. $\log_5 \frac{1}{5}=x$

36. $\log_6 \sqrt{6}=x$

37. $\log_3 \frac{1}{\sqrt{3}}=x$

38. $6=\log_2 64$

39. $\log_{\sqrt{3}} 81=8$

40. $\log_4 \frac{1}{64}=-3$

41. $\ln M=c$

$(\mathbf{42-50})$ Simplify.

42. $\log_5 1$

43. $\log_7 7^2$

44. $3^{\log_3 8}$

45. $10^{\log 3}$

46. $e^{2+\ln 3}$

47. $\ln e^{-3}$

48. $\ln e^{x-5}$

49. $\log_b b^n$

50. $\ln e^{x^2+3x}$

(51 – 67) Express the following in terms of sums and differences of logarithms.

51. $\log_5\left(\frac{125}{y}\right)$

52. $\log_b x^7$

53. $\ln\sqrt[7]{x}$

54. $\log_b\left(\frac{x^3y}{z^2}\right)$

55. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

56. $\log_a\sqrt[4]{\frac{m^8n^{12}}{a^3b^5}}$

57. $\log_a\sqrt[3]{\frac{a^2b}{c^5}}$

58. $\log_b\left(x^4\sqrt[3]{y}\right)$

59. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

60. $\ln\left(x^2\sqrt{x^2+1}\right)$

61. $\ln\frac{x^2}{x^2+1}$

62. $\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

63. $\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

64. $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

65. $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

66. $\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)$

67. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

(68 – 81) Write the expression as a single logarithm and simplify if necessary.

68. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

69. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

70. $4\ln x+7\ln y-3\ln z$

71. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

72. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln\left(x+2\right)\right]+\ln(x+y)$

73. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln\left(x+3\right)\right]+\ln\left(x+y\right)$

74. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

75. $2\ln\left(x+4\right)-\ln x-\ln\left(x^2-3\right)$

76. $\ln\left(x^2-25\right)-2\ln\left(x+5\right)+\ln\left(x-5\right)$

77. $5\log_a x-\frac{1}{2}\log_a\left(3x-4\right)-3\log_a\left(5x+1\right)$

(78 – 120) Solve the equations:

78. $2^x=128$

79. $3^x=243$

80. $5^x=70$

81. $2^{5x+3}=\frac{1}{16}$

82. $\left(\frac{2}{5}\right)^x=\frac{8}{125}$

83. $2^{3x-7}=32$

84. $4^{2x-1}=64$

85. $2^{x+4}=8^{x-6}$

86. $5^{x+4}=4^{x+5}$

87. $3^{x-1}=7^{2x+5}$

88. $3^{x+4}=2^{1-3x}$

89. $e^x=15$

90. $e^{x+1}=20$

91. $e^{x\ln 3} = 27$

92. $e^{x^2} = e^{7x-12}$

93. $f(x) = xe^x + e^x$

94. $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

95. $e^{2x} - 2e^x - 3 = 0$

96. $e^{2x+1} \cdot e^{-4x} = 3e$

97. $e^{2x} - 8e^x + 7 = 0$

98. $e^{2x} + 2e^x - 15 = 0$

99. $e^x + e^{-x} - 6 = 0$

100. $6\ln(2x) = 30$

101. $\log_4(5+x) = 3$

102. $\log(4x-18) = 1$

103. $\log_5 x + \log_5(4x-1) = 1$

104. $\log x - \log(x+3) = 1$

105. $\log x + \log(x-9) = 1$

106. $\ln(4x+6) - \ln(x+5) = \ln x$

107. $\ln(5+4x) - \ln(x+3) = \ln 3$

108. $\ln\sqrt[4]{x} = \sqrt{\ln x}$

109. $\sqrt{\ln x} = \ln\sqrt{x}$

110. $\log x^2 = (\log x)^2$

111. $\log x^3 = (\log x)^2$

112. $\ln(\ln x) = 2$

113. $\ln\left(e^{x^2}\right) = 64$

114. $e^{\ln(x-1)} = 4$

115. $\ln x^2 = \ln(12-x)$

116. $\ln x = 1 - \ln(x+2)$

117. $\ln x = 1 + \ln(x+1)$

118. $\log_3(x+3) + \log_3(x+5) = 1$

119. $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

120. $\ln(-4-x) + \ln 3 = \ln(2-x)$

121. Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate *x millimeters* of lead shielding is given by $I(x) = 100e^{-1.5x}$

a) What percentage of radiation will penetrate a lead shield that is 1 *millimeter* thick?

b) How many *millimeters* of lead shielding are required so that less than 0.02% of the radiations penetrates the shielding?

122. After a race, a runner's pulse rate *R*, in *beats* per *minute*, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \le t \le 15$$

Where *t* is measured in minutes.

a) Find the runner's pulse rate at the end of the race and 1 *minute* after the end of the race.

b) How long after the end of the race will the runner's pulse rate be 80 *beats* per *minute*?

123. A can of soda at $79°F$ is placed in a refrigerator that maintains a constant temperature of $36°F$. The temperature *T* of the soda *t minutes* after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

a) Find the temperature of the soda 10 *minutes* after it is placed in the refrigerator.

b) When will the temperature of the soda be $45°F$

124. During surgery, a patient's circulatory system requires at least 50 *milligrams* of an anesthetic. The amount of anesthetic present *t hours* after 80 *milligrams* of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

a) How much of the anesthetic is present in the patient's circulatory system 30 *minutes* after the anesthetic is administered?

b) How long can the operation last if the patient does not receive additional anesthetic?

125. The following function models the average typing speed *S*, in *words* per *minute*, for a student who has been typing for *t months*.

$$S(t) = 5 + 29\ln(t+1), \quad 0 \le t \le 9$$

Use *S* to determine how long it takes the student to achieve an average speed of 65 *words* per *minute*.

126. A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after *t days* is given by the function

$$P(t) = 1{,}200{,}000\left(1 - e^{-0.03t}\right)$$

a) How many days after a major crime has been reported has 40% of the population heard of the crime?

b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

127. Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature *A*, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after *t minutes* is given by $T(t) = A + \left(T_0 - A\right)e^{-kt}$, where *k* is a constant that depends on the object.

a) Determine the constant *k* for a canned soda drink that takes 5 *minutes* to cool from $75°F$ to $65°F$ after being placed in a refrigerator that maintains a constant temperature of $34°F$

b) What will be the temperature of the soda after 30 *minutes*?

c) When will the temperature of the soda drink be $36°F$?

128. Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6.0 billion in 1999 $(t=0)$ and 6.9 billion in 2009 $(t=10)$

a) Find an exponential growth function for the world population that fits the two data points.
b) Find the doubling time for the world population using the model in part (a).
c) Find the (absolute) growth rate $y'(t)$ and graph it, for $0 \le t \le 50$.
d) How fast was the population growing in 2014 $(t=15)$?

129. An exponential decay function $y(t)=y_0 e^{-kt}$ models he amount of drug in the blood *t hr* after an initial dose of $y_0 = 100\ mg$ is administred. Assume the half-life of the drug is 16 hours.

a) Find the exponential decay function that governs the amount of drug in the blood.
b) How much time is required for the drug to reach 1% of the initial dose (1 *mg*)?
c) If a second 100-*mg* dose is given 12 *hr* after the first dose, how much time is required for the drug level to reach 1 *mg*?

130. The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

131. How long will it take an initial deposit of $1500 to increase in value to $2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

132. The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

133. According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /*yr*.

a) Based on these figures, find the doubling time and project the population in 2050.
b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /yr.
c) Comment on th sensitivity of these projections to the growth rate.

134. The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

135. A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

136. The mass of radioactive material in a sample has decreased by 30% since the decay began. Assuming a half-life of 1500 *years*, how long ago did the decay begin?

137. Growing from an initial population of 150,000 at a constant annual growth rate of 4%/*yr*., how long will it take a city to reach a population of 1 million?

138. A savings account advertises an annual percentage yield (APY) of 5.4%, which means that the balance in the account increases at an annual growth rate of 5.4%/*yr*.

a) Find the balance in the account for $t \geq 0$ with an initial deposit of $1500, assuming the APY remains fixed and no additional deposits or withdrawals are made.

b) What is the doubling time of the balance?

c) After how many years does the balance reach $5,000?

139. A large die-casting machine used to make automobile engine blocks is purchased for $2.5 *million*. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 *years*?

b) After how many years is the value of the machine 10% of its original value?

140. Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \geq 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

141. City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500,000$ and a growth rate of $p > 3\% / yr$. What is the relationship between y_0 and p such that the Cities ***A*** and ***C*** have the same population in 10 years?

Section R.5 – Trigonometry

R.5–1 Degrees – Radians

θ measures one full rotation $\theta = 2\pi$ *The measure of θ in radians is* 2π

$$\boxed{1 = 1\ rad}$$

$$1° = 1\ degree$$

If no unit of angle measure is specified, then the angle is to be measured in radians.

$$Full\ Rotation:\ 360° = 2\pi\ rad$$

$$180° = \pi\ rad$$

Converting from Degrees to Radians

$$\frac{180°}{180} = \frac{\pi}{180}\ rad \qquad \Rightarrow 1° = \frac{\pi}{180}\ rad$$

Multiply a degree measure by $\frac{\pi}{180}\ rad$ and simplify to convert to radians.

Example 1

Convert 45° to radians

Solution

$$45° = 45\left(\frac{\pi}{180}\right) rad$$

$$= \frac{\pi}{4}\ rad$$

Converting from Radians to Degrees

Multiply a radian measure by $\frac{180°}{\pi}$ radian and simplify to convert to degrees.

$$\frac{180°}{\pi} = \frac{\pi}{\pi}\ rad$$

$$\boxed{\frac{180^\circ}{\pi} = 1\ rad}$$

Example 2

Convert 1 to *degrees*

Solution

$$1\ rad = 1\left(\frac{180^\circ}{\pi}\right)$$

$$= 1\left(\frac{180^\circ}{3.14}\right)$$

$$= 57.3^\circ$$

R.5–2 Arc Length

Definition

If a central angle θ, in a circle of a radius r, cuts off an arc of length s, then the measure of θ, in radians is:

$$\theta r = \frac{s}{r} r$$

$$s = r\theta \qquad (\theta \text{ in radians})$$

Note :*When applying the formula, the value of* θ ***must*** *be in* ***radian***

Example 3

A central angle θ in a circle of radius 3 *cm* cuts off an arc of length 6 *cm*.
What is the radian measure of θ.

Solution

$$\theta = \frac{6\ cm}{3\ cm} \qquad \theta = \frac{s}{r}$$

$$= 2\ rad$$

R.5–3 Area of a Sector

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.

$$\begin{array}{lcccl} \textit{Area of sector} & \rightarrow & \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} & \leftarrow & \textit{Central angle } \theta \\ \textit{Area of circle} & \rightarrow & & \leftarrow & \textit{One full rotation} \end{array}$$

Definition

If θ (in *radians*) is a central angle in a circle with radius *r*, then the area of the sector formed by an angle θ is given by

$$A = \frac{1}{2} r^2 \theta \qquad (\theta \textit{ in radian})$$

Example 4

Find the area of the sector formed by a central angle of 1.4 *radians* in a circle of radius 2.1 *meters*.

Solution

Given: $r = 2.1\ m,\ \ \theta = 1.4$

$$A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}\left(\frac{21}{10}\right)^2 \left(\frac{14}{10}\right)$$

$$= \frac{3,087}{1,000}$$

$$= 3.087 \ m^2$$

R.5–4 *Six* Trigonometry Functions

The relationships between side lengths and angles of triangles are defined as ***Trigonometry*** (from Greek trigōnon, "triangle" and metron, "measure").

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$

$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{opp}{hyp} = \frac{y}{r}$$

$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{adj}{hyp} = \frac{x}{r}$$

$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\cot\theta = \frac{adj}{opp} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y}$$

$$\sin A = \frac{a}{c} = \cos B$$

$$\csc A = \frac{c}{a} = \sec B$$

$$\cos A = \frac{b}{c} = \sin B$$

$$\sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot A = \frac{b}{a} = \tan B$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$	$\cos\theta = \pm\sqrt{1-\sin^2\theta}$	$1+\tan^2\theta = \sec^2\theta$
	$\sin\theta = \pm\sqrt{1-\cos^2\theta}$	$1+\cot^2\theta = \csc^2\theta$

Example 5

A circle has its center at *C* and a radius of 18 *inches*. If triangle *ADC* is a right triangle and *A* = 35°. Find *x*, the distance from *A* to *B*.

Solution

$$\sin 35° = \frac{18}{x+18}$$

$$(x+18)\sin 35° = 18$$

$$x+18 = \frac{18}{\sin 35°}$$

$$x = \frac{18}{\sin 35°} - 18$$

$$\approx 13 \ \ in$$

Example 6

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\tan x + \cos x = \frac{\sin x}{\cos x} + \cos x$$

$$= \sin x \frac{1}{\cos x} + \cos x \frac{\sin x}{\sin x}$$

$$= \sin x \sec x + \sin x \frac{\cos x}{\sin x}$$

$$= \sin x(\sec x + \cot x) \quad \checkmark$$

or

$$\sin x(\sec x + \cot x) = \sin x\left(\frac{1}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x}{\cos x} + \sin x \frac{\cos x}{\sin x}$$

$$= \tan x + \cos x$$

Example 7

Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1 - \tan^2 t \quad \checkmark$$

R.5–5 Sum and Difference Formulas

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ $\qquad$ $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ $\qquad$ $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\qquad$ $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 8

Show that $\quad \cos(x + 2\pi) = \cos x$

Solution

$$\cos(x+2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi$$

$$= \cos x \cdot (1) - \sin x \cdot (0)$$

$$= \cos x \quad \checkmark$$

Example 9

Establish the identity: $\quad \dfrac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

Solution

$$\frac{\cos(x-y)}{\sin x \sin y} = \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y}$$

$$= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y}$$

$$= \cot x \cot y + 1 \quad \checkmark$$

Example 10

Establish the identity: $\quad \cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

Solution

$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\dfrac{\cos x\cos y}{\sin x\sin y}-\dfrac{\sin x\sin y}{\sin x\sin y}}{\dfrac{\sin x\cos y}{\sin x\sin y}+\dfrac{\cos x\sin y}{\sin x\sin y}}$$

$$=\frac{\cot x\cot y-1}{\cot x+\cot y}$$ ✓

R.5–6 Double & Half –Angle

$$\cos 2A=\cos^2 A-\sin^2 A \qquad \sin 2A=2\sin A\cos A \qquad \tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

$$=2\cos^2 A-1$$

$$=1-2\sin^2 A$$

$$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}} \qquad \sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}} \qquad \tan\frac{A}{2}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$$

$$\sin\frac{x}{2}=\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{1+\tan^2\left(\frac{x}{2}\right)}} \qquad \cos\frac{x}{2}=\frac{1}{\sqrt{1+\tan^2\left(\frac{x}{2}\right)}}$$

Example 11

Prove $\dfrac{\tan(ax)}{\sqrt{1+\tan^2(ax)}}=\sin ax$

Solution

$$\frac{\tan(ax)}{\sqrt{1+\tan^2(ax)}}=\frac{\dfrac{\sin ax}{\cos ax}}{\sqrt{1+\dfrac{\sin^2 ax}{\cos^2 ax}}}$$

$$=\frac{\sin ax}{\sqrt{\cos^2 ax+\sin^2 ax}}$$

$$=\sin ax$$ ✓

$$\sin ax = \sin ax \ \frac{\sec ax}{\sec ax}$$

$$= \frac{\frac{\sin ax}{\cos ax}}{\frac{1}{\cos ax}}$$

$$= \frac{\tan ax}{\frac{\sqrt{\cos^2 ax + \sin^2 ax}}{\cos ax}}$$

$$= \frac{\tan ax}{\sqrt{\frac{\cos^2 ax + \sin^2 ax}{\cos^2 ax}}}$$

$$= \frac{\tan ax}{\sqrt{1 + \frac{\sin^2 ax}{\cos^2 ax}}}$$

$$= \frac{\tan ax}{\sqrt{1 + \tan^2 ax}} \quad \checkmark$$

Example 12

Prove $\quad (\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

Solution

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$$

$$= 1 + 2\sin\theta\cos\theta$$

$$= 1 + \sin 2\theta \quad \checkmark$$

Example 13

Prove $\quad \dfrac{2\cot x}{1 + \cot^2 x} = \sin 2x$

Solution

$$\frac{2\cot x}{1 + \cot^2 x} = \frac{2\frac{\cos x}{\sin x}}{1 + \frac{\cos^2 x}{\sin^2 x}}$$

$$= 2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x + \cos^2 x}$$

$$= 2\frac{\cos x}{1}\frac{\sin x}{1}$$
$$= 2\cos x \sin x$$
$$= \sin 2x \quad \checkmark$$

Example 14

Prove $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

Solution

$$\cos 4x = \cos(2.2x)$$
$$= 2\cos^2 2x - 1$$
$$= 2(\cos 2x)^2 - 1$$
$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$
$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$
$$= 8\cos^4 x - 8\cos^2 x + 2 - 1$$
$$= 8\cos^4 x - 8\cos^2 x + 1 \quad \checkmark$$

Example 15

Prove $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

Solution

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$
$$= \frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta \quad \checkmark$$

Example 16

Prove $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$

Solution

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$= \frac{1-\cos x}{2}\ \frac{\tan x}{\tan x}$$

$$= \frac{\tan x - \tan x\cos x}{2\tan x}$$

$$= \frac{\tan x - \frac{\sin x}{\cos x}\cos x}{2\tan x}$$

$$= \frac{\tan x - \sin x}{2\tan x}$$ ✓

R.5–7 *Solving* Trigonometry Equations

Example 17

Find the solutions of the equation $\sin\theta = \frac{1}{2}$ if

a) θ is in the interval $[0,\ 2\pi)$

b) θ is any real number

Solution

a) $\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b) Since the sine function has period 2π.

$\theta = \frac{\pi}{6} + 2\pi n$ *and* $\theta = \frac{5\pi}{6} + 2\pi n$

Example 18

Solve the equation $4\sin^2 x\tan x - \tan x = 0$ in the interval $[0,\ 2\pi)$.

Solution

$4\sin^2 x\tan x - \tan x = 0$

$\tan x\left(4\sin^2 x - 1\right) = 0$ *Factor out* **tan*x***

$\tan x = 0$ $\quad$ $4\sin^2 x - 1 = 0$

$\sin^2 x = \frac{1}{4}$

$\tan x = 0$ $\quad$ $\sin x = \frac{1}{2}$ $\quad$ $\sin x = -\frac{1}{2}$

$x = 0,\ \pi$ $\quad$ $x = \frac{\pi}{6},\ \frac{5\pi}{6}$ $\quad$ $x = \frac{7\pi}{6},\ \frac{11\pi}{6}$

Example 19

Solve the equation $2\sin^2 t - \cos t - 1 = 0$, and express the solutions both in radians and degrees.

Solution

$2\sin^2 t - \cos t - 1 = 0$

$2\left(1 - \cos^2 t\right) - \cos t - 1 = 0$ $\quad$ $\sin^2 t + \cos^2 t = 1$

$2 - 2\cos^2 t - \cos t - 1 = 0$

$-2\cos^2 t - \cos t + 1 = 0$ $\quad$ ***Multiply by –1***

$2\cos^2 t + \cos t - 1 = 0$ $\quad$ ***Factor or use quadratic formula***

$\left(2\cos t - 1\right)\left(\cos t + 1\right) = 0$

$2\cos t - 1 = 0$ $\quad$ $\cos t + 1 = 0$

$2\cos t = 1$ $\quad$ $\cos t = -1$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}$ *or* $t = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ $\quad$ $t = \pi$

$t = \frac{\pi}{3} + 2\pi n,\quad t = \frac{5\pi}{3} + 2\pi n,$ $\quad$ $t = \pi + 2\pi n$

$t = 60° + 360°n,\quad 300° + 360°n,\ \ and\quad 180° + 360°n$

R.5–8 Graphing *Sine* & Cosine

We consider graphs of the equation: $y = A\sin(Bx + C) + D \quad y = A\cos(Bx + C) + D$

Amplitude = $|A|$ Period = $\dfrac{2\pi}{|B|}$ Phase Shift $= \varphi = -\dfrac{C}{B}$

Vertical Shift: $y = D$

Example 20

Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ *for* $0 \le x \le 2\pi$

Solution

Amplitude: $|A| = 4$

Period: $P = \dfrac{2\pi}{2} = \pi$

Phase Shift:

$$\phi = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$$

VT: $y = 0$

	x	y
$0 + \frac{3\pi}{4}$	$\frac{3\pi}{4}$	4
$\frac{\pi}{4} + \frac{3\pi}{4}$	π	0
$\frac{\pi}{2} + \frac{3\pi}{4}$	$\frac{5\pi}{4}$	–4
$\frac{3\pi}{4} + \frac{3\pi}{4}$	$\frac{3\pi}{2}$	0
$\pi + \frac{3\pi}{4}$	$\frac{7\pi}{4}$	4

Example 21

Graph one complete cycle $y = 3 - 5\sin\left(\pi\, x + \frac{\pi}{4}\right)$

Solution

Amplitude: $|A| = 5$

Period: $P = \dfrac{2\pi}{\pi} = 2$

Phase Shift:

$$\phi = -\frac{\frac{\pi}{4}}{\pi} = -\frac{1}{4}$$

VT: $y = 3$

	x	y
$0 - \frac{1}{4}$	$-\frac{1}{4}$	3
$\frac{1}{2} - \frac{1}{4}$	$\frac{1}{4}$	–2
$1 - \frac{1}{4}$	$\frac{3}{4}$	3
$\frac{3}{2} - \frac{1}{4}$	$\frac{5}{4}$	8
$2 - \frac{1}{4}$	$\frac{7\pi}{4}$	3

To graph "Sine or *Cosine"*

The graphs of $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$, where $B > 0$, will have the following characteristics:

Amplitude = $|A|$ Period = $\frac{2\pi}{|B|}$ Phase Shift $= \varphi = -\frac{C}{B}$ Vertical Shift: $y = D$

1- Find the Amplitude.
2- Find the Period.
3- Construct a table.

x	$y = A\cos(Bx + C) + D$	$y = A\sin(Bx + C) + D$
$0 + \varphi$	$D + A$	D
$\frac{P}{4} + \varphi$	D	$D + A$
$\frac{P}{2} + \varphi$	$D - A$	D
$\frac{3P}{4} + \varphi$	D	$D - A$
$P + \varphi$	$D + A$	D

4- Graph *One Cycle*
5- Extend the graph, if necessary

To graph "Tangent or *Cotangent"*

The graphs of $y = A\tan(Bx + C) + D$ and $y = A\cot(Bx + C) + D$, where $B > 0$, will have the following characteristics:

No Amplitude Period $= \frac{\pi}{|B|}$ Phase Shift $= -\frac{C}{B}$ Vertical Shift: $y = D$

x	$y = A\tan(Bx + C) + D$	$y = A\cot(Bx + C) + D$
$0 + \varphi$	D	∞
$\frac{P}{4} + \varphi$	$D + A$	$D + A$
$\frac{P}{2} + \varphi$	∞	D
$\frac{3P}{4} + \varphi$	$D - A$	$D - A$
$P + \varphi$	D	∞

R.5–9 *Definition* of Polar Coordinates

To define polar coordinates, let an ***origin*** *O* (called the ***pole***) and an ***initial ray*** from *O*. Then each point *P* can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from *O* to *P* and θ gives the directed angle from the initial ray to yay *OP*.

R.5–10 *Definition* – Relationships between Rectangular and Polar Coordinates

The rectangular coordinates (*x*, *y*) and polar coordinates $(r,\ \theta)$ of a point P are related as follows:

1. $x = r\cos\theta, \quad y = r\sin\theta$
2. $r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x} \quad if\ \ x \neq 0$

Example 22

If $(r,\ \theta) = \left(4,\ \dfrac{7\pi}{6}\right)$ are polar coordinates of a point *P*, find the rectangular coordinates of *P*.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 4\cos\frac{7\pi}{6} \\ &= 4\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

$$y = r\sin\theta$$

$$= 4\sin\frac{7\pi}{6}$$
$$= 4\left(-\frac{1}{2}\right)$$
$$= -2$$

The rectangular coordinates of P are $(x, y) = \left(-2\sqrt{3},\ -2\right)$

Example 23

If $(x, y) = \left(-1,\ \sqrt{3}\right)$ are rectangular coordinates of a point P, find three different pairs the polar coordinates of P.

Solution

$$r = \pm\sqrt{x^2 + y^2}$$
$$= \pm\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$
$$= \pm\sqrt{1+3}$$
$$= \pm\sqrt{4}$$
$$= \pm 2$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1}$$
$$= -\sqrt{3}$$

$$\hat{\theta} = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{3\pi}{3}$$

$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2,\ \frac{2\pi}{3}\right)$, $\left(-2,\ \frac{5\pi}{3}\right)$, $\left(2,\ -\frac{4\pi}{3}\right)$, and $\left(-2,\ -\frac{\pi}{3}\right)$

R.5–11 Graphing Polar Coordinates

$(3,\ 45°)$

$\left(2,\ -\frac{4\pi}{3}\right)$

$\left(-4,\ \frac{\pi}{3}\right)$

$(-5,\ -210°)$

Example 24

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 16$$
$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 16$$
$$r^2\left(\cos 2\theta\right) = 16$$
$$r^2 = \frac{16}{\cos 2\theta} \qquad \cos 2\theta \neq 0$$
$$or \quad r^2 = 16\sec 2\theta$$

Example 25

Sketch the graph of the polar equation $r = 2 + 2\cos\theta$.

Solution

θ	r
0	4
$\frac{\pi}{4}$	$2+\sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$2-\sqrt{2}$
π	0
$\frac{3\pi}{2}$	2
2π	4

Exercises *Section R.5* – Trigonometry

1. Convert to radians.

a) $256°\ 20'$ *b)* $-78.4°$ *c)* $330°$ *d)* $-60°$ *e)* $-225°$

2. Convert to degrees.

a) $\frac{11\pi}{6}$ *c)* $\frac{\pi}{6}$ *e)* $\frac{\pi}{3}$ *g)* -4π

b) $-\frac{5\pi}{3}$ *d)* 2.4 *f)* $-\frac{5\pi}{12}$ *h)* $\frac{7\pi}{13}$

$(\mathbf{2-27})$ Verify the identity of each equation.

3. $\frac{\tan\theta\cot\theta}{\csc\theta}=\sin\theta$

4. $\frac{\sec^2\theta}{\tan\theta}=\sec\theta\csc\theta$

5. $\frac{\sin\theta}{\csc\theta}+\frac{\cos\theta}{\sec\theta}=1$

6. $\cot\theta+\tan\theta=\csc\theta\sec\theta$

7. $\tan x(\cos x+\cot x)=\sin x+1$

8. $\frac{\cos x}{1+\sin x}-\frac{1-\sin x}{\cos x}=0$

9. $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

10. $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

11. $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$

12. $\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{1}{\sin^2 x-\cos^2 x}$

13. $\sec x+\tan x=\frac{\cos x}{1-\sin x}$

14. $\frac{\cos x}{\cos x-\sin x}=\frac{1}{1-\tan x}$

15. $\frac{\cot^2 x}{\csc x-1}=\frac{1+\sin x}{\sin x}$

16. $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

17. $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$

18. $1-\frac{\cos^2 x}{1+\sin x}=\sin x$

19. $\frac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

20. $\frac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

21. $\cos 3x=\cos^3 x-3\cos x\sin^2 x$

22. $\cos^4 x-\sin^4 x=\cos 2x$

23. $\frac{\cos 2x}{\cos^2 x}=\sec^2 x-2\tan^2 x$

24. $\tan^2 x(1+\cos 2x)=1-\cos 2x$

25. $\frac{\cos 2x}{\sin^2 x}=2\cot^2 x-\csc^2 x$

26. $2\sin^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{1+\cos x}$

27. $\sec^2\left(\frac{x}{2}\right)=\frac{2\sec x+2}{\sec x+2+\cos x}$

$(\mathbf{28-37})$ Find the solutions of the equation that are in the interval $[0,\ 2\pi)$.

28. $2\sin^2 x=1-\sin x$

29. $\tan^2 x\sin x=\sin x$

30. $1-\sin x=\sqrt{3}\cos x$

31. $2\sin^2 x-\cos x-1=0$

32. $\sin x+\cos x\cot x=\csc x$

33. $2\sin^3 x+\sin^2 x-2\sin x-1=0$

34. $2\cos^2 t - 9\cos t = 5$

35. $\tan^2 x + \tan x - 2 = 0$

36. $\tan x + \sqrt{3} = \sec x$

37. $4\cos^2 x + 4\sin x - 5 = 0$

(38 – 60) Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation.

38. $y = 2\sin(x - \pi)$

39. $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right)$

40. $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

41. $y = 2\sin\left(x - \frac{\pi}{3}\right)$

42. $y = -\frac{2}{3}\sin\left(3x - \frac{\pi}{2}\right)$

43. $y = -2\sin(2\pi x + \pi)$

44. $y = 3\cos\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)$

45. $y = -\cos\pi\left(x - \frac{1}{3}\right)$

46. $y = \frac{5}{2} - 3\cos\left(\pi x - \frac{\pi}{4}\right)$

47. $y = -3 + \sin\left(\pi\, x + \frac{\pi}{2}\right)$

48. $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

49. $y = -2\sin(2x - \pi) + 3$

50. $y = 3\cos(x + 3\pi) - 2$

51. $y = 2\tan\left(2x + \frac{\pi}{2}\right)$

52. $y = -\frac{1}{4}\tan\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

53. $y = 2\cot\left(2x + \frac{\pi}{2}\right)$

54. $y = -\frac{1}{2}\cot\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

55. $y = 1 - 2\cot 2\left(x + \frac{\pi}{2}\right)$

56. $y = 3 + 2\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$

57. $y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$

58. $y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

59. $y = -3\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right)$

60. $y = 2\sec\left(2x - \frac{\pi}{2}\right)$

(61 – 66) Convert to rectangular coordinates.

61. $(4,\ 30°)$

62. $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$

63. $(3,\ 270°)$

64. $(2,\ 60°)$

65. $\left(\sqrt{2},\ -225°\right)$

66. $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

(67 – 72) Convert to polar coordinates.

67. $(3,\ 3)$

68. $(-2,\ 0)$

69. $(-1,\ \sqrt{3})$

70. $(-3,\ -3)\quad r \ge 0\quad 0° \le \theta < 360°$

71. $\left(2,\ -2\sqrt{3}\right)\quad r \ge 0\quad 0° \le \theta < 360°$

72. $(-2,\ 0)\quad r \ge 0\quad 0 \le \theta < 2\pi$

(73 – 85) Write the equation in rectangular coordinates.

73. $r^2 = 4$

74. $r = 6\cos\theta$

75. $r^2 = 4\cos 2\theta$

76. $r(\cos\theta - \sin\theta) = 2$

77. $r^2 = 4\sin 2\theta$

78. $r\sin\theta = -2$

79. $\theta = \frac{\pi}{4}$

80. $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

81. $r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16$

82. $r(\sin\theta - 2\cos\theta) = 6$

83. $r = 8\sin\theta - 2\cos\theta$

84. $r = \tan\theta$

85. $r\left(\sin\theta + r\cos^2\theta\right) = 1$

(86 – 94) Write the equation in polar coordinates.

86. $y^2 = 6x$

87. $xy = 8$

88. $x + y = 5$

89. $x^2 + y^2 = 9$

90. $(x+2)^2 + (y-3)^2 = 13$

91. $y^2 - x^2 = 4$

92. $x^2 + y^2 = 4x$

93. $y = -x$

94. $x + y = 4$

(95 – 105) Sketch the graph of the polar equation.

95. $r = 5$

96. $\theta = \frac{\pi}{4}$

97. $r = 4\cos\theta + 2\sin\theta$

98. $r = 2 + 4\sin\theta$

99. $r = 2 - \cos\theta$

100. $r = 4\csc\theta$

101. $r^2 = 4\cos 2\theta$

102. $r = 2^\theta \quad \theta \geq 0$

103. $r = e^{2\theta} \quad \theta \geq 0$

104. $r\theta = 1 \quad \theta > 0$

105. $r = 2 + 2\sec\theta$

106. A rope is being wound around a drum with radius 6 *inches*. How much rope will be wound around the drum if the drum is rotated through an angle of 30°?

107. The total arm and blade of a single windshield wiper was 10 *in*. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-*in*. wiper blade. What is the area of the region cleaned?

108. A conical paper cup is constructed by removing a sector from a circle of radius 5 *inches* and attaching edge *OA* to *OB*. Find angle *AOB* so that the cap has a depth of 4 *inches*.

109. The shadow of a vertical tower is 100 *feet* long when the angle of elevation of the sun is 60°. Find the height of the tower.

110. You were hiking directly toward a long straight road when you encountered a swamp. you turned 60° to the right and hiked 6 *mi* in that direction to reach the road. How far were you from the road when you encountered the swamp?

111. A helicopter hovers 1,000 *feet* above a small island. The angle of depression from the helicopter to point *P* on the coast is 30°. How far off the coast is the island?

112. A rectangular box has dimensions $8'' \times 6'' \times 4''$. Approximate, to the nearest tenth of a degree, the angle θ formed by a diagonal of the base and the diagonal of the box.

113. A conical paper cup has a radius of 2 *inches*, approximate, to the nearest degree, the angle β so that the cone will have a volume of $20\ in^3$.

114. As a hot-air balloon rises vertically, its angle of elevation from a point *P* on level ground 100 *km* from the point *Q* directly underneath the balloon changes from $19°\ 20'$ to $31°\ 50'$. Approximately how far does the balloon rise during this period?

115. On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston *P* from the center *O* of the crankshaft?

116. Rod *OA* rotates about the fixed-point *O* so that point *A* travels on a circle of radius *r*. Connected to point *A* is another rod *AB* of length $L > 2r$, and point *B* is connected to a piston. Show that the distance *x* between point *O* and point *B* is given by:

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$

Where θ is the angle of rotation of rod *OA*.

117. Shown in the figure is a design for a rain gutter.

a) Express the volume *V* as a function of θ.

b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

118. Shown in the figure is a plan for the top of a wing of a jet fighter.

a) Approximate angle ϕ.

b) If the fuselage is 4.80 *feet* wide, approximate the wingspan CC'.

c) Approximate the area of the triangle *ABC*.

119. Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'.

120. A mass attached to a spring oscillates upward and downward. The length L of the spring after t seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where L is measured in cm.

a) Sketch the graph of this function for $0 \le t \le 5$
b) What is the length the spring when it is at equilibrium?
c) What is the length the spring when it is shortest?
d) What is the length the spring when it is longest?

121. Based on years of weather data, the expected low temperature T (in °F) in Fairbanks, Alaska, can be approximated by

$$T = 36\sin\left(\frac{2\pi}{365}(t - 101)\right) + 14$$

a) Sketch the graph T for $0 \le t \le 365$
b) Predict when the coldest day of the year will occur.

122. To simulate the response of a structure to an earthquake, an engineer must choose a shape for the initial displacement of the beams in the building. When the beam has length L *feet* and the maximum displacement is a *ft*, the equation:

$$y = a - a\cos\frac{\pi}{2L}x$$

Has been used by engineers to estimate the displacement y. if $a = 1$ and $L = 10$, sketch the graph of the equation for $0 \le x \le 10$.

123. The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.

a) Verify that $d = h\csc\theta$

b) Determine θ when $d = 2h$

124. A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through *A* and *D* bisects θ and is perpendicular to the line through *B* and *C*.

a) Show that the length l of the artery from *A* to *B* is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10$ *mm*, $b = 6$ *mm*, and $\theta = 156°$.

125. Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, d, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.

a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0 .

126. The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.

If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.

The relationship between the cone's vertex angle θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by:

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Section R.6 – Inverse of a Function

R.6–1 Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x$$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x)$$

$$g\big(f(x)\big) = f^{-1}\big(f(x)\big) = \left(f^{-1} \circ f\right)(x) = x$$

If the inverse of a function f is also a function, it is named f^{-1} read "f – inverse"

The –1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

***Domain* and *Range* of** f *and* f^{-1}

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Graphing

R.6–2 *One-to-One* Functions

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then} \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$\text{if } f(a) = f(b), \quad \text{then } a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$

$\therefore f$ is one-to-one.

R.6–3 Finding the *Inverse* Function

	Example
Finding an Inverse Function	$f(x) = 2x + 7$
1. Replace $f(x)$ with y	$y = 2x + 7$
2. Interchange x and y	$x = 2y + 7$
3. Solve for y	$x - 7 = 2y$ $\frac{x-7}{2} = y$
4. Replace y with $f^{-1}(x)$	$f^{-1}(x) = \frac{x-7}{2}$

Example 1

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x+1}{4} = y^3$$

$$y = \left(\frac{x+1}{4}\right)^{1/3}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}}$$

R.6–4 The Inverse *Sine* Function

Definition

The inverse sine function, denoted by $\sin^{-1}$ *or* $\arcsin$, is defined by

$$y = \sin^{-1} x \quad or \quad y = \arcsin x \quad \textbf{\textit{iff}} \quad x = \sin y \quad for\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad and \quad -1 \le x \le 1$$

Properties of $\sin^{-1}$

$$\sin\left(\sin^{-1} x\right) = \sin\left(\arcsin x\right) = x \quad if \quad -1 \le x \le 1$$

$$\sin^{-1}\left(\sin y\right) = \arcsin\left(\sin y\right) = y \quad if \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

R.6–5 The Inverse *Cosine* Function

Definition

The inverse cosine function, denoted by $\cos^{-1}$ *or* $\arccos$, is defined by

$$y = \cos^{-1} x \quad \textit{iff} \quad x = \cos y \quad \textit{for } 0 \le y \le \pi \quad \textit{and} \quad -1 \le x \le 1$$

Notation	*Meaning*
$y = \cos^{-1} x$ *or* $y = \arccos x$	$x = \cos y$ *and* $0 \le y \le \pi$

R.6–6 The Inverse *Tangent* Function

Definition

The inverse cosine function, denoted by $\tan^{-1}$, is defined by:

$y = \tan^{-1} x$ *iff* $x = \tan y$ for any real number x and for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$y = \tan^{-1} x$ *or* $y = \arctan x$

$y = \tan x$ & $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$y = \tan^{-1} x$ & $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Example 3

Find the exact value: $\cos\left(\cos^{-1}(-0.5)\right)$, $\cos^{-1}(\cos(3.14))$, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$ $\arctan(\tan\pi)$

Solution

$\cos\left(\cos^{-1}(-0.5)\right) = -0.5$ Since $-1 \le -0.5 \le 1$

$\cos^{-1}(\cos(3.14)) = 3.14$ Since $0 \le 3.14 \le \pi$

$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\arctan(\tan\pi) = \arctan(0) = 0$ $\therefore \pi > \frac{\pi}{2}$

Example 4

Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution

$\alpha = \arctan\frac{2}{3} \rightarrow \tan\alpha = \frac{2}{3}$

$\sec\left(\arctan\frac{2}{3}\right) = \sec\alpha$

$= \frac{\sqrt{13}}{3}$

Example 5

If $-1 \le x \le 1$, rewrite $\cos\left(\sin^{-1}x\right)$ as an algebraic expression in x.

Solution

$\alpha = \sin^{-1}x \quad \rightarrow \quad \sin\alpha = x = \frac{x}{1}$

$\cos\left(\sin^{-1}x\right) = \cos\alpha$

$= \frac{\sqrt{1-x^2}}{1}$

$= \sqrt{1-x^2}$

Exercises *Section R.6* – Inverse of a Function

(1 – 12) For the given functions:

a) Is $f(x)$ one-to-one function.

b) Find $f^{-1}(x)$, if it exists.

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$.

d) Graph both functions (if $f^{-1}(x)$ exists).

1. $f(x)=\dfrac{x}{x-2}$

2. $f(x)=\dfrac{x+1}{x-1}$

3. $f(x)=\dfrac{2x+1}{x+3}$

4. $f(x)=\dfrac{3x-1}{x-2}$

5. $f(x)=\sqrt{x-1} \quad x\ge 1$

6. $f(x)=\sqrt{2-x} \quad x\le 2$

7. $f(x)=x^2+4x \quad x\ge -2$

8. $f(x)=3x+5$

9. $f(x)=2x^3-5$

10. $f(x)=\sqrt{3-x}$

11. $f(x)=\sqrt[3]{x}+1$

12. $f(x)=\left(x^3+1\right)^5$

(12 – 24) Find the exact value of the expression whenever it is defined.

13. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

14. $\arccos\left(\dfrac{\sqrt{2}}{2}\right)$

15. $\arctan\left(-\dfrac{\sqrt{3}}{3}\right)$

16. $\cos\left(\sin^{-1}\dfrac{1}{2}\right)$

17. $\arcsin\left[\sin\left(-\dfrac{\pi}{2}\right)\right]$

18. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$

19. $\arctan\left[\tan\left(-\dfrac{\pi}{4}\right)\right]$

20. $\sin\left[2\arccos\left(-\dfrac{3}{5}\right)\right]$

21. $\cot\left(\tan^{-1}\dfrac{1}{2}\right)$

22. $\tan\left(\cos^{-1}\dfrac{3}{5}\right)$

23. $\cos\left[\arctan\left(-\dfrac{3}{4}\right)-\arcsin\dfrac{4}{5}\right]$

24. $\tan\left[\cos^{-1}\left(\dfrac{1}{2}\right)+\sin^{-1}\left(-\dfrac{1}{2}\right)\right]$

(25 – 34) Write an equivalent expression that involves x only for:

25. $\cos\left(\cos^{-1}x\right)$

26. $\tan\left(\cos^{-1}x\right)$

27. $\csc\left(\sin^{-1}\dfrac{1}{x}\right)$

28. $\sin\left(\tan^{-1}x\right); \quad x>0$

29. $\sec\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) \quad x>0$

30. $\cot\left(\sin^{-1}\dfrac{\sqrt{x^2-9}}{x}\right) \quad x>0$

31. $\sin\left(2\sin^{-1}x\right) \quad x>0$

32. $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{1}{x}\right), \quad x>0$

33. $\sec\left(\tan^{-1}\dfrac{2}{\sqrt{x^2-4}}\right) \quad x>0$

34. $\sec\left(\sin^{-1}\dfrac{\sqrt{x^2-25}}{x}\right) \quad x>0$

$(\mathbf{35}-\mathbf{37})$ Sketch the graph of the equation:

35. $y=\sin^{-1}2x$

36. $y=\sin^{-1}(x-2)+\dfrac{\pi}{2}$

37. $y=\cos^{-1}\dfrac{1}{2}x$

Lecture One

Limits

Section 1.1 – Idea / Definition of Limits

We always learn to find the second coordinate (y) of any function/equation, we need the first coordinate (x). In this lecture, we are going to do that except that, we are going to add the ***limit*** to the function to analyze the behavior of the function at that point as *x* approach *a*; $x = a$, if it does exist or not. we will discuss the behavior to the left $x = a^-$ and to the right $x = a^+$ for any existence.

1.1−1 Position Function

An object that is falling or vertically projected into the air has its height above the ground, *s*(t), in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in *feet* per *second*

t is the time that the object is in motion, in *second*

s_0 is the original height (initial height) of the object, in *feet*

The average rate is given by: $\frac{\Delta s}{\Delta t}$

Example 1

A rock breaks loose from the top of a tall cliff. What is its average speed?

a) During the first 2 *sec* of fall?

b) During the 1-*sec* interval between second 1 and *second* 2?

Solution

Since the rock falls free (*down*) without any initial velocity or height. $\Rightarrow y(t) = 16t^2$

a) For the first 2 *sec*: *Average speed* $= \dfrac{\Delta y}{\Delta t}$

$$= \frac{y(2)-y(0)}{2-0}$$

$$= \frac{16(2)^2 - 16(0)^2}{2}$$

$$= \frac{64}{2}$$

$$= 32 \ \ ft/\sec$$

b) From 1 *sec* to 2 *sec*: *Average speed* $= \dfrac{y(2)-y(1)}{2-1}$

$$= \frac{16(2)^2 - 16(1)^2}{1}$$

$$= 48 \ \ ft/\sec$$

Example 2

Find the speed of a falling rock $\left(y(t) = 16t^2\right)$ over a time interval $\left[t_0,\ t_0 + h\right]$. Then find the average speed at 1 *sec* and 2 *sec*.

Solution

$$\frac{\Delta y}{\Delta t} = \frac{16\left(t_0 + h\right)^2 - 16\left(t_0\right)^2}{\left(t_0 + h\right) - t_0}$$

$$= \frac{16\left(t_0^2 + 2ht_0 + h^2\right) - 16t_0^2}{t_0 + h - t_0}$$

$$= \frac{16t_0^2 + 32ht_0 + 16h^2 - 16t_0^2}{h}$$

$$= 32\frac{ht_0}{h} + 16\frac{h^2}{h}$$

$$= 32t_0 + 16h$$

If $t_0 = 1$

$$\frac{\Delta y}{\Delta t} = 32(1) + 16h$$

$$= 32 + 16h$$

The average speed has the limiting value 32 *ft/sec* as h approaches 0.

If $t_0 = 2$

$$\frac{\Delta y}{\Delta t} = 32(2) + 16h$$
$$= 64 + 16h$$

The average speed has the limiting value 64 *ft/sec* as h approaches 0.

1.1−2 Average Rates of Changes and Secant Lines

Given the two pints $P\left(x_1,\ f\left(x_1\right)\right)$ and $Q\left(x_2,\ f\left(x_2\right)\right)$ and the differntence in the first coordinate:

$$\Delta x = x_1 - x_2$$
$$= h$$

The average rate of change of $y = f(x)$ with respect to x over the interval $\left[x_1,\ x_2\right]$ is given by this formula:

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x_2\right) - f\left(x_1\right)}{x_2 - x_1}$$
$$= \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}, \quad h \neq 0$$

A secant line is a line that instersects a graph into 2 points. when the secant line moves closer where the 2 points becomes one than the line is called tangent line $h = \Delta x = 0$.

This average rate is also called velocity or Difference Quotient.

1.1−3 Defining the Slope of a Curve

The slope of a line is the rate at which it rises or falls.

To define the tangency for general curves, we need an approach that makes the behavior of the secants through P and points Q as Q moves toward P along the curve:

1. Find the slope of the secant PQ.
2. Investigate the limiting value of the slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

$$m_{\tan} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

Example 3

Find the slope of the parabola $y = x^2$ at the point $P(2,\ 4)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\textbf{\textit{Secant slope}} = \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$\underline{= 4 + h}$$

As Q approaches P, h approaches 0.

Then the secant slope $h + 4 \to 4 = slope$

$$y = m\left(x - x_1\right) + y_1$$

$$y = 4(x-2) + 4$$

$$\underline{y = 4x - 4}$$

1.1−4 Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches x_0 from either side, then

$$\lim_{x \to x_0} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches x_0 is L."

Example 5

$$\left(\sqrt{x}\right)^{-} < \varepsilon^{2}$$

$$\Rightarrow x < \varepsilon^2 \quad if \quad 0 < x < \delta$$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon$$

According to the definition, this shows that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$

1.1–5 *Definitions*

We say that $f(x)$ has right-hand limit L at x_0 and $\lim\limits_{x \to x_0^+} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

We say that $f(x)$ has left-hand limit L at x_0 and $\lim\limits_{x \to x_0^-} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Notation	***Terminology***
$x \to a^-$	***x*** approaches ***a*** from the left (through values ***less*** than *a*)
$x \to a^+$	***x*** approaches ***a*** from the right (through values ***greater*** than *a*)

Example 4

How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

Solution

$$f(x) = \frac{(x-1)(x+1)}{x-1}$$

$$= x + 1 \quad \textit{for} \quad x \neq 1$$

For $x = 1$:

$$f(x = 1) = 1 + 1 = 2$$

x	.9	.99	.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$= 2$$

Example 5

Discuss the behavior of the following function as $x \to 0$.

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution

The unit step function $U(x)$ has no limit as $x \to 0$, it jumps, because the values jump at $x = 0$.

To the left of zero $\left(\textit{negative value } \mathbf{0^-}\right)$ $U(x) = 0$. For the positive values of x close to zero $\left(\mathbf{0^+}\right)$ $U(x) = 1$

1.1−6 One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on ***both sides*** of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called ***two-sided***.

If f fails to have two-sided limit at c, it may still have one-sided limit.

If the approach is from the *right*, the limit is a ***right-hand limit***. $\lim\limits_{x \to c^+} f(x) = L$

If the approach is from the *left*, the limit is a ***left-hand limit***. $\lim\limits_{x \to c^-} f(x) = M$

Example 6

The domain of $f(x)=\sqrt{4-x^2}$ is $[-2,\ 2]$; its graph is the semicircle.

We have: $\lim\limits_{x\to -2^+}\sqrt{4-x^2}=0$ *and* $\lim\limits_{x\to 2^-}\sqrt{4-x^2}=0$

The function doesn't have a left-hand limit at $x=-2$ or a right-hand limit at $x=2$.
It does not have ordinary two-sided limits at either –2 or 2.

1.1–7 *Theorem*

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x)=L \quad\Leftrightarrow\quad \lim_{x\to c^-} f(x)=L \quad \textit{and} \quad \lim_{x\to c^+} f(x)=L$$

Example 7

Given the function graphed:

At $x = 0$: $\lim\limits_{x\to 0^+} f(x)=1$

$\lim\limits_{x\to 0^-} f(x)$ *and* $\lim\limits_{x\to 0} f(x)$ don't exist.

The function is not defined to the left of $x = 0$

At $x = 1$: $\lim\limits_{x\to 1^-} f(x) = 0$ $\quad$ $\lim\limits_{x\to 1^+} f(x) = 1$

$\lim\limits_{x\to 1} f(x)$ doesn't exist.

The right-hand and left-hand limits are not equal.

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = 1$ $\quad$ $\lim\limits_{x\to 2^+} f(x) = 1$

$\lim\limits_{x\to 2} f(x) = 2$ even though $f(2) = 2$

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$

$= \lim\limits_{x\to 3} f(x)$

$= 2$

At $x = 4$: $\lim\limits_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$

$\lim\limits_{x\to 4^+} f(x)$ *and* $\lim\limits_{x\to 4} f(x)$ do not exist.

The function is not defined to the right of $x = 4$.

Example 8

Discuss the behavior of the following function as $x \to 0$.

a) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\qquad$ *b)* $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Solution

a)

$g(x)$ has *no limit* as $x \to 0$ because the values of $g(x)$ grow arbitrary large (negative and positive) value as $x \to 0$ and do not stay close.

b)

$f(x)$ has *no limit* as $x \to 0$ because the function's values oscillate between -1 and $+1$ in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

Exercises *Section* 1.1 – Idea of Limits

(1 – 10) Find the average rate of change of the function with given interval.

1. $f(x) = x^3 + 1 \quad 2 \le x \le 3$

2. $f(x) = x^2 \quad -1 \le x \le 1$

3. $f(x) = x^2 + 1 \quad 0 \le x \le 2$

4. $f(x) = x^2 + 1 \quad -2 \le x \le 2$

5. $f(x) = x^3 + 2 \quad 0 \le x \le 2$

6. $f(x) = \sqrt{x+1} \quad 0 \le x \le 3$

7. $f(x) = \frac{1}{x} \quad 1 \le x \le 2$

8. $f(t) = \cos t \quad 0 \le t \le \frac{\pi}{2}$

9. $f(t) = \sin t \quad 0 \le t \le \frac{\pi}{2}$

10. $f(t) = 2 + \cos t \quad -\pi \le t \le \pi$

(11 – 18) Find the slope of the given function and a point and an equation of the tangent line at that point.

11. $y = x^2 - 3 \quad @ \quad P(2,\ 1)$

12. $y = x^2 - 2x - 3 \quad @ \quad P(2,\ -3)$

13. $y = x^3 \quad @ \quad P(2,\ 8)$

14. $y = 2 - \sin x \quad @ \quad P(0,\ 2)$

15. $y = x^2 + 1 \quad @ \quad x = 1$

16. $y = \frac{1}{x} \quad @ \quad x = 1$

17. $y = \sin x \quad @ \quad x = 0$

18. $y = 2 - \cos x \quad @ \quad x = -\pi$

(19 – 22) Evaluate the limit using the form $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for:

19. $f(x) = x^2, \quad x = 1$

20. $y = x^2 - 1 \quad x = -1$

21. $f(x) = \sqrt{3x+1}, \quad x = 0$

22. $f(x) = \dfrac{1}{x+1}, \quad x = 0$

23. Make a table of values for the function $f(x) = \dfrac{x+2}{x-2}$ at the points.

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \textit{and } x = 1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \ne 1$ in the table.

b) Extending the table, if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

24. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.

a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ and\ t=8$

25. A rock is tossed into the air from ground level with an initial velocity of 15 $m/\sec$. Its height in meters at time t seconds is given by the function $h(t)=15t-4.9t^2$

a) Find the average velocity during the first 2 *sec*?

b) Find the average velocity of the rock over the time interval $[1,\ 1.05]$?

26. A rock is free dropped from an initial height of 25 ft. Its height in feet at time t seconds is given by the function $h(t)=25+16t^2$

a) Find the average velocity during the first 1 *sec*?

b) Find the average velocity of the rock over the time intervals $[1,\ 2]$?

c) Find the average velocity of the rock over the time intervals $[1.4,\ 1.5]$?

d) Find the average velocity of the rock over the time intervals $[1.5,\ 1.6]$?

e) Interpret the results.

27. A rocket is launched into the air from ground level. The height, in feet, is given by the function $h(t)=860+130t-16t^2$

a) Find the average velocity during the first 1 *sec*?

b) Find the average velocity of the rocket over the time interval $[4.99,\ 5]$?

c) Find the average velocity of the rocket over the time interval $[4,\ 5]$?

28. An athlete running a 40-*m* dash. The position of the athlete is given by $d(t)=\frac{1}{6}t^3+4t$, where *d* is the position in meters and *t* is the time elapsed, measured in seconds.

a) Find the average velocity during the first 1 *sec*?

b) Find the average velocity between second 1 and *second* 2?

c) Find the average velocity of the rocket over the time interval $[1.95,\ 2.05]$?

29. An object dropped from certain height will free fall and is given by the function $h(t)=16t^2$, where *t* in seconds.

a) Find the displacement from 3 to 5 seconds?

b) Find the average rate of change of the object over the time interval $[3,\ 5]$?

30. A function given by the function $f(x)=\frac{11}{100}x^{1.36}$

Find the average rate of change of the function over the interval $[200,\ 300]$?

31. A function given by the function $f(x)=3x^2+56x+863$

Find the average rate of change of the function over the interval $[4,\ 17]$?

32. The cost of sending an envelope first-class is $\$1.00$ for the first ounce and $\$0.25$ for any additional ounce. If *x* represents the weight of the envelope, in ounces, the $C(x)$ is the cost of mailing it, where

$$C(x)=\begin{cases} \$1.00 & \text{if} \quad 0<x\le 1 \\ \$1.25 & \text{if} \quad 1<x\le 2 \\ \$1.50 & \text{if} \quad 2<x\le 3 \end{cases}$$

a) Graph the cost function in the intervals $[0,\ 5]$

b) $\lim\limits_{x\to 1^-} C(x)$, $\lim\limits_{x\to 1^+} C(x)$, $\lim\limits_{x\to 1} C(x)$

c) $\lim\limits_{x\to 2^-} C(x)$, $\lim\limits_{x\to 2^+} C(x)$, $\lim\limits_{x\to 2} C(x)$

d) $\lim\limits_{x\to 3^-} C(x)$, $\lim\limits_{x\to 3^+} C(x)$, $\lim\limits_{x\to 3} C(x)$

e) $\lim\limits_{x\to 3.5^-} C(x)$, $\lim\limits_{x\to 3.5^+} C(x)$, $\lim\limits_{x\to 3.5} C(x)$

f) $\lim\limits_{x\to 4^+} C(x)$

g) $\lim\limits_{x\to 4.5} C(x)$

33. For the function $f(x)=x^2$, find the different quotient when

a) $x=3$ & $h=2$

b) $x=3$ & $h=0.1$

c) $x=3$ & $h=0.01$

34. For the function $f(x)=x^3$, find the different quotient when

a) For any x and h.

b) $x=3$ & $h=2$

c) $x=3$ & $h=0.1$

d) $x=3$ & $h=0.01$

35. For the function $f(t)$ graphed, find the following limit, or explain why they do not exist.

a) $\lim\limits_{t\to -2^-} f(t)$ *b)* $\lim\limits_{t\to -2^+} f(t)$ *c)* $\lim\limits_{t\to -2} f(t)$ *d)* $\lim\limits_{t\to -1^-} f(t)$

e) $\lim\limits_{t\to -1^+} f(t)$ *f)* $\lim\limits_{t\to -1} f(t)$ *g)* $\lim\limits_{t\to 0^-} f(t)$ *h)* $\lim\limits_{t\to 0^+} f(t)$

i) $\lim\limits_{t\to 0} f(t)$ *j)* $\lim\limits_{t\to -\frac{1}{2}} f(t)$ *k)* $\lim\limits_{t\to \frac{1}{2}} f(t)$

36. For the function $f(x)$ graphed, find the following limit, or explain why they do not exist.

a) $\lim\limits_{x\to -2} f(x)$ *b)* $\lim\limits_{x\to 2} f(x)$ *c)* $\lim\limits_{x\to 1^-} f(x)$

d) $\lim\limits_{x\to 1^+} f(x)$ *e)* $\lim\limits_{x\to 1} f(x)$

37. For the function $f(x)$ graphed, find the following limit, or explain why they do not exist.

a) $\lim\limits_{x\to -5} f(x)$ *b)* $\lim\limits_{x\to -3^-} f(x)$ *c)* $\lim\limits_{x\to -3^+} f(x)$ *d)* $\lim\limits_{x\to -3} f(x)$

e) $\lim\limits_{x\to -1^-} f(x)$ *f)* $\lim\limits_{x\to -1^+} f(x)$ *g)* $\lim\limits_{x\to -1} f(x)$ *h)* $\lim\limits_{x\to 0} f(x)$

i) $\lim\limits_{x\to 1^-} f(x)$ *j)* $\lim\limits_{x\to 1^+} f(x)$ *k)* $\lim\limits_{x\to 1} f(x)$ *l)* $\lim\limits_{x\to 2} f(x)$

38. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a) $\lim\limits_{x\to -1^+} f(x) = 1$

b) $\lim\limits_{x\to 0^-} f(x) = 0$

c) $\lim\limits_{x\to 0^-} f(x) = 1$

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

e) $\lim\limits_{x\to 0} f(x)$ *exists*

f) $\lim\limits_{x\to 0} f(x) = 0$

g) $\lim\limits_{x\to 0} f(x) = 1$

h) $\lim\limits_{x\to 1} f(x) = 1\backslash$

i) $\lim\limits_{x\to 1} f(x) = 0$

j) $\lim\limits_{x\to 2^-} f(x) = 2$

k) $\lim\limits_{x\to -1^-} f(x) = 0$ *does not exist*

l) $\lim\limits_{x\to 2^+} f(x) = 0$

Section 1.2 – Techniques of Limits

In this section, we are going to evaluate the limit and using different techniques to find the limit at a certain point.

1.2−1 Properties of Limits

Constant function $\left(f(x)=k\right)$: $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} k = k$

Identity function $\left(f(x)=x\right)$: $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} x = x_0$

1.2−2 *Limit* Laws

If $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$

Constant Multiple Rule: $\lim\limits_{x\to c} \left[bf(x)\right] = b \lim\limits_{x\to c} f(x)$

$$= bL$$

Sum and Difference Rules: $\lim\limits_{x\to c} \left[f(x) \pm g(x)\right] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$

$$= L \pm M$$

Product Rule: $\lim\limits_{x\to c} \left[f(x)\cdot g(x)\right] = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

$$= L.M$$

Quotient Rule: $\lim\limits_{x\to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$

$$= \frac{L}{M} \qquad M \neq 0$$

Power Rule: $\lim\limits_{x\to c} \left(f(x)\right)^n = \left[\lim\limits_{x\to c} f(x)\right]^n$

$$= L^n$$

Root Rule: $$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$

$$= \sqrt[n]{L} \qquad n > 0, \quad L > 0,\ n \text{ is even}$$

Example 1

Find the following limits:

$a)\ \lim_{x\to c}\left(x^3+4x^2-3\right)$ $\qquad b)\ \lim_{x\to c}\dfrac{x^4+x^2-1}{x^2+5}$ $\qquad c)\ \lim_{x\to -2}\sqrt{4x^2-3}$

Solution

a) $$\lim_{x\to c}\left(x^3+4x^2-3\right) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c}(3)$$ ***Sum and Difference Rules***

$$= c^3+4c^2-3$$

b) $$\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to c}\left(x^4+x^2-1\right)}{\lim_{x\to c}\left(x^2+5\right)}$$ ***Quotient Rule***

$$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$$ ***Sum and Difference Rules***

$$= \frac{c^4+c^2-1}{c^2+5}$$

c) $$\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}\left(4x^2-3\right)}$$ ***Root Rule***

$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$$ ***Difference Rule***

$$= \sqrt{4(-2)^2-3}$$

$$= \sqrt{16-3}$$

$$= \sqrt{13}$$

1.2–3 *Theorem* – Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$$

1.2–4 *Theorem* – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

Example 2

Find the limit: $\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

Solution

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5}$$

$$= \frac{0}{6}$$

$$= 0$$

1.2–5 Eliminating *Zero* Denominators Algebraically

1.2–6 Evaluating a limit by Factoring

Example 3

Evaluate: $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

Solution

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{1^2 + 1 - 2}{1^2 - 1} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)}$$

$$= \lim_{x \to 1} \frac{(x+2)}{x}$$

$$= \frac{1+2}{1}$$

$$= 3$$

1.2−7 Evaluating a limit by Multiplying by a conjugate

Example 4

Evaluate: $\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{0+100}-10}{0} = \frac{0}{0}$$

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \qquad (a-b)(a+b) = a^2 - b^2; \quad \left(\sqrt{a}\right)^2 = a$$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{1}{\sqrt{x^2+100}+10}$$

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0+100}+10}$$

$$= \frac{1}{10+10}$$

$$= \frac{1}{20}$$

1.2−8 The Sandwich (Squeeze) *Theorem*

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L \quad \textit{then} \quad \lim_{x \to c} g(x) = L$$

Example 5

Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ *for all* $x \ne 0$.

Find the $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution

$$\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) = 1 - \frac{0}{4}$$

$$= 1$$

$$\lim_{x \to 0} \left(1 + \frac{x^2}{2}\right) = 1$$

The Sandwich theorem implies that $\lim_{x \to 0} u(x) = 1$

1.2–9 *Theorem*

Suppose that $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$$

Example 6

Show that $y = \sin\left(\frac{1}{x}\right)$ has ***no*** limit as x approaches zero from either side.

Solution

As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from –1 to 1.

There is no single number L that the function's values stay increasingly close to as x approaches zero.

The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

1.2−10 Limit Involving $\frac{\sin\theta}{\theta}$

A central fact about $\frac{\sin\theta}{\theta}$ is that in radian measure it limit as $\theta \to 0$ is **1**.

1.2−11 *Theorem*

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \quad (\theta \text{ in rad.})$$

Proof

We need to show that the right-hand limit is 1, $\theta < \frac{\pi}{2}$

Notice that:

$$Area\ \Delta OAP < Area\ Sector\ OAP < Area\ \Delta OAT$$

$$Area\ \Delta OAP = \frac{1}{2} base \times height = \frac{1}{2}(1)(\sin\theta)$$

$$Area\ Sector\ \Delta OAP = \frac{1}{2} r^2 \times \theta = \frac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$Area\ \Delta OAP = \frac{1}{2} base \times height = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta$$

$$\Rightarrow \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\frac{2}{\sin\theta}\frac{1}{2}\sin\theta < \frac{1}{2}\theta\frac{2}{\sin\theta} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}\frac{2}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ ***Taking reciprocals reverses the inequalities.***

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0^+} \cos\theta = 1$, then

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$$

So $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Example 7

Show that $\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$

Solution

Using the half-angle formula: $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{1 - 2\sin^2\left(\frac{x}{2}\right) - 1}{x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2\left(\frac{x}{2}\right)}{x}$$ ***Let*** $\theta = \frac{x}{2}$

$$= -\lim_{\theta\to 0} \frac{2\sin^2(\theta)}{2\theta}$$

$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta$$

$$= -(1)(0)$$

$$= 0$$

Example 8

Show that $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x} = \dfrac{2}{5}$

Solution

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{\left(\frac{2}{5}\right)\sin 2x}{\left(\frac{2}{5}\right)5x}$$

Since we need 2x in the denominator

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1)$$

$$= \frac{2}{5}$$

1.2−12 Limit of *sine* as approach 0

$$\lim_{x\to 0} \frac{\sin ax}{bx} = \lim_{x\to 0} \frac{\sin ax}{bx}\frac{a}{a}$$

$$= \frac{a}{b} \lim_{ax\to 0} \frac{\sin ax}{ax}$$

Since $\lim\limits_{ax\to 0} \dfrac{\sin ax}{ax} = 1$

Then

$$\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

Example 9

Evaluate $\lim\limits_{x\to 0} \dfrac{\sin \pi x}{3x}$

Solution

$$\lim_{x\to 0} \frac{\sin \pi x}{3x} = \frac{\pi}{3}$$

Example 10

Show that $\lim\limits_{x\to 0} \dfrac{\tan x \ \sec 2x}{3x} = \dfrac{1}{3}$

Solution

$$\lim_{x\to 0} \frac{\tan x \ \sec 2x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos 2x} = 1$$

$$= \frac{1}{3}(1)(1)(1)$$

$$= \frac{1}{3}$$

Example 11

Evaluate $\lim\limits_{t\to 0} \dfrac{\sqrt{t^2+25}-5}{t^2}$

Solution

$$\lim_{t\to 0} \frac{\sqrt{t^2+25}-5}{t^2} = \frac{5-5}{0} = \frac{0}{0}$$

$$= \lim_{t\to 0} \frac{\sqrt{t^2+25}-5}{t^2} \cdot \frac{\sqrt{t^2+25}+5}{\sqrt{t^2+25}+5}$$

$$= \lim_{t\to 0} \frac{t^2+25-25}{t^2\left(\sqrt{t^2+25}+5\right)}$$

$$= \lim_{t\to 0} \frac{t^2}{t^2\left(\sqrt{t^2+25}+5\right)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+25}+5}$$

$$= \frac{1}{5+5}$$

$$= \frac{1}{10}$$

Example 12

Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{2}$

Solution

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2} = \frac{1-1}{2}$$

$$= 0$$

Example 13

Evaluate $\lim\limits_{x \to 1} \left(e^x - \ln x\right)$

Solution

$$\lim_{x \to 1} \left(e^x - \ln x\right) = e - \ln 1$$

$$= e$$

Exercises *Section* 1.2 – Techniques of Limits

$(\mathbf{1-150})$ Find the limit:

1. $\lim\limits_{x\to 3} (-1)$

2. $\lim\limits_{x\to -1} 3$

3. $\lim\limits_{x\to 1{,}000} 18\pi^2$

4. $\lim\limits_{x\to 3} 4t^2$

5. $\lim\limits_{x\to 2} (6x-2)$

6. $\lim\limits_{x\to 1} \sqrt{5x+6}$

7. $\lim\limits_{x\to 9} \sqrt{x}$

8. $\lim\limits_{x\to -3} \left(x^2+3x\right)$

9. $\lim\limits_{x\to -4} |x-4|$

10. $\lim\limits_{x\to 4} (x+2)$

11. $\lim\limits_{x\to 4} (x-4)$

12. $\lim\limits_{x\to 2} (5x-6)^{3/2}$

13. $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

14. $\lim\limits_{x\to 1} (2x+4)$

15. $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

16. $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$

17. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

18. $\lim\limits_{x\to 3} \dfrac{x^2-x-1}{\sqrt{x+1}}$

19. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

20. $\lim\limits_{x\to 0} (3x-2)$

21. $\lim\limits_{x\to 1} \left(2x^2-x+4\right)$

22. $\lim\limits_{x\to -2} \left(x^3-2x^2+4x+8\right)$

23. $\lim\limits_{x\to 1} \left(x^4-3x^3+7x^2+9x-5\right)$

24. $\lim\limits_{x\to 0} \left(x^4-3x^3+7x^2+9x-5\right)$

25. $\lim\limits_{x\to -1} \left(2x^6-2x^3+7x^2+4x-5\right)$

26. $\lim\limits_{x\to 2} \left(-2x^5-\frac{3}{2}x^3+\frac{1}{4}x^2+5x-5\right)$

27. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

28. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

29. $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x-3}$

30. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

31. $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

32. $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

33. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

34. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

35. $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

36. $\lim\limits_{x\to 0} (2x-8)^{1/3}$

37. $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

38. $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

39. $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

40. $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

41. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

42. $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

43. $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

44. $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

45. $\lim\limits_{x\to 0} \dfrac{\sqrt{9+x}-\sqrt{9-x}}{x}$

46. $\lim\limits_{x\to 0} \dfrac{\sqrt{9+2x}-\sqrt{9}}{x}$

47. $\lim\limits_{x\to 1} \dfrac{x-\sqrt[4]{x}}{x-1}$

48. $\lim\limits_{x\to 0} \dfrac{6-\sqrt{36-x^2}}{x}$

49. $\lim\limits_{x\to 2} \dfrac{x^2-4}{\sqrt{x^2+5}-3}$

50. $\lim\limits_{x\to 0} (2\sin x-1)$

51. $\lim\limits_{x\to 0} \sin^2 x$

52. $\lim\limits_{x\to 0} \sec x$

53. $\lim\limits_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

54. $\lim\limits_{x\to -\pi} \sqrt{x+4}\,\cos(x+\pi)$

55. $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

56. $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

57. $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

58. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

59. $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$

60. $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

61. $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

62. $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

63. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

64. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

65. $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

66. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

67. $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

68. $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

69. $\lim\limits_{\theta\to \pi/4} \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

70. $\lim\limits_{x\to \pi/2} \dfrac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}$

71. $\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x}$

72. $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

73. $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

74. $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

75. $\lim\limits_{x\to 3} \dfrac{1}{x-3}\left(\dfrac{1}{\sqrt{x+1}}-\dfrac{1}{2}\right)$

76. $\lim\limits_{x\to 1/3} \dfrac{x-\frac{1}{3}}{(3x-1)^2}$

77. $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

78. $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

79. $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{x-81}$

80. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{x-1}$

81. $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$

82. $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1}$

83. $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$

84. $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$

85. $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} \quad n\in\mathbb{Z}^+$

86. $\lim\limits_{h\to 0} \dfrac{100}{(10h-1)^{11}+2}$

87. $\lim\limits_{h\to 0} \dfrac{(5+h)^2-25}{h}$

88. $\lim\limits_{x\to 3} \dfrac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}$

89. $\lim\limits_{x\to 1} \dfrac{\sqrt{10x-9}-1}{x-1}$

90. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$

91. $\lim\limits_{x\to c} \dfrac{x^2-2cx+c^2}{x-c}$

92. $\lim\limits_{x\to -c} \dfrac{x^2+5cx+4c^2}{x^2+cx}$

93. $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$

94. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

95. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

96. $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

97. $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{ax+1}-1} \quad (a\neq 0)$

98. $\lim\limits_{x\to \pi} \dfrac{\cos^2 x+3\cos x+2}{\cos x+1}$

99. $\lim\limits_{x\to \frac{3\pi}{2}} \dfrac{\sin^2 x+6\sin x+5}{\sin^2 x-1}$

100. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x-1}{\sqrt{\sin x}-1}$

101. $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+\sin x}-\frac{1}{2}}{\sin x}$

102. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{e^x-1}$

103. $\lim\limits_{x\to \frac{\pi}{4}} \csc x$

104. $\lim\limits_{x\to 4} \dfrac{x-5}{\left(x^2-10x+24\right)^2}$

105. $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin^2 x}$

106. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\sin x}$

107. $\lim\limits_{x\to 0} \dfrac{x^3 - 5x^2}{x^2}$

108. $\lim\limits_{x\to 5} \dfrac{4x^2 - 100}{x-5}$

109. $\lim\limits_{x\to 3} \dfrac{\sqrt{9-6x+x^2}}{x-3}$

110. $\lim\limits_{x\to 3} \dfrac{\sqrt{9+6x+x^2}}{x-3}$

111. $\lim\limits_{x\to 3} \dfrac{\sqrt{x^2-9}}{x-3}$

112. $\lim\limits_{x\to \frac{4\pi}{3}} \sin x$

113. $\lim\limits_{x\to \frac{2\pi}{3}} \cos x$

114. $\lim\limits_{x\to \frac{7\pi}{4}} \sin x$

115. $\lim\limits_{x\to 1} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

116. $\lim\limits_{x\to 2} \dfrac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

117. $\lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

118. $\lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

119. $\lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

120. $\lim\limits_{x\to 1} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

121. $\lim\limits_{x\to \pi} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

122. $\lim\limits_{\theta\to \frac{\pi}{2}} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)$

123. $\lim\limits_{\theta\to -\pi} \ln\left|2+\cos\theta\right|$

124. $\lim\limits_{x\to 0} e^{x^3}$

125. $\lim\limits_{x\to 1} e^{x^2}$

126. $\lim\limits_{x\to 1} e^{x^3-1}$

127. $\lim\limits_{x\to -1} e^{x^3-1}$

128. $\lim\limits_{x\to 2} \left(e^{x^2} - \ln x\right)$

129. $\lim\limits_{x\to 1} \left(e^{x^2} - \ln x\right)$

130. $\lim\limits_{x\to e} \ln x$

131. $\lim\limits_{x\to e} \ln x^2$

132. $\lim\limits_{x\to 0^+} \ln x$

133. $\lim\limits_{x\to 1} \dfrac{1}{\ln x}$

134. $\lim\limits_{x\to e} \ln e^{2x}$

135. $\lim\limits_{x\to 1} \ln e^{x^2}$

136. $\lim\limits_{x\to -1} \ln e^{x^2}$

137. $\lim\limits_{x\to 1} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

138. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

139. $\lim\limits_{x\to 3} \frac{1}{3}\ln\left|x^3+3x^2-6x\right|$

140. $\lim\limits_{x\to \ln 2} \arctan e^x$

141. $\lim\limits_{x\to 0} \arctan e^x$

142. $\lim\limits_{x\to 1} \ln\left|2x^3+9x^2+12x+36\right|$

143. $\lim\limits_{x\to \frac{\pi}{4}} e^{\sin^2 x}$

144. $\lim\limits_{x\to 9} \dfrac{x+9}{\sqrt{x}}$

145. $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}}$

146. $\lim\limits_{x\to -2} \dfrac{2x^3+5x}{3x-2}$.

147. $\lim\limits_{x\to 2} \dfrac{2x^4+3}{2x-5}$

148. $\lim\limits_{x\to -4} \left(\dfrac{2x}{x+4}+\dfrac{16}{x+4}\right)$

149. $\lim\limits_{x\to 4} \dfrac{\sqrt{x+5}-3}{x^2-3x-4}$

150. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{(x-3)(x+1)}$

(151 – 154) Evaluate the limit in terms of the constant a.

151. $\lim\limits_{x\to 5} (5x+2ax+4a)$

152. $\lim\limits_{x\to 3} \dfrac{x-a}{x^2+a}$

153. $\lim\limits_{t\to 4} (5a+t)$

154. $\lim\limits_{t\to 4} \sqrt{4a+t}$

(155 – 156) Find all values of a such that the limit exists

155. $\lim\limits_{x\to 1} \dfrac{x^2+4x+a}{x-1}$

156. $\lim\limits_{x\to 1} \dfrac{x^2+2x+a}{x-a}$

157. Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

158. Explain why the limits do not exist for $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

159. If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

160. If $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$

161. If $x^4 \le f(x) \le x^2$ $-1 \le x \le 1$ and $x^2 \le f(x) \le x^4$ $x<-1$ *and* $x>1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

162. Given the piecewise function: $f(x)=\begin{cases}3x-1 & for \quad x\le -1\\ x^2 & for \quad x>-1\end{cases}$

Find

a) $\lim\limits_{x\to -5} f(x)$ *b)* $\lim\limits_{x\to -1} f(x)$ *c)* $\lim\limits_{x\to 1} f(x)$

163. Find the limit: $\lim\limits_{x\to 0} f(x)$ $f(x)=\begin{cases}x^2+1 & x<0\\ 2x+1 & x>0\end{cases}$

164. Let $f(x)=\begin{cases}0, & x\le 0\\ \sin\frac{1}{x}, & x>0\end{cases}$

a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

165. Let $g(x)=\sqrt{x}\ \sin\frac{1}{x}$

a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

Section 1.3 – Infinite Limits

1.3−1 *Definitions*

We say that $f(x)$ has the **limit *L* as *x* approaches infinity** and write $\lim_{x\to\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \quad x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

We say that $f(x)$ has the **limit *L* as *x* approaches *minus* infinity** and write $\lim_{x\to-\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \quad x < M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

***Basic Facts*:** $\lim_{x\to\pm\infty} k = k$ *and* $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

Example 1

Find $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}$$ *Divide by* x^2

$$= \frac{5+0-0}{3+0}$$ $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

$$= \frac{5}{3}$$

Example 2

Find $\lim_{x\to\infty} \frac{11x+2}{2x^3-1}$

Solution

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$$ $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

$$= \frac{0+0}{2-0}$$

$$= 0$$

1.3−2 *Definition*

We say $\lim\limits_{x\to 0^+} f(x) = \infty$

That $\lim\limits_{x\to 0^+} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x \to 0^+$.

We say $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$

That $\lim\limits_{x\to 0^-} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x \to 0^-$.

The limit has a value of infinity or minus infinity, such a function $f(x) = \frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$.

1.3−3 Vertical Asymptote (*VA*) - Think Domain

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$\lim_{x\to a^+} f(x) \to \pm\infty \quad or \quad \lim_{x\to a^-} f(x) \to \pm\infty$$

As ***x*** approaches ***a*** from either the left or the right

$$\lim_{x\to 0^+} \frac{1}{x} \to \infty \quad or \quad \lim_{x\to 0^-} \frac{1}{x} \to -\infty$$

Example 3

Find $\lim\limits_{x\to 3^+} \frac{2-5x}{x-3}$ *and* $\lim\limits_{x\to 3^-} \frac{2-5x}{x-3}$

Solution

$$\lim_{x\to 3^+} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^+-3} \begin{array}{l} \to -13 \\ \to \textit{positive and approaches } 0 \end{array}$$

$$= \frac{-13}{0^+}$$

$$= -\infty$$

$$\lim_{x\to 3^-} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^- - 3} \begin{array}{l} \to -13 \\ \to \textit{negative and approaches } 0 \end{array}$$

$$= \frac{-13}{0^-}$$

$$= \infty$$

***Example* 4**

Find $\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2}$

Solution

$$\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{168}{0^-} \qquad \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{(x-2)(x-3)}{x(x+4)} \begin{array}{l} \to \textit{positive} \\ \to \textit{negative and approaches } 0 \end{array}$$

$$= -\infty$$

1.3−4 Limits at Infinity

Notation	***Terminology***
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

1.3−5 Horizontal Asymptote (*HA*)

The line $y = b$ is a ***horizontal asymptote*** for the graph of a function f if

$$\lim_{x\to\infty} f(x) = b \quad \textit{or} \quad \lim_{x\to -\infty} f(x) = b$$

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$

$= \frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is ***less*** than of denominator ($n < m$) $\Rightarrow$ $y = 0$

$$\lim_{x\to\infty} \frac{2x+1}{4x^2+5} = \lim_{x\to\infty} \frac{2x}{4x^2}$$

$$= 0$$

$$HA: \quad y = 0$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow$ $y = \frac{a_n}{b_m}$

$$\lim_{x\to\infty} \frac{2x^2+1}{4x^2+5} = \lim_{x\to\infty} \frac{2x^2}{4x^2}$$

$$= \frac{1}{2}$$

$$HA: \quad y = \frac{1}{2}$$

3. If the degree of numerator is ***greater*** than of denominator ($n > m$) $\Rightarrow$ ***No*** horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow \quad No \;\; HA$$

$$\lim_{x\to\infty} \frac{2x^3+1}{4x^2+5} = \lim_{x\to\infty} \frac{2x^3}{4x^2}$$

$$= \infty$$

Example 5

Find the limit of $\displaystyle\lim_{x\to\infty} \frac{2-x^4}{2x^5+1}$

Solution

$$\lim_{x\to\infty} \frac{2-x^4}{2x^5+1} = \lim_{x\to\infty} \frac{-x^4}{2x^5}$$

$$= 0$$

Example 6

Find the horizontal asymptotes of the graph of $f(x)=\dfrac{x^3-2}{|x|^3+1}$

Solution

For $x \geq 0$

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to\infty}\frac{x^3}{x^3}$$

$$=1$$

For $x \leq 0$

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{-x\to-\infty}\frac{x^3}{(-x)^3}$$

$$=-1$$

The ***HA*** are $y=-1$ and $y=1$.

Example 7

Find $\lim\limits_{x\to\infty}\sin\left(\dfrac{1}{x}\right)$

Solution

Let $t=\dfrac{1}{x}$

$t\to 0 \quad as \quad x\to\infty$

$$\lim_{x\to\infty}\sin\left(\frac{1}{x}\right)=\lim_{t\to 0}\sin t$$

$$=0$$

Example 8

Let $f(x)=\dfrac{x^2-4x+3}{x^2-1}$, determine the following limits and find the vertical asymptotes of $f(x)$.

a) $\lim\limits_{x\to 1} f(x)$ *b)* $\lim\limits_{x\to -1^-} f(x)$ *c)* $\lim\limits_{x\to -1^+} f(x)$

Solution

$x^2 - 1 = 0 \quad \Rightarrow \quad \underline{x = \pm 1}$

a) $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 - 1} = \dfrac{0}{0}$

$$= \lim_{x \to 1} \frac{(x-1)(x-3)}{(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{x-3}{x+1}$$

$$\underline{= -1}$$

The ***vertical asymptote***: $\underline{x = -1}$, while the ***hole*** is $\underline{(1, \ -1)}$

b) $f(x) = \dfrac{(x-1)(x-3)}{(x-1)(x+1)}$

$$\underline{= \frac{x-3}{x+1}}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x-3}{x+1} \quad \begin{matrix} \to \textit{negative} \\ \to \textit{negative and approaches } 0 \end{matrix}$$

$$\underline{= \infty}$$

c) $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} \dfrac{x-3}{x+1} \quad \begin{matrix} \to \textit{negative} \\ \to \textit{positive and approaches } 0 \end{matrix}$

$$\underline{= -\infty}$$

Example 9

Find $\lim\limits_{\theta \to 0^+} \cot\theta$ *and* $\lim\limits_{\theta \to 0^-} \cot\theta$

Solution

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \Rightarrow \quad \cot 0 = \frac{1}{0},$$

As $\theta \to 0^+$ $\cos\theta > 0;$ $\sin\theta > 0$

$$\lim_{\theta \to 0^+} \cot\theta \underline{= \infty}$$

As $\theta \to 0^-$ $\cos\theta > 0;$ $\sin\theta < 0$

$$\lim_{\theta \to 0^+} \cot\theta \underline{= -\infty}$$

Example 10

Find $\lim\limits_{x \to \pm\infty} x\sin\left(\frac{1}{x}\right)$

Solution

Let $t = \frac{1}{x} \quad \Rightarrow \quad x = \frac{1}{t}$

$$\lim_{x \to \infty} x\sin\left(\frac{1}{x}\right) = \lim_{t \to 0^+} \frac{\sin t}{t}$$

$$\underline{= 1}$$

$$\lim_{x \to -\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t \to 0^-} \frac{\sin t}{t}$$

$$\underline{= 1}$$

Example 11

Find the horizontal asymptote of $y = 2 + \frac{\sin x}{x}$

Solution

Since $0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$

$$\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0$$

$$\lim_{x\to\pm\infty}\frac{\sin x}{x} = 0$$

$$\lim_{x\to\pm\infty}\left(2+\frac{\sin x}{x}\right) = 2+0$$

$$= 2$$

***HA*:** $y = 2$

Example 12

Find $\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$

Solution

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\cdot\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}} \qquad (a-b)(a+b) = a^2-b^2$$

$$= \lim_{x\to\infty}\frac{x^2-\left(x^2+16\right)}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2-x^2-16}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{-\frac{16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{-\frac{16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$= \frac{0}{1+\sqrt{1+0}}$$

$$= 0$$

1.3−6 *Slant* or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a ***slant*** or ***oblique*** asymptote, and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{\big) 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2}$$

$$= (3x - 6) + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example 13

Find the horizontal and vertical asymptotes of the curve $y = \dfrac{x + 3}{x + 2}$

Solution

HA: $y \to \dfrac{x}{x} = 1 \quad \Rightarrow \quad y = 1$

VA: $x + 2 = 0 \quad \Rightarrow \quad x = -2$

Example 14

Find the horizontal and vertical asymptotes of the curve $f(x) = -\dfrac{8}{x^2 - 4}$

Solution

HA: $y \quad \to \quad \lim\limits_{x \to \infty} -\dfrac{8}{x^2} = 0$

$\underline{y = 0}$

VA: $x^2 - 4 = 0 \quad \Rightarrow \quad \underline{x = \pm 2}$

$$\lim_{x \to 2^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to 2^-} f(x) = \infty$$

Example 15

Find $\displaystyle\lim_{x \to 1^+} \frac{1}{x-1}$ *and* $\displaystyle\lim_{x \to 1^-} \frac{1}{x-1}$

Solution

As $x \to 1^+ \quad \Rightarrow \quad x - 1 \to 0^+$

$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

Example 16

Find $\displaystyle\lim_{x \to 2} \frac{(x-2)^2}{x^2 - 4}$

Solution

$$\lim_{x \to 2} \frac{(x-2)^2}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)^2}{(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{(x-2)}{(x+2)}$$

$$= \frac{0}{4}$$

$$\underline{= 0}$$

Example 17

Find $\displaystyle\lim_{x \to 2} \frac{x-2}{x^2 - 4}$

Solution

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{1}{x+2}$$

$$=\frac{1}{4}$$

Example 18

Find $a)\ \lim_{x\to 2^+}\frac{x-3}{x^2-4}$ $\quad b)\ \lim_{x\to 2^-}\frac{x-3}{x^2-4}$ $\quad c)\ \lim_{x\to 2}\frac{x-3}{x^2-4}$

Solution

a) $\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}$

$$=-\infty$$

b) $\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}$

$$=\infty$$

c) $\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}$

$$=\text{doesn't exist}$$

Exercises *Section* 1.3 – Infinite Limits

$(\mathbf{1-295})$ Find the limit.

1. $\lim\limits_{x\to 5} \dfrac{x-7}{x(x-5)^2}$

2. $\lim\limits_{x\to -5^+} \dfrac{x-5}{x+5}$

3. $\lim\limits_{x\to 3^-} \dfrac{x-4}{x^2-3x}$

4. $\lim\limits_{x\to 0^+} \dfrac{1}{3x}$

5. $\lim\limits_{x\to -5^-} \dfrac{3x}{2x+10}$

6. $\lim\limits_{x\to 0} \dfrac{1}{x^{2/3}}$

7. $\lim\limits_{x\to 0^-} \dfrac{1}{3x^{1/3}}$

8. $\lim\limits_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

9. $\lim\limits_{\theta\to 0^-} (1+\csc\theta)$

10. $\lim\limits_{\theta\to 0^+} \csc\theta$

11. $\lim\limits_{x\to 0^+} (-10\cot x)$

12. $\lim\limits_{\theta\to\frac{\pi}{2}^+} \frac{1}{3}\tan\theta$

13. $\lim\limits_{x\to 2^+} \dfrac{1}{x-2}$

14. $\lim\limits_{x\to 2^-} \dfrac{1}{x-2}$

15. $\lim\limits_{x\to 2} \dfrac{1}{x-2}$

16. $\lim\limits_{x\to 3^+} \dfrac{2}{(x-3)^3}$

17. $\lim\limits_{x\to 3^-} \dfrac{2}{(x-3)^3}$

18. $\lim\limits_{x\to 3} \dfrac{2}{(x-3)^3}$

19. $\lim\limits_{x\to 4^+} \dfrac{x-5}{(x-4)^2}$

20. $\lim\limits_{x\to 4^-} \dfrac{x-5}{(x-4)^2}$

21. $\lim\limits_{x\to 4} \dfrac{x-5}{(x-4)^2}$

22. $\lim\limits_{x\to 1^+} \dfrac{x-2}{(x-1)^3}$

23. $\lim\limits_{x\to 1^-} \dfrac{x-2}{(x-1)^3}$

24. $\lim\limits_{x\to 1} \dfrac{x-2}{(x-1)^3}$

25. $\lim\limits_{x\to 3^+} \dfrac{(x-1)(x-2)}{x-3}$

26. $\lim\limits_{x\to 3^-} \dfrac{(x-1)(x-2)}{x-3}$

27. $\lim\limits_{x\to 3} \dfrac{(x-1)(x-2)}{x-3}$

28. $\lim\limits_{x\to 2^+} \dfrac{x-4}{x(x+2)}$

29. $\lim\limits_{x\to 2^-} \dfrac{x-4}{x(x+2)}$

30. $\lim\limits_{x\to 2} \dfrac{x-4}{x(x+2)}$

31. $\lim\limits_{x\to 2^+} \dfrac{x^2-4x+3}{(x-2)^2}$

32. $\lim\limits_{x\to 2^-} \dfrac{x^2-4x+3}{(x-2)^2}$

33. $\lim\limits_{x\to 2} \dfrac{x^2-4x+3}{(x-2)^2}$

34. $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

35. $\lim\limits_{x\to -2^-} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

36. $\lim\limits_{x\to -2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

37. $\lim\limits_{u\to 0^+} \dfrac{u-1}{\sin u}$

38. $\lim\limits_{x\to 0^-} \dfrac{2}{\tan x}$

39. $\lim\limits_{x\to 1^+} \dfrac{x^2-5x+6}{x-1}$

40. $\lim\limits_{x\to 4} \dfrac{x-5}{\left(x^2-10x+24\right)^2}$

41. $\lim\limits_{x\to 2\pi^-} \csc x$

42. $\lim\limits_{x\to 0^+} e^{\sqrt{x}}$

43. $\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{1+\sin x}{\cos x}$

44. $\lim\limits_{x\to\frac{\pi}{2}^+} \dfrac{1+\sin x}{\cos x}$

45. $\lim\limits_{x\to 0^-} \dfrac{e^x}{1-e^x}$

46. $\lim\limits_{x\to 0^+} \dfrac{e^x}{1-e^x}$

47. $\lim\limits_{x\to 1^-} \dfrac{x}{\ln x}$

48. $\lim\limits_{x\to 0^+} \dfrac{x}{\ln x}$

49. $\lim\limits_{x\to 0^-} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

50. $\lim\limits_{x\to 0^+} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

51. $\lim\limits_{x\to 1^-} \dfrac{\ln x}{\sin^{-1} x}$

52. $\lim\limits_{x\to 0} \dfrac{e^x}{\sin x}$

53. $\lim\limits_{x\to 0^-} \dfrac{e^x}{\sin x}$

54. $\lim\limits_{x\to 2^-} \dfrac{|x-3|}{x-2}$

55. $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$

56. $\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2}$

57. $\lim\limits_{x\to 3^-} \dfrac{1}{(x-3)^4}$

58. $\lim\limits_{x\to 3^+} \dfrac{1}{(x-3)^4}$

59. $\lim\limits_{x\to -2^-} \dfrac{1}{(x+2)^3}$

60. $\lim\limits_{x\to -2^+} \dfrac{1}{(x+2)^3}$

61. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{e^x}{\cos x}$

62. $\lim\limits_{x\to 0} \dfrac{e^x}{\ln(x)}$

63. $\lim\limits_{x\to 1} \dfrac{1}{\ln x}$

64. $\lim\limits_{x\to \pi} \dfrac{x}{\sin x}$

65. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln\left(\frac{1}{x}\right)}{\cos x}$

66. $\lim\limits_{x\to 0^+} \left(\dfrac{4}{\sqrt{x}} - \dfrac{4}{\sqrt{x^2+x}}\right)$

67. $\lim\limits_{x\to 0^-} \left(\dfrac{4}{\sqrt{x}} - \dfrac{4}{\sqrt{x^2+x}}\right)$

68. $\lim\limits_{x\to 0} \left(\dfrac{4}{\sqrt{x}} - \dfrac{4}{\sqrt{x^2+x}}\right)$

69. $\lim\limits_{x\to 3^+} \dfrac{x^2+x+2}{x^2-2x-3}$

70. $\lim\limits_{x\to 3^-} \dfrac{x^2+x+2}{x^2-2x-3}$

71. $\lim\limits_{x\to 3} \dfrac{x^2+x+2}{x^2-2x-3}$

72. $\lim\limits_{x\to \infty} x^{12}$

73. $\lim\limits_{x\to -\infty} 3x^9$

74. $\lim\limits_{x\to -\infty} x^{-8}$

75. $\lim\limits_{x\to -\infty} x^{-9}$

76. $\lim\limits_{x\to -\infty} 2x^{-6}$

77. $\lim\limits_{x\to \infty} \left(3x^{12}-9x^7\right)$

78. $\lim\limits_{x\to -\infty} \left(3x^7+x^2\right)$

79. $\lim\limits_{x\to -\infty} \left(-2x^{16}+2\right)$

80. $\lim\limits_{x\to -\infty} \left(2x^{-6}+4x^5\right)$

81. $\lim\limits_{x\to -\infty} \dfrac{\cos x}{3x}$

82. $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{2x+7-5\sin x}$

83. $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

84. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

85. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

86. $\lim\limits_{x\to\infty} \dfrac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

87. $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

88. $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+10}$

89. $\lim\limits_{x\to\infty} \dfrac{x^4-1}{x^5+2}$

90. $\lim\limits_{x\to-\infty} \left(-3x^3+5\right)$

91. $\lim\limits_{x\to\infty} \left(e^{-2x}+\dfrac{2}{x}\right)$

92. $\lim\limits_{x\to\infty} \dfrac{1}{\ln x+1}$

93. $\lim\limits_{x\to\infty} \left(3+\dfrac{10}{x^2}\right)$

94. $\lim\limits_{x\to\infty} \left(5+\dfrac{1}{x}+\dfrac{10}{x^2}\right)$

95. $\lim\limits_{x\to\infty} \dfrac{4x^2+2x+3}{x^2}$

96. $\lim\limits_{x\to\infty} \left(5+\dfrac{100}{x}+\dfrac{\sin^4 x^3}{x^2}\right)$

97. $\lim\limits_{\theta\to\infty} \dfrac{\cos\theta}{\theta^2}$

98. $\lim\limits_{\theta\to\infty} \dfrac{\cos\theta^5}{\sqrt{\theta}}$

99. $\lim\limits_{x\to\infty} \dfrac{4x}{20x+1}$

100. $\lim\limits_{x\to-\infty} \dfrac{4x}{20x+1}$

101. $\lim\limits_{x\to\infty} \dfrac{3x^2-7}{x^2+5x}$

102. $\lim\limits_{x\to-\infty} \dfrac{3x^2-7}{x^2+5x}$

103. $\lim\limits_{x\to\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

104. $\lim\limits_{x\to-\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

105. $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

106. $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

107. $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

108. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

109. $\lim\limits_{x\to\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

110. $\lim\limits_{x\to-\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

111. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)$

112. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

113. $\lim\limits_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

114. $\lim\limits_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

115. $\lim\limits_{x\to-\infty} \left(x+\sqrt{x^2-4x+2}\right)$

116. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

117. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

118. $\lim\limits_{x\to\infty}\left(x+\sqrt{x^2-4x+1}\right)$

119. $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)$

120. $\lim\limits_{x\to-\infty}\left(\sqrt{4x^2-2x+1}-2x\right)$

121. $\lim\limits_{x\to\infty}\left(\sqrt{4x^2-2x+1}-2x\right)$

122. $\lim\limits_{x\to\infty}x\left(\sqrt{4x^2+1}-2x\right)$

123. $\lim\limits_{x\to-\infty}x\left(\sqrt{4x^2+1}+2x\right)$

124. $\lim\limits_{x\to-\infty}x\left(\sqrt{x^2+4}+x\right)$

125.

126. $\lim\limits_{x\to-\infty}\left(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3}\right)$

127. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

128. $\lim\limits_{x\to\infty}\dfrac{x-1}{x^{2/3}-1}$

129. $\lim\limits_{x\to-\infty}\dfrac{x-1}{x^{2/3}-1}$

130. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+2x+6}-3}{x-1}$

131. $\lim\limits_{x\to\infty}\dfrac{\left|1-x^2\right|}{x(x+1)}$

132. $\lim\limits_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

133. $\lim\limits_{x\to\infty}\dfrac{\tan^{-1}x}{x}$

134. $\lim\limits_{x\to\infty}\dfrac{\cos x}{e^{3x}}$

135. $\lim\limits_{x\to0}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

136. $\lim\limits_{x\to\infty}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

137. $\lim\limits_{x\to-\infty}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

138. $\lim\limits_{x\to0}\dfrac{x^2-4x+4}{x^3+5x^2-14x}$

139. $\lim\limits_{x\to2}\dfrac{x^2-4x+4}{x^3+5x^2-14x}$

140. $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x^4-a^4}$

141. $\lim\limits_{x\to0}\dfrac{(x+h)^2-x^2}{h}$

142. $\lim\limits_{h\to0}\dfrac{(x+h)^2-x^2}{h}$

143. $\lim\limits_{x\to1}\dfrac{1-\sqrt{x}}{1-x}$

144. $\lim\limits_{x\to0}\dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

145. $\lim\limits_{x\to1}\dfrac{x^{1/3}-1}{\sqrt{x}-1}$

146. $\lim\limits_{x\to64}\dfrac{x^{2/3}-16}{\sqrt{x}-8}$

147. $\lim\limits_{x\to0}\dfrac{\tan(2x)}{\tan(\pi x)}$

148. $\lim\limits_{x\to\pi^-}\csc x$

149. $\lim\limits_{x\to\pi}\sin\left(\frac{x}{2}+\sin x\right)$

150. $\lim\limits_{x\to\pi}\cos^2(x-\tan x)$

151. $\lim\limits_{x\to0}\dfrac{8x}{3\sin x-x}$

152. $\lim\limits_{x\to0}\dfrac{\cos 2x-1}{\sin x}$

153. $\lim\limits_{x\to-\infty}\dfrac{4-3x^3}{\sqrt{x^6+9}}$

154. $\lim\limits_{x\to-\infty}\dfrac{x^2-4x+8}{3x^3}$

155. $\lim\limits_{x\to-\infty} \dfrac{2x^2+3}{5x^2+7}$

156. $\lim\limits_{x\to\infty} \dfrac{x^4+x^3}{12x^3+128}$

157. $\lim\limits_{x\to-\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$

158. $\lim\limits_{x\to\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$

159. $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

160. $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}$

161. $\lim\limits_{x\to\infty} \dfrac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

162. $\lim\limits_{x\to2^+} \ln(x-2)$

163. $\lim\limits_{x\to1} x^2\ln\left(2-\sqrt{x}\right)$

164. $\lim\limits_{\theta\to0^+} \sqrt{\theta}e^{\cos\frac{\pi}{\theta}}$

165. $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x+6}$

166. $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{5x^2+6}$

167. $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x^3+6}$

168. $\lim\limits_{x\to\infty} \dfrac{1}{5x^2-3x+6}$

169. $\lim\limits_{\theta\to0} \dfrac{\theta\ \cot 4\theta}{\sin^2\theta\ \cot^2 2\theta}$

170. $\lim\limits_{x\to0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

171. $\lim\limits_{x\to2} \dfrac{x^4-16}{x-2}$

172. $\lim\limits_{x\to2} \dfrac{x^3-8}{x-2}$

173. $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

174. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

175. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

176. $\lim\limits_{x\to\infty} \dfrac{x-3}{\sqrt{4x^2+25}}$

177. $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

178. $\lim\limits_{x\to\infty} \dfrac{x^4-x}{15x^3+4}$

179. $\lim\limits_{x\to\infty} \dfrac{x+\sin x+2\sqrt{x}}{x+\sin x}$

180. $\lim\limits_{x\to\infty} \dfrac{x^{2/3}-x^{-1}}{x^{2/3}+\cos^2 x}$

181. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

182. $\lim\limits_{x\to0} \dfrac{\sin 5x}{3x}$

183. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{2x}$

184. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

185. $\lim\limits_{x\to-1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

186. $\lim\limits_{x\to-\infty} \left(\dfrac{1-x^3}{x^2+7x}\right)^5$

187. $\lim\limits_{x\to\infty} \sqrt{\dfrac{x^2-5x}{x^3+x-2}}$

188. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

189. $\lim\limits_{x\to-5^-} \dfrac{3x}{2x+10}$

190. $\lim\limits_{x\to-8^+} \dfrac{3x}{x+8}$

191. $\lim\limits_{x\to 0} \dfrac{-1}{x^2(x+1)}$

192. $\lim\limits_{x\to 7} \dfrac{4}{(x-7)^2}$

193. $\lim\limits_{x\to 0} \dfrac{1}{x^{2/3}}$

194. $\lim\limits_{x\to 2} \dfrac{x-2}{6x^2-10x-4}$

195. $\lim\limits_{x\to 6} \dfrac{\sqrt{x-2}-2}{x^2-36}$

196. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$

197. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin x}$

198. $\lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{\sin\frac{\pi}{x}}$

199. $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2\sin x}{\sin 3x - 3\sin x}$

200. $\lim\limits_{x\to 1} \dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

201. $\lim\limits_{x\to 0} \left(\dfrac{1}{x}+\dfrac{5}{x(x-5)}\right)$

202. $\lim\limits_{x\to 3} \left(\dfrac{1}{x-3}-\dfrac{4}{x^2-2x-3}\right)$

203. $\lim\limits_{x\to 2^-} \sqrt{x-2}$

204. $\lim\limits_{x\to 2^+} \sqrt{x-2}$

205. $\lim\limits_{x\to -2} \sqrt{x^2-6x+3}$

206. $\lim\limits_{x\to \infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$

207. $\lim\limits_{x\to \infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

208. $\lim\limits_{x\to -\infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

209. $\lim\limits_{x\to \infty} \dfrac{1+2x-3x^2}{x^2+x^3}$

210. $\lim\limits_{x\to \infty} \dfrac{1-2x+3x^5}{x^2-3x^3}$

211. $\lim\limits_{x\to 1} \dfrac{x-3}{x^2-2x+1}$

212. $\lim\limits_{x\to \infty} \dfrac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{4x^2-6x+7}$

213. $\lim\limits_{x\to \infty} \left(\dfrac{x^2}{x+1}-\dfrac{x^2}{x-1}\right)$

214. $\lim\limits_{x\to -\infty} \dfrac{2x-5}{|3x+2|}$

215. $\lim\limits_{x\to \infty} \dfrac{2x-5}{|3x+2|}$

216. $\lim\limits_{x\to \infty} \dfrac{3x+2\sqrt{x}}{1-x}$

217. $\lim\limits_{x\to \infty} \dfrac{2x+1}{\sqrt{3x^2+x-1}}$

218. $\lim\limits_{x\to \infty} \dfrac{1-x^2}{3x^2-x-1}$

219. $\lim\limits_{x\to \infty} \dfrac{x^3-2}{x^2+5}$

220. $\lim\limits_{x\to \infty} \sqrt{x}$

221. $\lim\limits_{x\to \infty} \dfrac{1}{\sqrt{x^2+1}}$

222. $\lim\limits_{x\to \infty} \dfrac{x^3+\sin x}{x^3+\cos x}$

223. $\lim\limits_{x\to 1^-} \dfrac{1}{x-1}$

224. $\lim\limits_{x\to 1^+} \dfrac{1}{x-1}$

225. $\lim\limits_{x\to 0} \sin\dfrac{1}{x^2}$

226. $\lim\limits_{\theta\to \infty} \sin\theta$

227. $\lim\limits_{\theta\to \infty} \dfrac{\cos\theta}{\theta}$

228. $\lim\limits_{\theta\to 0} \theta^2 \cos(2\pi\theta)$

229. $\lim\limits_{\theta\to \frac{\pi}{2}} \dfrac{\cot\theta}{\cos\theta}$

230. $\lim\limits_{\theta\to \pi} \dfrac{\sin\theta}{\tan\theta}$

231. $\lim\limits_{x\to -\infty} e^{x^2}$

232. $\lim\limits_{x\to -\infty} e^{x^3}$

233. $\lim\limits_{x\to -\infty} \ln|x|$

234. $\lim\limits_{x\to \frac{2\pi}{3}} \sin x$

235. $\lim\limits_{x\to \frac{5\pi}{4}} \cos x$

236. $\lim\limits_{x\to 0} \dfrac{\sin(2\pi x)}{\sin(3\pi x)}$

237. $\lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

238. $\lim\limits_{x\to 0^+} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

239. $\lim\limits_{x\to 0} e^{x^2}$

240. $\lim\limits_{x\to 1} e^{x^2-1}$

241. $\lim\limits_{x\to 1} \ln x$

242. $\lim\limits_{x\to 2} \left(e^x - \ln x\right)$

243. $\lim\limits_{x\to 1} \dfrac{1}{\ln x}$

244. $\lim\limits_{x\to 4} \left(x^2 - 4x + 1\right)$

245. $\lim\limits_{x\to 1} \dfrac{x+3}{x+6}$

246. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x+1}$

247. $\lim\limits_{x\to 3} \dfrac{x^2-6x+9}{x^2-9}$

248. $\lim\limits_{x\to 2} \dfrac{1}{4-x^2}$

249. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

250. $\lim\limits_{x\to \pi} \dfrac{(x-\pi)^2}{\pi x}$

251. $\lim\limits_{x\to 2} \dfrac{\sqrt{4-4x+x^2}}{x-2}$

252. $\lim\limits_{x\to 2} \dfrac{(x-2)^2}{x^2-4}$

253. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9}$

254. $\lim\limits_{x\to 2} \dfrac{x+2}{x^2+5x+6}$

255. $\lim\limits_{x\to -3} (5-x)^{4/3}$

256. $\lim\limits_{x\to 0} \sqrt{7+\sec^2 x}$

257. $\lim\limits_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

258. $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

259. $\lim\limits_{x\to 0} \dfrac{x^2+4x}{\sqrt{x^3+x^2}}$

260. $\lim\limits_{x\to 0} \sin e^{-1/x^2}$

261. $\lim\limits_{x\to \infty} \sin e^{-1/x^2}$

262. $\lim\limits_{x\to 2^-} \dfrac{-x}{\sqrt{4-x^2}}$

263. $\lim\limits_{x\to -\infty} \dfrac{-x}{\sqrt{4+x^2}}$

264. $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{4+x^2}}$

265. $\lim\limits_{x\to-\infty} \dfrac{\ln(x+2)}{\ln\left(x^2+x-2\right)}$

266. $\lim\limits_{x\to\infty} \dfrac{\ln(x+2)}{\ln\left(x^2+x-2\right)}$

267. $\lim\limits_{x\to-1^-} \left(x^2-2x-3\right)^{-2/3}$

268. $\lim\limits_{x\to1^+} \dfrac{1}{\sqrt[3]{x^2-3x+2}}$

269. $\lim\limits_{x\to-1^-} \dfrac{1}{\sqrt[4]{2x^2+5x+3}}$

270. $\lim\limits_{x\to\infty} \dfrac{x^3-x+6}{3x^3+2x-5}$

271. $\lim\limits_{\theta\to\infty} \cot^{-1}\theta$

272. $\lim\limits_{\theta\to\infty} \sec^{-1}\theta$

273. $\lim\limits_{x\to\infty} \left(e^{-3x}\cos 2x\right)$

274. $\lim\limits_{x\to-\infty} \left(e^{-3x}\cos 2x\right)$

275. $\lim\limits_{x\to\infty} \sin\left(\tan^{-1}x\right)$

276. $\lim\limits_{x\to0} \dfrac{1-\cos x}{x^2}$

277. $\lim\limits_{x\to0} \dfrac{1-\cos x^3}{x^6}$

278. $\lim\limits_{x\to0} \dfrac{1-\cos x^4}{x^8}$

279. $\lim\limits_{x\to\infty} \dfrac{\ln\left(3+e^{2x}\right)}{\ln\left(1+e^x\right)}$

280. $\lim\limits_{x\to\frac{\pi}{2}} e^{-\tan x}$

281. $\lim\limits_{x\to0^+} \tan^{-1}(\ln x)$

282. $\lim\limits_{x\to-\infty} 4\tan^{-1}x-1$

283. $\lim\limits_{x\to\infty} 4\tan^{-1}x-1$

284. $\lim\limits_{x\to\infty} \ln(1-\cos x)$

285. $\lim\limits_{x\to\infty} \dfrac{x-\cos\pi x}{x+1}$

286. $\lim\limits_{x\to-1} \dfrac{x-\cos\pi x}{x+1}$

287. $\lim\limits_{x\to\infty} \dfrac{60x^{-0.4}+50}{3x^{-0.4}+5}$

288. $\lim\limits_{t\to2^-} \dfrac{t^2}{t^2-4}$

289. $\lim\limits_{t\to2^-} \dfrac{t}{t^2-4}$

290. $\lim\limits_{t\to2^+} \dfrac{t^2}{t^2-4}$

291. $\lim\limits_{t\to-2^-} \dfrac{t^2}{t^2-4}$

292. $\lim\limits_{t\to-2^+} \dfrac{t^2}{t^2-4}$

293. $\lim\limits_{t\to1} \dfrac{t^3-1}{t^2+t+1}$

294. $\lim\limits_{t\to1^-} \dfrac{t^2+t+1}{t^3-1}$

295. $\lim\limits_{t\to1^+} \dfrac{t^2+t+1}{t^3-1}$

296. Analyze $\lim\limits_{x\to1^+} \sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x\to1^-} \sqrt{\dfrac{x-1}{x-3}}$

297. Let $f(x)=\dfrac{x^2-7x+12}{x-a}$

a) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=\infty$?

c) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=-\infty$?

$(\mathbf{298-304})$ Find the limit as $x\to\infty$ and as $x\to-\infty$ of:

298. $h(x)=\dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

299. $f(x)=\dfrac{2x+3}{5x+7}$

300. $f(x)=\dfrac{2x^3+7}{x^3-x^2+x+7}$

301. $f(x)=\dfrac{x+1}{x^2+3}$

302. $f(x)=\dfrac{7x^3}{x^3-3x^2+6x}$

303. $f(x)=\dfrac{9x^4+x}{2x^4+5x^2-x+6}$

304. $f(x)=\dfrac{-2x^3-2x+3}{3x^3+3x^2-5x}$

$(\mathbf{305-308})$ Graph the rational function and include the equations of the asymptotes.

305. $y=\dfrac{1}{2x+4}$ **306.** $y=\dfrac{2x}{x+1}$ **307.** $y=\dfrac{x^2}{x-1}$ **308.** $y=\dfrac{x^3+1}{x^2}$

309. Let $f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

$(\mathbf{310-329})$ Find the vertical, horizontal, hole, and oblique asymptotes (if any) of:

310. $y=\dfrac{3x}{1-x}$

311. $y=\dfrac{x^2}{x^2+9}$

312. $y=\dfrac{x-2}{x^2-4x+3}$

313. $y=\dfrac{5x-1}{1-3x}$

314. $y=\dfrac{3}{x-5}$

315. $y=\dfrac{x^3-1}{x^2+1}$

316. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

317. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

318. $y=\dfrac{x-3}{x^2-9}$

319. $y=\dfrac{6}{\sqrt{x^2-4x}}$

320. $f(x)=\dfrac{4x^3+1}{1-x^3}$

321. $f(x)=\dfrac{x+1}{\sqrt{9x^2+x}}$

322. $f(x)=1-e^{-2x}$

323. $f(x)=\dfrac{1}{\ln x^2}$

324. $f(x)=\dfrac{1}{\tan^{-1}x}$

325. $f(x)=\dfrac{2x^2+6}{2x^2+3x-2}$

326. $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

327. $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

328. $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

329. $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

330. A 25-*foot* ladder is leaning against a house. If the base of the ladder is pulled away from the house at the rate of $2\ ft/\sec$, the top will move down the wall at a rate of:

$$h(x)=\frac{2x}{\sqrt{625-x^2}}\quad ft/\sec$$

a) Find the rate when *x* is 7 *feet*.

b) Find the rate when *x* is 15 *feet*.

c) Find the limit of *h* as $x\to 25^-$.

331. After an injection, the concentration of a drug in a muscle varies according to a function of time $f(t)$. Suppose that t is measured in hours and $f(t)=e^{-0.02t}-e^{-0.42t}$.

a) Find the $\lim\limits_{t\to 0} f(t)$

b) Find the $\lim\limits_{t\to\infty} f(t)$

c) Interpret both limits in terms of the concentration of the drug.

332. Suppose an object with initial velocity $v_0 = 0$ $ft/\sec$ and a mass m slugs (constant) is accelerated by a constant force F (pounds) for t seconds.

According to Newton's laws of motion, the object's speed will ne $v_N = \frac{Ft}{m}$.

According to Einstein's theory of relativity, the object's speed will be $v_E = \frac{Fct}{\sqrt{mc^2 + F^2t^2}}$

Where c is the speed of light.

a) Compute $\lim_{t\to\infty} v_N$

b) Compute $\lim_{t\to\infty} v_E$

333. In relativity theory, the length of an object, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the object's length as L_0 at rest. Then at speed v the length will appear to be:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

This equation is the Lorentz contraction formula. Where c is the speed of light in a vacuum, $\approx 3\times 10^8$ $m/\sec$.

a) What happens to L as v increases?

b) Find the $\lim_{v\to c^-} L$

c) Find the $\lim_{v\to c^+} L$

334. A quadratic equation is given by $ax^2 + 2x - 1 = 0$, where a is a constant and r_i are the roots.

a) Find the limit for each root as $a \to -1^+$.

b) Find the limit for each root as $a \to 0$.

Section 1.4 – Continuity

1.4–1 *Definition* of Continuity

Let c be a number in the interval $(a,\ b)$, and let f be a function whose domain contains the interval $(a,\ b)$. The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim\limits_{x \to c} f(x)$ *exists*
3. $\lim\limits_{x \to c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b), then it is continuous on an open interval (a, b)

1.4–2 *Definition*

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if:

$$\lim_{x \to c} f(x) = f(c)$$

Endpoint: A function $y = f(x)$ is **continuous at a left point *a*** or is **continuous at a right point *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad or \quad \lim_{x \to b^-} f(x) = f(b), \quad respectively$$

If a function f is not continuous at a point c, we say that f is ***discontinuous*** at c. (is a ***point of discontinuity***)

Example 1

Find the points at which the function f is continuous and the points at which f is not continuous.

Solution

The function f is continuous at every point in its domain [0, 4] except at $x=1,\ x=2,$ and $x=4$.

At these points, there are breaks in the graph.

$x=0$	$\lim\limits_{x\to 0^+} f(x)=f(0)=1$	f is continuous $@x=0$
$x=1$	$\lim\limits_{x\to 1} f(x)$ *doesn't exist*	f is discontinuous $@x=1$
$x=2$	$\lim\limits_{x\to 2} f(x)=1,\ but\ \ 1\neq f(2)$	f is discontinuous $@x=2$
$x=3$	$\lim\limits_{x\to 3} f(x)=f(3)=2$	f is continuous $@x=3$
$x=4$	$\lim\limits_{x\to 4^-} f(x)=1,\ but\ \ 1\neq f(4)$	f is discontinuous $@x=4$
$c<0,\ c>4$	These points are not in the domain of f.	f is discontinuous
$0<c<4,\ \ c\neq 1,2$	$\lim\limits_{x\to c} f(x)=f(c)$	

Example 2

At what points the function $f(x)=\sqrt{4-x^2}$ is continuous?

Solution

The function is continuous at every point of its domain [–2, 2].
Including $x=-2$, where f is right-continuous, and $x=2$, where f is left-continuous.

1.4–3 Continuous Functions

A function is ***continuous on an interval*** *iff* it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

Example 3

Determine at which points do the function $f(x)=\frac{1}{x}$ is *continuous* and *discontinuous.*

Solution

The function $f(x)$ is a *continuous* everywhere except at $x=0$.

It has a point of *discontinuity* at $x=0$, however, because it is not defined.

It is discontinuous on any interval containing $x=0$

1.4−4 *Theorem* – Properties of Continuous Functions

If the functions f *and* g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

Sums and Differences	$f \pm g$
Constant multiples	$k \cdot g$, for any number k.
Products	$f \cdot g$
Quotients	$\frac{f}{g}$
Powers	f^n ***n*** *a positive integer*
Roots	$\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer.

Proof

$$\begin{aligned}\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}\left(f(x)+g(x)\right)\\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x)\\ &= f(c)+g(c)\\ &= (f+g)(c)\end{aligned}$$

This shows that $f+g$ is continuous.

1.4−5 Composites

All composites of continuous functions are continuous.
If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$

Example 4

Show that $y=\sqrt{x^2-3x+2}$ is continuous everywhere on its domain

Solution

Let $\begin{cases} f(x)=x^2-3x+2, & Domain:\mathbb{R} \\ g(x)=\sqrt{f(x)} & Domain:[0,\ \infty) \end{cases}$

$\therefore$ The function y is continuous on $[0,\ \infty)$

$x^2-3x+2=0 \quad \Rightarrow \quad x=1,\ 2$

The function is continuous everywhere except $1<x<2$

Example 5

Show that $y=\left|\dfrac{x\sin x}{x^2+2}\right|$ is continuous everywhere on its domain.

Solution

Let $\begin{cases} x\sin x & Domain:\mathbb{R} \\ x^2+2 & Domain:\mathbb{R} \end{cases}$

$\therefore$ The function is the composite of a quotient continuous functions with the continuous absolute value function.

Example 6

At what points is the function $f(x)=\dfrac{x+5}{x^2-3x+2}$ continuous.

Solution

The function $f(x)$ is a *continuous* everywhere except at $x=1,\ 2$.

1.4−6 *Theorem*

If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then:

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right)$$

Proof

Let $\varepsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that:

$$|g(y) - g(b)| < \varepsilon \quad \textit{whenever} \quad 0 < |y - b| < \delta_1$$

$$\lim_{x\to c} f(x) = b,\ \exists\ \delta > 0 \ni \ |f(x) - b| < \delta_1 \quad \textit{whenever} \quad 0 < |x - c| < \delta$$

If we let $y = f(x)$, we then have that $|y - b| < \delta_1$ *whenever* $0 < |x - c| < \delta$

Which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of the limit, this proves that $\lim_{x\to c} g(f(x)) = g(b)$.

Example 7

Find the $\lim_{x\to\frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

Solution

$$\lim_{x\to\frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x\to\frac{\pi}{2}} 2x + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2} + x\right)\right)$$

$$= \cos(\pi + \sin 2\pi)$$

$$= \cos(\pi + 0)$$

$$= \cos(\pi)$$

$$= -1$$

Example 8

Show that $f(x)=\dfrac{x^2+x-6}{x^2-4}$, $x \neq 2$ has a continuous extension to $x=2$, and find that extension.

Solution

$$f(x)=\frac{x^2+x-6}{x^2-4}$$
$$=\frac{(x-2)(x+3)}{(x-2)(x+2)}$$
$$=\frac{x+3}{x+2}$$

After simplification, the function is continuous at $x=2$

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}=\lim_{x\to 2}\frac{x+3}{x+2}$$
$$=\frac{5}{4}$$

The new function is the function f with its point of discontinuity at $x=2$ removed.

1.4–7 *Theorem* – the Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

1.4−8 A Consequence for *Root* Finding

We call a solution of the equation $f(x)=0$ a ***root*** of the equation or zero of the function *f*. The Intermediate Value Theorem said that if ***f*** is continuous, then any interval on which *f* changes sign contains a zero of the function.

1.4−9 *Definition*: Intermediate Value Theorem

For any polynomial function $f(x)$ with real coefficients and $f(a)\neq f(b)$ for $a<b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a,\ b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between a and b.

Otherwise, cannot be determined from the interval $[a,\ b]$ because could more that on zero or none.

Example 9

Show that there is a root of the equation x^3-x-1 between 1 and 2.

Solution

$$f(1)=1^3-1-1=-1<0$$

$$f(2)=2^3-2-1=5>0$$

Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example 10

Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

Solution

The function $g(x) = \sqrt{2x+5}$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$ since it is the composite of the square root function with nonnegative linear function $y = 2x+5$. Then the function $f(x) = \sqrt{2x+5} + x^2$ is the sum of the function $g(x)$ and $y = x^2$. It follows that $f(x)$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$.

By trial and error:

$$f(0) = \sqrt{2(0)+5} + 0^2 = \sqrt{5} > 0$$

$$f(2) = \sqrt{2(2)+5} + 2^2 = \sqrt{9} + 4 = 7 > 0$$

f is continuous on the interval $[0,\ 2] \subset \left[-\frac{5}{2},\ \infty\right)$.

Since the value $y_0 = 4$ is between $\sqrt{5}$ and 7, by the Intermediate Value Theorem there is a number $c \in [0,\ 2] \ni f(c) = 4$. That is, the number c solves the original equation.

Example 11

Use the Intermediate Value Theorem to show that the equation $f(x) = x^3 + x^2 - 2x - 1$ has three real zeros between the interval $[-3,\ 3]$

Solution

x	$f(x)$
-3	-13
-2	-1
-1	1
0	-1
1	-1
2	7
3	29

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $(-2,\ -1)$, $(-1,\ 0)$, and $(1,\ 2)$

1.4−10 The Greatest Integer Function

The limit of the greatest integer function $f(x) = \ x$ *or* $f(x) = \lfloor x \rfloor$ as x approaches from the left, and $f(x) = \lceil x \rceil$ as x approaches from the right.

The greatest integer function is called a step function and presents a piece wise function.

In general, the function can be presented as $\lfloor x \rfloor$ *or* x *means* $n \le x < n+1$, where $n \in \mathbb{N}$

$f(x) = \ x$

Example: $1.2 \ = 1$ $\lfloor -1.2 \rfloor = -2$

Properties

- $\lfloor x \rfloor = x, \quad x \in \mathbb{Z}$
- $\lfloor x + n \rfloor = x + n, \quad n \in \mathbb{Z}$
- $\lfloor -x \rfloor = -\lfloor x \rfloor, \quad x \in \mathbb{Z}$
- $\lfloor -x \rfloor = -\lfloor x \rfloor - 1, \quad x \notin \mathbb{Z}$
- *If* $\lfloor f(x) \rfloor \ge y, \quad f(x) \ge y$

Example 12

For the function $f(x) = \lceil x \rceil$, find: $\lim\limits_{x \to 1^+} \lceil x \rceil$, $\lim\limits_{x \to 1^-} \lceil x \rceil$

Solution

$\lim\limits_{x \to 1^+} \lceil x \rceil \ = 2$ $\lim\limits_{x \to 1^-} \lceil x \rceil \ = 1$

Example 13

Graph the function $f(x)=\left\lfloor \frac{1}{2}x \right\rfloor$, find:

Solution

Example 14

For the function $f(x)=\ x\ $, find:

$$\lim_{x\to 2^+} x\ ,\quad \lim_{x\to 2^-} x\ ,\quad \lim_{x\to 0^+} x\ ,\quad \lim_{x\to 0^-} x\ ,$$

Solution

$$\lim_{x\to 2^+} x\ =2 \qquad \lim_{x\to 2^-} x\ =1$$

$$\lim_{x\to 0^+} x\ =0 \qquad \lim_{x\to 0^-} x\ =-1$$

1.4−11 Heaviside Function

The Heaviside step function $H(x)$, sometimes called the Heaviside theta function. it is a function written as piecewise function:

$$H(x)=\begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$$

The Heaviside step function or ***unit*** step function $u(t)$, named after Oliver Heaviside (1850–1925)

$$u(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$

The Heaviside step function plays a fundamental role in Laplace Transforms, Fourier Series, control theory and signal processing, but also in network reliability.

Example 15

Graph the function $f(x)=H(x)\left(x^2+1\right)$

Solution

$$f(x)=H(x)\left(x^2+1\right)$$
$$=\begin{cases}0 & x<0\\ x^2+1 & x\geq 0\end{cases}$$

The function $f(x)$ is discontinuous at $x=0$.

Exercises *Section* 1.4 – Continuity

1. Given the graphed function $f(x)$

a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x)=f(-1)$?

d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x)=f(1)$?

h) Is f continuous at $x=1$?

(2 – 21) At what point(s) is the given function continuous?

2. $y=\dfrac{1}{x-2}-3x$

3. $y=\dfrac{x+3}{x^2-3x-10}$

4. $f(x)=\dfrac{x^2}{x^2+4}$

5. $f(x)=\dfrac{x+1}{x^2-5x+4}$

6. $f(x)=\dfrac{2}{(x+3)(x-3)}$

7. $f(x)=\dfrac{1}{x^2-4}$

8. $f(x)=x^2-3x+4$

9. $f(x)=\cos(\pi x)$

10. $y=|x-1|+\sin x$

11. $y=\dfrac{x+2}{\cos x}$

12. $y=\tan\dfrac{\pi x}{2}$

13. $y=\dfrac{x\tan x}{x^2+1}$

14. $y=\dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$

15. $y=\sqrt{2x+3}$

16. $y=\sqrt[4]{3x-1}$

17. $y=(2-x)^{1/5}$

18. $f(x)=\sqrt{x^2-9}$

19. $f(x)=\sqrt{9-x^2}$

20. $f(x)=\dfrac{1}{\sqrt{4-x^2}}$

21. $f(x)=\dfrac{1}{\sqrt{x^2-4}}$

22. Find $\lim\limits_{x\to\pi} \sin(x-\sin x)$, then is the function continuous at the point being approached?

23. Find $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

24. Find $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

25. Explain why the equation $\cos x = x$ has at least one solution.

(26 – 33) Show that the equation has three solutions in the given interval

26. $x^3 - 15x + 1 = 0$; $[-4, 4]$

27. $x^3 + 10x^2 - 100x + 50 = 0$; $(-20, 10)$

28. $70x^3 - 87x^2 + 32x - 3 = 0$; $(0, 1)$

29. $x^3 - 3x - 1 = 0$; $[-2, 2]$

30. $x^3 - 4x^2 + 2 = 0$; $(-3, 5)$

31. $3x^3 - 10x + 2 = 0$; $[-4, 4]$

32. $4x^3 - x^2 - 6x + 1 = 0$, $(-3, 3)$

33. $3x^3 + 7x^2 - 5x - 3 = 0$, $(-4, 2)$

(34 – 48) Use the Intermediate Value Theorem to find interval the zeros in the given interval

34. $x^2 - 5x + 4 = 0$; $[-3, 3]$

35. $x^3 + 3x - 2 = 0$; $[-2, 2]$

36. $2x^4 - 4x^2 + 1 = 0$; $[-2, 2]$

37. $x^4 - x^2 - x - 4 = 0$; $(-3, 3)$

38. $x^5 + 2x^4 - 6x^3 + 2x - 3 = 0$; $[-4, 2]$

39. $x^6 - 8x^4 + 10x^2 - 1 = 0$; $[-3, 3]$

$f(x) = \frac{x}{(x+1)^2}$; $[1, 8]$

40. $\frac{x^2 + x}{x - 1} = 0$; $[-4, 1)$

41. $f(x) = x^3 - 3x$; $[-2, 2]$

42. $f(x) = x^3 - 2x^2 - x + 2$; $[-2, 3]$

43. $f(x) = x^4 - 1$; $[-2, 2]$

44.

45. $f(x) = \frac{x^3 - 1}{x - 1}$; $[-2, 2]$

46. $f(x) = e^x + x - 2$; $[0, 2]$

47. $f(x) = \sin x + x - 3$; $[0, \pi]$

48. $f(x) = \sqrt{x^2 + 4x} - 2$; $[0, 3]$

49. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\frac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

50. Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (***c*** is called a ***fixed point*** of f).

51. Use the Intermediate Value Theorem to show that the equation $x^5+7x+5=0$ has a solution in the interval $(-1,\ 0)$.

$(\mathbf{52-60})$ Determine whether the following functions are continuous at a.

52. $f(x)=\dfrac{1}{x-5};\quad a=5$

53. $h(x)=\sqrt{x^2-9};\quad a=3$

54. $g(x)=\begin{cases}\dfrac{x^2-16}{x-4} & if \quad x\neq 4\\ 8 & if \quad x=4\end{cases};\quad a=4$

55. $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & if \quad x\neq 2\\ 5 & if \quad x=2\end{cases};\quad a=2$

56. $f(x)=\begin{cases}\dfrac{1}{2}x+1 & if \quad x<4\\ -x+7 & if \quad x>4\end{cases};\quad a=4$

57. $f(x)=\begin{cases}2x+5 & if \quad x\leq 2\\ 4x+1 & if \quad x>2\end{cases};\quad a=2$

58. $f(x)=\begin{cases}2x & if \quad x\leq 0\\ 2x+1 & if \quad x>0\end{cases};\quad a=0$

59. $f(x)=\begin{cases}3x-1 & if \quad x<1\\ 4 & if \quad x=1\\ 2x & if \quad x>1\end{cases};\quad a=1$

60. $f(x)=\begin{cases}x^2 & if \quad x<-1\\ 2 & if \quad x=-1\\ -3x+2 & if \quad x>-1\end{cases};\quad a=-1$

$(\mathbf{61-64})$ Find the intervals on which the following functions are continuous. Specify right- or left-continuity at the endpoints.

61. $f(x)=\sqrt{x^2-5}$

62. $f(x)=e^{\sqrt{x-2}}$

63. $f(x)=\dfrac{2x}{x^3-25x}$

64. $f(x)=\cos e^x$

$(\mathbf{65-72})$ Determine the points of discontinuity. Specify right- or left- continuous.

65. $f(x)=\dfrac{1}{x}$

66. $f(x)=|x|$

67. $f(x)=\dfrac{x-2}{|x-1|}$

68. $f(x)=\dfrac{x-1}{|1-x|}$

69. $f(x)=\lfloor x\rfloor$

70. $f(x)=\left\lfloor \dfrac{x}{2}\right\rfloor$

71. $f(x)=\dfrac{1}{x^2-1}$

72. $f(x)=\dfrac{1}{2-|x|}$

73. Let $g(x)=\begin{cases}5x-2 & if \quad x<1\\ a & if \quad x=1\\ ax^2+bx & if \quad x>1\end{cases}$

Determine the values of the constants a and b for which $g(x)$ is continuous at $x=1$.

74. Let $f(x)=\begin{cases} \dfrac{x^3-3x^2-4x+12}{x-3} & \text{if} \quad x\neq 3 \\ a & \text{if} \quad x=3 \end{cases}$

Determine the value of the constant a for which $f(x)$ is continuous at $x=3$.

75. Let $f(x)=\begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{if} \quad x\geq -\dfrac{5}{2},\ x\neq 2 \\ a & \text{if} \quad x=2 \end{cases}$

Determine the value of the constant a for which $f(x)$ is continuous at $x=2$.

76. Let $f(x)=\begin{cases} x^2-a & \text{if} \quad x<7 \\ 8x+3a & \text{if} \quad x\geq 7 \end{cases}$

Determine the value of the constant a for which $f(x)$ is continuous.

77. Let $f(x)=\begin{cases} \dfrac{3}{x} & \text{for} \quad x<-1 \\ ax+b & \text{for} \quad -1\leq x\leq \dfrac{1}{2} \\ \dfrac{5}{x} & \text{for} \quad x>\dfrac{1}{2} \end{cases}$

Determine the values of the constants a and b for which $f(x)$ is continuous.

78. A Factory sells candy by the pound, charging \$1.50 per pound for the quantities up to and including 20 pounds. Above 20 pounds, the Factory charges \$1.25 per pound for the entire quantity, plus a quantity surcharge a. If x represents the number of pounds, the price function is

$$p(x)=\begin{cases} 1.5x & \text{if} \quad x\leq 20 \\ 1.25x+a & \text{if} \quad x>20 \end{cases}$$

Determine the value of the constant a for which $p(x)$ is continuous at $x=20$.

79. The amount of an antibiotic (in *mg*) in the blood t hours after an intravenous line is opened is given by

$$m(t)=100\left(e^{-0.1t}-e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m=30$ *mg*

c) Is the amount of drug in the blood ever 50 *mg*?

$(\mathbf{80-88})$ Determine of the functions f and g if $\lim\limits_{x\to 0} f(x)$, $\lim\limits_{x\to 0} g(x)$, $\lim\limits_{x\to 0} (f\cdot g)(x)$, and $\lim\limits_{x\to 0}\left(\frac{f}{g}\right)(x)$ does not exist or exist.

80. $f(x)=\lfloor 2x\rfloor$ & $g(x)=\lceil 2x\rceil$

81. $f(x)=\lfloor 2x\rfloor$ & $g(x)=\lfloor 2x\rfloor$

82. $f(x)=\ x$ & $g(x)=-\ x$

83. $f(x)=\frac{2}{x}$ & $g(x)=\frac{2}{x}$

84. $f(x)=\frac{2}{x}$ & $g(x)=-\frac{2}{x}$

85. $f(x)=\frac{2}{x^2}$ & $g(x)=\frac{2}{x+1}$

86. $f(x)=x+3$ & $g(x)=\frac{3}{x}$

87. $f(x)=\begin{cases}-1, & x<0\\ 1, & x\geq 0\end{cases}$ & $g(x)=\begin{cases}1, & x<0\\ -1, & x\geq 0\end{cases}$

88. $f(x)=H(x)$ & $g(x)=1-H(x)$

89. The Heaviside function is given the unit step function $u(t)$

$$u(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$

It used in the study if electric to represent the sudden surge of electric voltage, or current, when a switch turned on.

a) If the switched turned on at time $t=0$ and the voltage $V(t)$ in a circuit of 120 volts are applied instantaneously to the circuit. Write the formula for $V(t)$ in terms of $u(t)$

b) Graph the voltage $V(t)$.

c) If the switched turned on at time $t=0$ and the voltage $V(t)$ is gradually increased to 120 volts over a 60 second time interval for 1 minute. Write the formula for $V(t)$ in terms of $u(t)$.

d) Graph the voltage $V(t)$

e) If the switched turned on at time $t=5$ (time corresponds to translation) and the voltage $V(t)$ in a circuit of 240 volts are applied instantaneously to the circuit. Write the formula

for $V(t)$ in terms of $u(t)$

f) Graph the voltage $V(t)$.

g) If the switched turned on at time $t = 8$ and the voltage $V(t)$ is gradually increased to 120 volts over a 30 second time interval. Write the formula for $V(t)$ in terms of $u(t)$ for $t \leq 38$

h) Graph the voltage $V(t)$.

Section 1.5 – Precise Definition of Limits

1.5-1 *Definition*

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that **the limit of** $f(x)$ **as *x* approaches** x_0 **is the number *L***, and write

$$\lim_{x \to x_0} f(x) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all *x*,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

This definition was discovered by Augustin-Louis Cauchy (1789 – 1857).

ε : epsilon (lowercase *Greek* letter).

δ : delta (lowercase *Greek* letter).

Example 1

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that *y* is close to 7 when *x* is close to 4, so $\lim_{x \to 4} (2x - 1) = 7$. However, how close to $x_0 = 4$ does *x* have to be so that $y = 2x - 1$ differs from 7 by, say less than 2 units?

Solution

We need to find the values of *x* for $|y - 7| < 2$.

$$\begin{aligned} |y - 7| &= |2x - 1 - 7| \\ &= |2x - 8| \end{aligned}$$

$$|2x-8|<2$$

$$-2<2x-8<2$$

$$-2+8<2x-8+8<2+8$$

$$6<2x<10$$

$$\frac{6}{2}<\frac{2x}{2}<\frac{10}{2}$$

$$3<x<5$$

$$3-4<x-4<5-4$$

$$-1<x-4<1$$

Keeping x within 1 unit of $x_0=4$ will keep y within 2 units of $y_0=7$

Example 2

Show that $\lim\limits_{x\to 1}(5x-3)=2$

Solution

Let $x_0=1,\quad f(x)=5x-3,\quad and\ \ L=2$.

For any given $\varepsilon>0$, there exists a $\delta>0$ so that $x\neq 1$ and x is within distance δ of $x_0=1$, that is:

$$0<|x-1|<\delta \quad\Rightarrow\quad |f(x)-2|<\varepsilon$$

$$\left|(5x-3)-2\right| < \varepsilon$$

$$\left|5x-5\right| < \varepsilon$$

$$5\left|x-1\right| < \varepsilon$$

$$\left|x-1\right| < \frac{\varepsilon}{5}$$

Thus, we can take: $\delta = \frac{\varepsilon}{5}$

If $0 < \left|x-1\right| < \delta = \frac{\varepsilon}{5}$

$$\begin{aligned}\left|(5x-3)-2\right| &= \left|5x-5\right| \\ &= 5\left|x-1\right| \\ &= 5\frac{\varepsilon}{5} \\ &= \varepsilon\end{aligned}$$

Which proves that $\lim\limits_{x\to 1}(5x-3) = 2$

Example 3

Prove the results presented graphically $\lim\limits_{x\to x_0} x = x_0$

Solution

Let $\varepsilon > 0$ be given, we must find $\delta > 0$ such that for all x.

$$0 < \left|x - x_0\right| < \delta \quad \Rightarrow \quad \left|x - x_0\right| < \varepsilon$$

This implication will hold if $\delta = \varepsilon$ or any smaller number.

Example 4

For the limit $\lim_{x\to 5}\sqrt{x-1}=2$, find a $\delta>0$ that works for $\varepsilon=1$.

That is, find a $\delta>0$ such that for all x: $\quad 0<|x-5|<\delta \;\Rightarrow\; \left|\sqrt{x-1}-2\right|<1$

Solution

$$\left|\sqrt{x-1}-2\right|<1$$

$$-1<\sqrt{x-1}-2<1$$

$$-1+2<\sqrt{x-1}-2+2<1+2$$

$$1<\sqrt{x-1}<3 \qquad \textit{Square all sides}$$

$$1<x-1<9$$

$$1+1<x-1+1<9+1$$

$$2<x<10$$

The inequality holds for all x in the open interval $(2,\ 10)$.

So, it holds for all $x\neq 5$ in the interval as well.

Finding δ value.

$$5-\delta<x<5+\delta$$

Centered at $x_0=5$ inside the interval $(2,\ 10)$

$$\begin{cases}5-\delta=2\\5+\delta<10\end{cases}\rightarrow \delta=3 \ (\textit{to be centered})$$

$$0<|x-5|<3 \;\Rightarrow\; \left|\sqrt{x-1}-2\right|<1$$

1.5-2 How to Find Algebraically a δ for a Given *f*, *L*, x_0, and $\varepsilon>0$

The process of finding a $\delta>0$ such that for all x:

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Can be accomplished in two steps

1. Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval $(a,\ b)$ containing x_0 on which the inequality holds for all $x \neq x_0$.
2. Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta,\ x_0 + \delta)$ centered at x_0 inside the interval $(a,\ b)$. The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq x_0$ in this δ–interval.

Example 5

Prove that $\lim\limits_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

1. *Solve the inequality* $|f(x) - 4| < \varepsilon$ *to find an open interval containing* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.

For $x \neq x_0 = 2$, $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$|x^2 - 4| < \varepsilon$$

$$-\varepsilon < x^2 - 4 < \varepsilon \qquad \textit{Add 4 to all sides}$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon \qquad \textit{Square root}$$

$$\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon} \qquad \textit{Assume } \varepsilon < 4$$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \varepsilon},\ \sqrt{4 + \varepsilon})$

2. *Find a value of* $\delta > 0$ *that places the open interval* $(2-\delta,\ 2+\delta)$ *inside the interval* $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.

$\Rightarrow \delta = \min\left(2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\right)$.

$$0 < |x-2| < \delta$$

$$-\left(2-\sqrt{4-\varepsilon}\right) < x-2 < \sqrt{4+\varepsilon}-2$$

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2$$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$\therefore\ 0 < |x-2| < \delta \quad \Rightarrow \quad |f(x)-4| < \varepsilon$$

Example 6

Given that $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$.

Prove that $\lim\limits_{x\to c}\left(f(x)+g(x)\right) = L+M$.

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x-c| < \delta \quad \Rightarrow \quad |f(x)+g(x)-(L+M)| < \varepsilon$$

$$|f(x)+g(x)-(L+M)| = |f(x)+g(x)-L-M|$$

$$= \left|\left(f(x)-L\right)+\left(g(x)-M\right)\right| \qquad \textit{Triangle Inequality } |a+b| \le |a|+|b|$$

$$\le \left|\left(f(x)-L\right)\right|+\left|\left(g(x)-M\right)\right|$$

Since $\lim\limits_{x\to c} f(x)=L$, there exists a number $\delta_1>0$ such that for all x:

$$0<|x-c|<\delta_1 \quad \Rightarrow \quad \left|f(x)-L\right|<\frac{\varepsilon}{2}$$

Similarly, since $\lim\limits_{x\to c} g(x)=M$, there exists a number $\delta_2>0$ such that for all x:

$$0<|x-c|<\delta_2 \quad \Rightarrow \quad \left|g(x)-M\right|<\frac{\varepsilon}{2}$$

Let $\delta=\min\left\{\delta_1,\ \delta_2\right\}$, the smaller of δ_1 and δ_2. If $0<|x-c|<\delta$ then $0<|x-c|<\delta_1$, so $\left|f(x)-L\right|<\frac{\varepsilon}{2}$ and $|x-c|<\delta_2$, so $\left|g(x)-M\right|<\frac{\varepsilon}{2}$. Therefore

$$\left|f(x)+g(x)-(L+M)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

This show that $\lim\limits_{x\to c}\left(f(x)+g(x)\right)=L+M$

Exercises *Section* 1.5 – Precise Definition of Limits

(1 – 2) Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for

1. $a = 1, \quad b = 7, \quad x_0 = 5$

2. $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

3. Use the graph to find a $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$f(x) = -\frac{3}{2}x + 3 \quad x_0 = -3 \quad L = 7.5 \quad \varepsilon = 0.15$

(4 – 25) Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

4. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$

5. $f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$

6. $f(x) = x + 2, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.001$

7. $f(x) = 3x + 2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$

8. $f(x) = 2x - 4, \quad L = -2, \quad x_0 = 1, \quad \varepsilon = 0.1$

9. $f(x) = 2x + 1, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.01$

10. $f(x) = 3 - 4x, \quad L = -1, \quad x_0 = 1, \quad \varepsilon = 0.05$

11. $f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$

12. $f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$

13. $f(x)=\sqrt{x-4}$, $L=2$, $x_0=8$, $\varepsilon=0.1$

14. $f(x)=\sqrt{x+3}$, $L=2$, $x_0=1$, $\varepsilon=0.05$

15. $f(x)=x^2$, $L=3$, $x_0=\sqrt{3}$, $\varepsilon=0.1$

16. $f(x)=x^2+1$, $L=5$, $x_0=2$, $\varepsilon=0.1$

17. $f(x)=x^2+1$, $L=5$, $x_0=2$, $\varepsilon=0.05$

18. $f(x)=x^3$, $L=1$, $x_0=1$, $\varepsilon=0.01$

19. $f(x)=\frac{120}{x}$, $L=5$, $x_0=24$, $\varepsilon=1$

20. $f(x)=\frac{x+2}{x^2}$, $L=3$, $x_0=1$, $\varepsilon=0.1$

21. $f(x)=\frac{x^2}{x+2}$, $L=1$, $x_0=2$, $\varepsilon=0.1$

22. $f(x)=\sin 2x$, $L=\frac{1}{2}$, $x_0=\frac{\pi}{12}$, $\varepsilon=0.1$

23. $f(x)=\cos x$, $L=\frac{\sqrt{3}}{2}$, $x_0=\frac{\pi}{6}$, $\varepsilon=0.05$

24. $f(x)=\cos 3x$, $L=\frac{1}{2}$, $x_0=\frac{\pi}{9}$, $\varepsilon=0.01$

25. $f(x)=e^x$, $L=1$, $x_0=0$, $\varepsilon=0.01$

$(\mathbf{26}-\mathbf{40})$ Give a formal proof that

26. $\lim\limits_{x\to 4}(9-x)=5$

27. $\lim\limits_{x\to 1}\frac{1}{x}=1$

28. $\lim\limits_{x\to 5}\frac{x^2-25}{x-5}=10$

29. $\lim\limits_{x\to 3}\frac{x^2-9}{x-3}=6$

30. $\lim\limits_{x\to 2}\frac{2x^2-3x-2}{x-2}=5$

31. $\lim\limits_{x\to 1}\frac{2x^2+2x-4}{x-1}=6$

32. $\lim\limits_{x\to 2}(5x+8)=18$

33. $\lim\limits_{x\to 1}(5x-2)=3$

34. $\lim\limits_{x\to 0}x^4=0$

35. $\lim\limits_{x\to 2}x^2=4$

36. $\lim\limits_{x\to 2}\frac{1}{(x-2)^4}=\infty$

37. $\lim\limits_{x\to 2}\left(x^2+2x\right)=8$

38. $\lim\limits_{x\to 0}f(x)=0$ *if* $f(x)=\begin{cases}2x, & x<0\\ \frac{x}{2}, & x\geq 0\end{cases}$

39. $\lim\limits_{x\to 0^+} f(x) = -2 \quad if \quad f(x) = \begin{cases} 8x-3, & x<0 \\ 4x-2, & x\geq 0 \end{cases}$

40. $\lim\limits_{x\to 1^-} f(x) = 3 \quad if \quad f(x) = \begin{cases} 5x-2, & x<1 \\ 7x-1, & x\geq 1 \end{cases}$

41. Prove that $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Summary

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x_2\right) - f\left(x_1\right)}{x_2 - x_1} = \lim_{h \to 0} \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}$$

Limit definition: $\lim\limits_{x \to x_0} f(x) = L$

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0\ \ni\ \forall x \qquad 0 < \left|x - x_0\right| < \delta \quad \Rightarrow \quad \left|f(x) - L\right| < \varepsilon$$

$f(x)$ is ***continuous***, if:

1. $f(c)$ is defined
2. $\lim\limits_{x \to c} f(x)$ *exists*
3. $\lim\limits_{x \to c} f(x) = f(c)$

$$\lim_{x \to x_0} k = k \qquad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\sin a\,\theta}{b\,\theta} = \frac{a}{b}$$

$$\lim_{x \to x_0} f(x) = \frac{0}{\neq 0} = 0 \qquad \lim_{x \to x_0} f(x) = \frac{\neq 0}{0} = \infty \qquad \lim_{x \to x_0} f(x) = \frac{0}{0} \quad \to \quad \begin{cases} \textit{factoring} \\ \textit{conjugate} \end{cases}$$

$$\lim_{x \to \infty} \frac{a_n\, x^n}{b_m\, x^m} \quad \to \quad \begin{cases} n < m & \lim\limits_{x \to \infty} \dfrac{a_n\, x^n}{b_m\, x^m} = 0 \\ n = m & \lim\limits_{x \to \infty} \dfrac{a_n\, x^n}{b_m\, x^m} = \dfrac{a_n}{b_m} \\ n > m & \lim\limits_{x \to \infty} \dfrac{a_n\, x^n}{b_m\, x^m} = \infty \end{cases}$$

Exercises Lecture 1 – Review

1. Find the slope of the parabola $y = x^2 + 3$ at the point $P(3,\ 12)$. Write an equation for the tangent to the parabola at this point.

2. Find the slope of the parabola $y = e^x$ @ $x = 1$. Write an equation for the tangent to the parabola at this point.

3. Prove that $\lim\limits_{x \to 0} \dfrac{x}{\sin x} = 1$

4. Prove that $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

5. Prove that $\lim\limits_{x \to 0} x^2 \cos(2\pi x) = 0$

6. Prove that $\lim\limits_{x \to 0} x^3 \sin\left(\dfrac{\pi}{x}\right) = 0$

7. Prove that $\lim\limits_{x \to \infty} \dfrac{1}{x \sin x} = 0$

8. Prove that $\lim\limits_{x \to \infty} \dfrac{\cos x}{x} = 0$

9. Prove that $\lim\limits_{x \to \infty} \dfrac{\sin^2 x}{2^x} = 0$

10. Prove that $\lim\limits_{x \to \infty} \dfrac{\cos \pi x}{5^x} = \dfrac{5}{6}$

$(\mathbf{11} - \mathbf{79})$ Find the limits

11. $\lim\limits_{x \to 2} \left(t^2 - 3t + 5\right)$

12. $\lim\limits_{x \to 2} \left(\pi^2\right)$

13. $\lim\limits_{\theta \to \frac{\pi}{6}} \cos\theta$

14. $\lim\limits_{\theta \to \frac{\pi}{3}} \sin\theta$

15. $\lim\limits_{x \to 4} \left(x^2 - 3x + 5\right)$

16. $\lim\limits_{t \to 2} (2t - 3)$

17. $\lim\limits_{x \to -2} (3x + 2)$

18. $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x - 2}$

19. $\lim\limits_{x \to 1} \dfrac{x^2 - 6x + 5}{x + 2}$

20. $\lim\limits_{x \to 1} \dfrac{x^2 + 6x + 5}{x - 1}$

21. $\lim\limits_{x \to 0^+} \dfrac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$

22. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{3x}$

23. $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

24. $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

25. $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$

26. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

27. $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

28. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

29. $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

30. $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

31. $\lim\limits_{x\to 0} \dfrac{\tan(\pi x)}{\tan(3x)}$

32. $\lim\limits_{x\to 0} \dfrac{\cos 2x-1}{\sin x}$

33. $\lim\limits_{x\to 0^+} \left(x-\dfrac{1}{x^3}\right)$

34. $\lim\limits_{x\to -2^+} \dfrac{2x^2+x+1}{x+2}$

35. $\lim\limits_{x\to -2^-} \dfrac{2x^2+x+1}{x+2}$

36. $\lim\limits_{x\to 1^+} \dfrac{x^2-3x}{x-1}$

37. $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$

38. $\lim\limits_{x\to 4} \dfrac{16\left(\sqrt{x}-2\right)}{x-4}$

39. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

40. $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

41. $\lim\limits_{x\to \infty} \dfrac{\sqrt{x+1}-2}{x-3}$

42. $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x+1}-2}{x-3}$

43. $\lim\limits_{x\to \infty} \dfrac{x+\sin x+2\sqrt{x}}{x+\sin x}$

44. $\lim\limits_{x\to -\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

45. $\lim\limits_{x\to -\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

46. $\lim\limits_{x\to -\infty} \dfrac{x^2-4x+8}{3x^3}$

47. $\lim\limits_{x\to -\infty} \dfrac{2x^2+3}{5x^2+7}$

48. $\lim\limits_{x\to \infty} \dfrac{x^4+x^3}{12x^3+128}$

49. $\lim\limits_{x\to \infty} \dfrac{\sqrt[5]{x^{15}-2x^4+x^3}}{12x^3+128}$

50. $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta}$

51. $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$

52. $\lim\limits_{\theta\to \frac{\pi}{4}} \dfrac{\cos\theta}{\cot\theta}$

53. $\lim\limits_{\theta\to \frac{\pi}{2}} \dfrac{\cos\theta}{\cot\theta}$

54. $\lim\limits_{\theta\to \frac{\pi}{2}} (\sec\theta+\tan\theta)$

55. $\lim\limits_{\theta\to \frac{\pi}{2}} \dfrac{\sec\theta}{\tan\theta}$

56. $\lim\limits_{x\to \infty} \ln x$

57. $\lim\limits_{x\to e} \ln x$

58. $\lim\limits_{x\to-\infty} e^{x^2}$

59. $\lim\limits_{x\to\infty} e^{x^2}$

60. $\lim\limits_{x\to3} e^{\ln x}$

61. $\lim\limits_{x\to-1} \ln\left(e^{x^2}\right)$

62. $\lim\limits_{x\to1} \dfrac{e^x-e^{-x}}{2}$

63. $\lim\limits_{x\to0} \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

64. $\lim\limits_{x\to\infty} \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

65. $\lim\limits_{x\to0} \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

66. $\lim\limits_{x\to1} \ln\left(e^x-e^{-x}\right)$

67. $\lim\limits_{x\to1} \ln\left(e^x+e^{-x}\right)$

68. $\lim\limits_{x\to0} \left(\ln\left(e^x\right)+e^{-x}\right)$

69. $\lim\limits_{x\to0} \ln\left(\sec x\right)$

70. $\lim\limits_{x\to2} \dfrac{3-\sqrt{x^2+5}}{x^2-4}$

71. $\lim\limits_{x\to-2} \dfrac{3-\sqrt{x^2+5}}{x^2-4}$

72. $\lim\limits_{x\to0} \dfrac{(x+2)^2-4}{x}$

73. $\lim\limits_{x\to-2^+} \dfrac{x^2+5x+6}{x+2}$

74. $\lim\limits_{x\to3^+} \dfrac{|x-3|}{x-3}$

75. $\lim\limits_{x\to-3^+} \dfrac{|x+3|}{x+3}$

76. $\lim\limits_{x\to3^+} \sqrt{x-3}$

77. $\lim\limits_{x\to3^-} \sqrt{x-3}$

78. $\lim\limits_{x\to\frac{3}{5}^+} (5x)$

79. $\lim\limits_{x\to1} \left(x^2-3x+\dfrac{1}{x}\right)^{21}$

80. If $\lim\limits_{x\to4} \dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to4} f(x)$

81. Find the limit as $x\to\infty$ and as $x\to-\infty$ of $h(x)=\dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

82. At what points is the function $f(x)=|x-1|+\sin x$ continuous?

83. At what points is the function $f(x)=\dfrac{x-2}{x^2-5x+4}$ continuous?

84. At what points is the function $f(x)=\dfrac{x-2}{x^2+3x+2}$ continuous?

85. At what points is the function $f(x)=\ln(x)$ continuous?

86. Show that the equation $x^3-15x+1=0$ has ***three*** solutions in the interval [–4, 4]

87. Show that the equation $2x^3 - x^2 - 3x + 1 = 0$ has ***three*** solutions in the interval $(-3,\ 3)$

88. Show that the equation $x^3 + 3x - 5 = 0$ has ***one*** solution in the interval $[-2,\ 2]$

89. Show that the equation $x^4 - x^3 + x - 2 = 0$ has ***two*** solutions in the interval $[-2,\ 2]$

90. Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$$

91. Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = 4x - 3, \quad L = 1, \quad x_0 = 1, \quad \varepsilon = .01$$

92. Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x^2 - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = .1$$

93. Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \frac{2}{x}, \quad L = 1, \quad x_0 = 2, \quad \varepsilon = .1$$

(94 – 99) Find the vertical, horizontal, hole and oblique asymptotes (if any) of

94. $y = \dfrac{x-2}{x^2 - 4x + 3}$

95. $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1}$

96. $f(x) = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

97. $f(x) = \dfrac{x^3 - 2x^2 - 4x + 8}{x - 2}$

98. $f(x) = \dfrac{x^2 + 4}{x - 3}$

99. $y = \dfrac{\sqrt{x^2 + 4}}{x}$

100. The motion of a spring is the result by given by the steady-state function.

$$x(t) = -\frac{1}{625}e^{-t}\left(24\cos 4t + 7\sin 4t\right)$$

a) Find the limit as *t* approaches infinity
b) Graph the steady-state function.
c) Compare the part(a) with the graph.

101. The motion of a spring is the result by given by the transient function.

$$x(t) = \frac{1}{625}\left(24 + 100t\right)e^{-t}$$

a) Find the limit as *t* approaches infinity.
b) Graph the steady-state function.
c) Compare the part(a) with the graph.

102. A mass of 64 *pounds* is attached to a spring with a spring constant 32 *lb/ft* and then comes to rest in the equilibrium position. Neglect the damping. The results are given by the two given functions

Steady-State: $x_p(t) = e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)$

Transient: $x_h(t) = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t$

a) Find $\lim\limits_{t\to\infty} x_p(t)$

b) Find $\lim\limits_{t\to\infty} x_h(t)$

c) Find $\lim\limits_{t\to\infty} \left(x_h(t) + x_p(t)\right)$

103. Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 *grams* of the compound *C* is formed in 10 *minutes*. If the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*

The amount of *C* at time *t* is given by the function:

$$C(t) = 1{,}000\,\frac{e^{0.126t} - 1}{25e^{0.126t} - 4}$$

Find the $\lim\limits_{t\to\infty} C(t)$.

104. A thermometer is removed from a room where the temperature is $70° F$ and is taken outside, where the air temperature is $10° F$. After one-half minute the thermometer reads $50° F$. The temperature is given by the function

$$T(t) = 10 + 60e^{-.811t}$$

Find the $\lim_{t \to \infty} T(t)$.

105. At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 80° *F*. the liquid bath has a controlled temperature given by $T_m(t) = 100 - 40e^{-0.1t}$, $t \geq 0$, where *t* is measured in *minutes*. What the temperature as time approaches infinity.

106. Stacy weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient. Her weight is given as a function of time.

$$m(t) = 10e^{-\frac{3}{2000}t} + 50$$

Does her weight ultimately approach an equilibrium weight?

107. A 200–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 *ohms* and the capacitance is 5×10^{-6} *farad*. With the initial value $i(0) = 0.4$, the charge equation is given by

$$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}$$

Determine the charge as $t \to \infty$

Practice *Exam*

Directions: You must show all your work, a correct answer is worth **0** point without any supporting work.

1. *a*) Write the ***limit definition***.

b) Then find the interval of x and give a value for δ that f satisfies the limit to the given values.

$$f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$$

2. At what points is the function $y = \dfrac{2x}{(x+2)(x-1)}$ continuous?

(3 – 28) Find the limits.

3. $\lim\limits_{x \to 1{,}000} 8\pi^3$

4. $\lim\limits_{x \to 1} \sqrt{5x+3}$

5. $\lim\limits_{x \to 4} \left(x^2 + 4x - 1\right)$

6. $\lim\limits_{x \to -2} \left(x^3 - 4x - 1\right)$

7. $\lim\limits_{x \to -2} \left(x^3 - 2x^2 + 4x + 4\right)$

8. $\lim\limits_{x \to 0} \dfrac{\sin \pi x}{4x}$

9. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - 2x + 3}$

10. $\lim\limits_{x \to 0} \left(2\sin x - 1\right)$

11. $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x+3} - 2}$

12. $\lim\limits_{x \to 4} \dfrac{x^3 - 7x^2 + 12x}{4 - x}$

13. $\lim\limits_{x \to \frac{\pi}{4}} \left(2x^2 + \cot x\right)$

14. $\lim\limits_{\theta \to 0} \dfrac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

15. $\lim\limits_{x \to \pi} \sin\left(\dfrac{x}{2} + \sin x\right)$

16. $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 - 5x + 6}$

17. $\lim\limits_{x \to 2} \dfrac{x-2}{x+4}$

18. $\lim\limits_{x \to -2} \dfrac{x-3}{x+2}$

19. $\lim\limits_{x \to -5^+} \dfrac{x-5}{x+5}$

20. $\lim\limits_{x \to 3^-} \dfrac{x-4}{x^2 - 3x}$

21. $\lim\limits_{x \to \infty} \left(x + \sqrt{x^2 - 2x + 2}\right)$

22. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\sin x}$

23. $\lim\limits_{x \to \infty} \dfrac{5x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

24. $\lim\limits_{x \to \infty} \dfrac{3 - x}{2x^2 + 5}$

25. $\lim\limits_{x \to \infty} \dfrac{x^4 + x^3}{12x^3 + 128}$

26. $\lim\limits_{x \to -\infty} e^{x^3}$

27. $\lim\limits_{x \to 0^+} \ln x$

28. $\lim\limits_{x \to e} \ln x^3$

Prove (***Show all steps*** and formula(s))

29. $\lim\limits_{x \to 0^+} \dfrac{\sin x}{\sin \sqrt{x}} = 0$

30. $\lim\limits_{\theta \to 0} \dfrac{\cos \theta - 1}{\theta^2} = -\dfrac{1}{2}$

Exam B

1. *a*) Write the ***limit definition***.

b) Then find the interval of x and give a value for δ that f satisfies the limit to the given values.

$$f(x) = x^2 - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = .1$$

2. At what points is the function $f(x) = |x-1| + \sin x$ continuous?

(3 – 28) **Find the limits.**

3. $\lim\limits_{x \to 5} 8\sqrt{\pi}$

4. $\lim\limits_{x \to 2} \sqrt{4x+3}$

5. $\lim\limits_{x \to -3} \left(-2x^2 - 4x + 8\right)$

6. $\lim\limits_{x \to 1} \dfrac{x^3 - 7x^2 + 12x}{4 - x}$

7. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - 2x + 3}$

8. $\lim\limits_{x \to 0} \left(2\sin x + \pi\right)$

9. $\lim\limits_{x \to 2} \dfrac{x-2}{\sqrt{x+2} - 2}$

10. $\lim\limits_{x \to 4} \dfrac{x^3 - 7x^2 + 12x}{4 - x}$

11. $\lim\limits_{x \to \frac{\pi}{4}} \left(2x^2 + \tan x\right)$

12. $\lim\limits_{\theta \to 0} \dfrac{1 - \cos\theta}{\sin\theta}$

13. $\lim\limits_{x \to \pi/2} \dfrac{\dfrac{1}{\sqrt{\sin x}} - 1}{x + \dfrac{\pi}{2}}$

14. $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 + 5x - 6}$

15. $\lim\limits_{x \to 2} \dfrac{x-2}{x+4}$

16. $\lim\limits_{x \to -2} \dfrac{x-3}{x+2}$

17. $\lim\limits_{x \to -5^+} \dfrac{x-5}{x+5}$

18. $\lim\limits_{x \to \infty} \dfrac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

19. $\lim\limits_{x \to \infty} \dfrac{6x^4 - 3x^2 + 8}{\sqrt{36x^8 + x^4 + 2}}$

20. $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2 - 2x + 2}\right)$

21. $\lim\limits_{x \to \infty} \dfrac{x^5 - 2x^3}{8x^4 + 128}$

22. $\lim\limits_{x \to -\infty} \sqrt{x}$

23. $\lim\limits_{x \to \pi} \cos^2\left(x - \tan x\right)$

24. $\lim\limits_{x \to -\infty} e^{x^2}$

25. $\lim\limits_{x \to e} \ln x^2$

26. $\lim\limits_{x \to 0^-} \ln x$

27. $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x}$

28. $\lim\limits_{\theta \to \frac{\pi}{4}} \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$

Prove (***Show all steps*** and formula)

29. $\lim\limits_{x \to \infty} \dfrac{1}{x \sin x} = 0$

30. $\lim\limits_{x \to 0} x^2 \cos\left(2\pi x\right) = 0$

Lecture Two

Differentiation

Section 2.1 – Introducing the Derivative

2.1−1 Definition

The slope of the curve $y = f(x)$ at the point $P\left(x_0, f\left(x_0\right)\right)$ is the number:

$$m = \lim_{h \to 0} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}, \qquad (\lim \exists)$$

The tangent line to the curve at P is the line through P with this slope.

***Example* 1**

a) Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal $-\frac{1}{4}$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) The slope of $f(x) = \frac{1}{x}$ at $\left(a, \frac{1}{a}\right)$ is:

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-a-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)}$$

$$= -\frac{1}{a^2}$$

The slope at $x = -1$ is: $= -\frac{1}{(-1)^2}$ $= -1$

b) The slope equals to $x = -\frac{1}{4}$

$$\Rightarrow -\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \quad \rightarrow \quad a = \pm 2$$

$$\begin{cases} x = -2 \Rightarrow y = -\frac{1}{2} \\ x = 2 \Rightarrow y = \frac{1}{2} \end{cases} \Rightarrow \quad \left(-2,\ -\frac{1}{2}\right) \textit{ and } \left(2,\ \frac{1}{2}\right)$$

c) The slope $\left(-\frac{1}{a^2}\right)$ is always negative if $a \neq 0$

$\lim_{a\to\pm\infty}\left(-\frac{1}{a^2}\right) = 0$ The slope approaches 0 and the tangent becomes horizontal.

$\lim_{a\to 0^-}\left(-\frac{1}{a^2}\right) = -\infty$ The slope approaches $-\infty$ and the tangent increasingly steep.

2.1–2 *Definition* of the Derivative

The derivative of a function f at a point x_0, denoted $f'(x_0)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \qquad (\lim \exists)$$

If f' exists at a particular x, we say that f is ***differentiable*** (has a ***derivative***) at x.

If f' exists at every point in the domain of f, we call f ***differentiable.***

The process of finding derivatives is called ***differentiation***.

Notations

Some common alternative notations for the derivative are:

$$f'(x), \quad f', \quad \frac{d}{dx}[f(x)], \quad \frac{d}{dx}f, \quad \frac{dy}{dx}, \quad y', \quad \dot{y}, \text{ and } D_x[y]$$

Example 2

Differentiate $\qquad f(x) = \dfrac{x}{x-1}$

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad\qquad f(x+h) = \frac{(x+h)}{(x+h)-1}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{x^2 - x + hx - h - x^2 - hx + x}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

Example 3

Find the derivative of $f(x) = x^2$

Solution

$$f(x+h) = (x+h)^2$$

$$= x^2 + 2hx + h^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} (2x + h)$$

$$= 2x$$

Example 4

a) Find the derivative of $f(x)=\sqrt{x}$ for $x>0$

b) Find the tangent line to the curve $y=\sqrt{x}$ *at* $x=4$

Solution

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$

b) The slope of the curve at $x = 4$ is:

$$f'(4)=\frac{1}{2\sqrt{4}}$$

$$=\frac{1}{4}$$

The tangent is the line through the point (4, 2) with slope $\frac{1}{4}$:

$$y-2=\frac{1}{4}(x-4)$$

$$y=\frac{1}{4}x-1+2$$

$$y=\frac{1}{4}x+1$$

2.1–3 Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

2.1–4 *Theorem* – Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof

Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c+h) = f(c)$. If $h \neq 0$, then:

$$\begin{aligned} f(c+h) &= f(c) + \left(f(c+h) - f(c) \right) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \end{aligned}$$

Take the limits as $h \to 0$.

$$\begin{aligned} \lim_{h \to 0} f(c+h) &= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) \end{aligned}$$

Summary

The following are all interpretations for the limit of the difference quotient, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

1. The slope of the graph of $y = f(x)$ @ $x = x_0$.
2. The slope of the tangent to the curve $y = f(x)$ @ $x = x_0$.
3. The rate of change of $f(x)$ with respect to x @ $x = x_0$.
4. The derivative $f'(x_0)$ at a point.

Exercises *Section* 2.1 – Introducing the Derivative

(1 – 10) Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

1. $y = x^2 - 3$ @ $P(2,\ 1)$

2. $y = x^2 - 2x - 3$ @ $P(2,\ -3)$

3. $y = 4 - x^2$; $P(-1,\ 3)$

4. $y = \dfrac{1}{x^2}$; $P(-1,\ 1)$

5. $f(x) = 2\sqrt{x}$; $P(1,\ 2)$

6. $f(x) = x^3 + 3x$; $P(1,\ 4)$

7. $f(x) = 4x^2 - 7x + 5$; $P(2,\ 7)$

8. $f(x) = 5x^3 + x$; $P(1,\ 6)$

9. $f(x) = \dfrac{x+3}{2x+1}$; $P(0,\ 3)$

10. $f(x) = \dfrac{1}{2\sqrt{3x+1}}$; $P\left(0,\ \frac{1}{2}\right)$

(11 – 16) Find the slope of the given curve

11. $y = 1 - x^2$ @ $x = 2$

12. $y = \dfrac{1}{x-1}$ @ $x = 3$

13. $y = \dfrac{x-1}{x+1}$ @ $x = 0$

14. $f(x) = x^2$ @ $x = 1$

15. $f(x) = \sqrt{3x+1}$ @ $x = 0$

16. $f(x) = \dfrac{1}{x+1}$ @ $x = 0$

17. Find equations of all lines having slope –1 that are tangent to the curve $y = \dfrac{1}{x-1}$

18. What is the rate of change of the area of a circle $\left(A = \pi r^2\right)$ with respect to the radius when the radius is $r = 3$?

19. Find the slope of the tangent to the curve $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$

20. Find the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3),\ f'(0),\ f'(1)$

21. Find the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0),\ r'\left(\frac{1}{2}\right),\ r'(1)$

22. Find the derivative of $f(x) = 3x^2 - 2x$

23. Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

24. Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

25. Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

26. Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

27. Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}}$

28. Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

29. Suppose the height *s* of an object (in *m*) above the ground after *t* seconds is approximated by the function $s = -4.9t^2 + 25t + 1$

a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01,\ 0.001,\ 0.0001,$ and 0.00001 .

b) Use the table in part (*a*) to estimate the instantaneous velocity of the object at $t = 1$.

c) Use limits to verify your estimate in part (*b*).

30. Suppose the following graph represents the number of bacteria in a culture *t* hours after the start of an experiment.

a) At approximately what time is the instantaneous growth rate the greatest, for $0 \le t \le 36$? Estimate the growth rate at this time.

b) At approximately what time is the instantaneous growth rate the least, for $0 \le t \le 36$? Estimate the growth rate at this time.

c) What is the average growth rate over the interval $0 \le t \le 36$?

Section 2.2 – Rules of Differentiation

2.2–1 Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st *derivative*	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$

2.2–2 Derivative of a *constant Function*

If f has the constant value $f(x) = c$

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant.}$$

Proof

Let $f(x) = c$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

***Example* 1**

Find the derivative:

a) $f(x) = 9$

$$\underline{f'(x) = 0}$$

b) $h(t) = \pi$

$$\underline{D_t[h(t)] = 0}$$

2.2–3 *Power* Rule

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \qquad \textbf{\textit{n}} \text{ is any real number}$$

Proof

Let $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \left(nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1} \right)$$

$$= nx^{n-1}$$

Example 2

Find the derivative of: $a)\ y = x^3 \quad b)\ y = x^{2/3} \quad c)\ y = \frac{1}{x^4} \quad d)\ D_x\left(x^{\sqrt{2}}\right) \quad e)\ \frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right)$

Solution

a) $y = x^3$

$$\frac{dy}{dx} = 3x^{3-1}$$

$$= 3x^2$$

b) $y = x^{2/3}$

$$y' = \frac{2}{3}x^{2/3-1}$$

$$= \frac{2}{3}x^{-1/3}$$

c) $y = \frac{1}{x^4} = x^{-4}$

$$y' = -4x^{-4-1}$$

$$= -4x^{-5}$$

$$= -\frac{4}{x^5}$$

d) $D_x\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

e) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{(2+\pi)/2}\right)$

$$= \left(\frac{2+\pi}{2}\right)x^{1+\pi/2-1}$$

$$= \frac{1}{2}(2+\pi)\sqrt{x^{\pi}}$$

2.2–4 Derivative *Constant* Multiple Rule

If f is a differentiable function of x, and c is a real number (constant), then $\frac{d}{dx}(cf) = c\frac{df}{dx}$

If n is any real number, then $\frac{d}{dx}\left(c\,x^n\right) = c\,n\,x^{n-1}$

Proof

$$\frac{d}{dx}(cf) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \textit{Factor c}$$

$$= c\frac{df}{dx}$$

Example 3

If $y = 8x^4$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 8\left(4x^3\right)$$

$$= 32x^3$$

Example 4

If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = -\frac{3}{4}\left(12x^{11}\right)$$

$$= -9x^{11}$$

2.2–5 *Sum* or *Difference* Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \qquad \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$= u' + v' \qquad = u' - v'$$

Proof

$$f(x) = u(x) + v(x)$$

$$\frac{d}{dx}\left[u(x)+v(x)\right] = \lim_{h\to 0}\frac{\left[u(x+h)+v(x+h)\right]-\left[u(x)+v(x)\right]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-u(x)-v(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\right) + \lim_{h\to 0}\left(\frac{v(x+h)-v(x)}{h}\right)$$

$$= \frac{du}{dx} + \frac{dv}{dx} \quad \checkmark$$

Example 5

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{8}{3}x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

Example 6

Find the derivative of $y = x^{5/2} + x^3 + \frac{1}{2}x^2 + 4$

Solution

$$y' = \frac{5}{2}x^{3/2} + 3x^2 + x$$

Example 7

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution

$$y' = 4x^3 - 4x$$

$$y' = 0 \quad \Rightarrow \quad 4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0$$

$$x = 0, \ \pm 1$$

The curve has horizontal tangents at $x = 0,\ 1,\ and\ \ -1$.

The corresponding points on the curve are: (0, 2), (1, 1) and (–1, 1)

2.2–6 *Product* Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example 8

Find the derivative of $f(x)=(2x+3)\left(3x^2\right)$

Solution

$$f'=(2x+3)\left(3x^2\right)'+(2x+3)'\left(3x^2\right)$$

$$=(2x+3)(6x)+(2)\left(3x^2\right)$$

$$=12x^2+18x+6x^2$$

$$=18x^2+18x$$

$$=18x(x+1)$$

Or distribute $f(x)=(2x+3)\left(3x^2\right)$

$$f(x)=6x^3+9x^2$$

$$f'(x)=18x^2+18x$$

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv)=\lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$$

$$=\lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{u(x+h)v(x+h)-u(x+h)v(x)}{h}+\frac{u(x+h)v(x)-u(x)v(x)}{h}\right]$$

$$=\lim_{h\to 0}\left(u(x+h)\frac{v(x+h)-v(x)}{h}+v(x)\frac{u(x+h)-u(x)}{h}\right)$$

$$=\lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$=u\cdot\frac{dv}{dx}+v\cdot\frac{du}{dx}\quad\checkmark$$

Example 9

Find the derivative of $y=\left(3x^2+1\right)\left(x^3+3\right)$

Solution

$$u=3x^2+1 \quad v=x^3+3$$

$$u'=6x \qquad v'=3x^2$$

$$y'=(6x)\left(x^3+3\right)+\left(3x^2\right)\left(3x^2+1\right)$$

$$=6x^4+18x+9x^4+3x^2$$

$$=15x^4+3x^2+18x$$

$$y=\left(3x^2+1\right)\left(x^3+3\right)$$

$$=3x^5+9x^2+x^3+3$$

$$y'=15x^4+18x+3x^2$$

Example 10

Find the derivative of $y=\left(3x^3+2x+5\right)\left(x^2-2x+4\right)$

Solution

$$y'=\underbrace{\left(9x^2+2\right)}_{u'}\underbrace{\left(x^2-2x+4\right)}_{v}+\underbrace{(2x-2)}_{v'}\underbrace{\left(3x^3+2x+5\right)}_{u}$$

$$=9x^4-18x^3+36x^2+2x^2-4x+8+6x^4+4x^2+10x-6x^3-4x-10$$

$$=15x^4-24x^3+42x^2+2x-2$$

2.2–7 *Quotient* Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$$

$$=\frac{f'g-g'f}{g^2}$$

Formulas (the proof is in the Appendix ***A***)

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2+2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x+\begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

Example 11

Find $f'(x)$ if $f(x)=\dfrac{2x-1}{4x+3}$

Solution

$$f'(x)=\frac{2(3)-(-1)(4)}{4x+3}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\frac{10}{(4x+3)^2}$$

$$f'=\frac{(2x-1)'(4x+3)-(2x-1)(4x+3)'}{(4x+3)^2}$$

$u=2x-1$ $\quad v=4x+3$

$u'=2$ $\quad v'=4$

$$=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$= \frac{8x+6-8x+4}{(4x+3)^2}$$

$$= \frac{10}{(4x+3)^2}$$

Example 12

Find the derivative of $y = \dfrac{(x-1)\left(x^2-2x\right)}{x^4}$

Solution

$$y = \frac{x^3-2x^2-x^2+2x}{x^4}$$

$$= \frac{x^3-3x^2+2x}{x^4}$$

$$= \frac{x^3}{x^4} - \frac{3x^2}{x^4} + \frac{2x}{x^4}$$

$$= x^{-1} - 3x^{-2} + 2x^{-3}$$

$$y' = -x^{-2} + 6x^{-3} - 6x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

2.2−8 Combining the product and Quotient Rules

Example 13

Find the derivative of $y=\dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$y=\frac{(1+x)(2x-1)}{x-1}$$

$$=\frac{2x^2+x-1}{x-1}$$

$$y'=\frac{\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}x^2+2\begin{vmatrix}2 & -1\\ 0 & -1\end{vmatrix}x+\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix}}{x-1}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2+2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x+\begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$y'=\frac{2x^2-4x}{(x-1)^2}$$

Or

$$y'=\frac{(x-1)\frac{d}{dx}[(1+x)(2x-1)]-(1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$=\frac{(x-1)[(1)(2x-1)+2(1+x)]-(1+x)(2x-1)(1)}{(x-1)^2}$$

$$=\frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{(x-1)^2}$$

$$=\frac{(x-1)(4x+1)-2x+1-2x^2+x}{(x-1)^2}$$

$$=\frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$=\frac{2x^2-4x}{(x-1)^2}$$

2.2–9 *Second*– and *Higher*–Order Derivatives

Notation for Higher-Order Derivatives							
1.	***1st derivative***	y'	***y*** prime	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
2.	***2nd derivative***	y''	***y*** double prime	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
3.	***3rd derivative***	y'''	***y*** triple prime	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
4.	***4th derivative***	$y^{(4)}$		$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
5.	***nth derivative***	$y^{(n)}$		$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \Rightarrow \qquad \underline{f^{(n)}(x) = n!a_n}$$

Example 14

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

Solution

$$y' = 3x^2 - 6x$$
$$y'' = 6x - 6$$
$$y''' = 6$$
$$\underline{y^{(4)} = 0}$$

Example 15

Find the derivatives of $f(x) = 4x^5 - x^4 + 3x^3 - 2x + 5, \quad f^{(5)}(x)$

Solution

$$f^{(5)}(x) = 4(5!)$$
$$\underline{= 480}$$

Example 16

Find the derivatives of $f(x) = 4x^5 - x^4 + 3x^3 - 2x + 5$, $f^{(6)}(x)$

Solution

$$\boxed{f^{(6)}(x) = 0}$$

Exercises *Section* 2.2 – Rules of Differentiation

$(\mathbf{1}-\mathbf{140})$ Find the derivative of each function

1. $f(x)=100\pi^2$

2. $f(x)=5t^2$

3. $f(x)=\frac{1}{2}\theta$

4. $f(x)=\sqrt{5}$

5. $f(x)=100\pi^2 x$

6. $y=\frac{1}{x^3}$

7. $D_x\left(x^{4/3}\right)$

8. $y=\sqrt{z}$

9. $D_t(-8t)$

10. $y=\frac{9}{4x^2}$

11. $y=6x^3+15x^2$

12. $y=3x^4-6x^3+\frac{x^2}{8}+5$

13. $p(t)=12t^4-6\sqrt{t}+\frac{5}{t}$

14. $f(x)=3x-4$

15. $f(x)=2x^3-3x^2-5$

16. $f(x)=x^3+x-\sqrt{x}$

17. $f(x)=x^2+3x-x^{-1}-1+x^{-1/2}$

18. $f(t)=4\sqrt{t}-\frac{1}{4}t^4+t+1+\frac{1}{t}$

19. $f(x)=x^{0.35}+x^{-\pi^2}+x^{\sqrt{7}}$

20. $f(x)=4x^{5/3}+6x^{-3/2}-11x$

21. $f(x)=2x^{5/4}+4x^{-2}-6x$

22. $f(t)=7t^{-5/14}+2t^{-6}+6$

23. $f(x)=\sqrt[4]{x}+\sqrt[3]{x}+\sqrt{2}\,x^{\sqrt{2}}$

24. $f(x)=6\sqrt{x}-\frac{1}{\sqrt{x}}$

25. $f(x)=\frac{x^2+4x^{1/2}}{x^2}$

26. $f(x)=\sqrt{x}\left(\sqrt{x}-1\right)$

27. $f(x)=\frac{1-2x}{x^{1/2}}$

28. $f(y)=16y^{0.2}+8y^{-0.8}$

29. $f(x)=\frac{x^3+3\sqrt{x}}{x}$

30. $y=\frac{x^3-4x}{\sqrt{x}}$

31. $f(x)=\left(4x^2-3x\right)^2$

32. $y=3x\left(2x^2+5x\right)$

33. $y=3\left(2x^2+5x\right)$

34. $y=(3x-2)(2x+3)$

35. $y=\frac{x^2+4x}{5}$

36. $y=\frac{3x^4}{5}$

37. $g(s)=\frac{s^2-2s+5}{\sqrt{s}}$

38. $f(x)=\frac{x+1}{\sqrt{x}}$

39. $f(x)=4x^{5/3}+6x^{-3/2}-11x$

40. $f(x)=\frac{2}{3}x^3+\pi x^2+7x+1$

41. $f(x)=\frac{x^5-x^3}{15}$

42. $f(x)=x^{1/3}+2x^{1/4}-3x^{1/5}$

43. $f(t)=3\sqrt[3]{t^2}-\dfrac{2}{\sqrt{t^3}}$

44. $f(t)=\sqrt{t}\left(5-t-\frac{1}{3}t^2\right)$

45. $f(x)=\frac{3}{5}x^{5/3}+\frac{5}{3}x^{-3/5}$

46. $f(x)=x^{23}-x^{-23}$

47. $g(t)=3t^2+\dfrac{6}{t^7}$

48. $y=(x+1)\left(\sqrt{x}+2\right)$

49. $y=\left(4x+3x^2\right)(6-3x)$

50. $y=\left(\frac{1}{x}+1\right)(2x+1)$

51. $y=\dfrac{3-\frac{2}{x}}{x+4}$

52. $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

53. $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

54. $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

55. $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

56. $y=(2x+3)\left(5x^2-4x\right)$

57. $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

58. $y=\dfrac{x+4}{5x-2}$

59. $z=\dfrac{4-3x}{3x^2+x}$

60. $y=(2x-7)^{-1}(x+5)$

61. $y=\left(2\sqrt{x}-1\right)(4x+1)^{-1}$

62. $f(x)=\left(2+x^{-1}\right)\left(x^{3/2}+1\right)$

63. $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

64. $y=\dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

65. $f(x)=\dfrac{x^{3/2}\left(x^2+1\right)}{x+1}$

66. $f(x)=\dfrac{x^3-4x^2+x}{x-2}$

67. $g(x)=\dfrac{x(3-x)}{2x^2}$

68. $y=\dfrac{2x^2}{3x+1}$

69. $f(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$

70. $f(x)=\dfrac{x}{1+x^2}$

71. $y=\dfrac{x^2-2ax+a^2}{x-a}$

72. $f(x)=\dfrac{x^2+4x^{1/2}}{x^2}$

73. $f(x)=(2x+1)\left(3x^2+2\right)$

74. $f(x)=\dfrac{x^2-1}{x^2+1}$

75. $y=\dfrac{4x^3+3x+1}{2x^5}$

76. $y=\dfrac{4}{3-x}$

77. $y=\dfrac{2}{1-x^2}$

78. $f(x)=\dfrac{\pi}{2-\pi x}$

79. $f(x)=(1-2x)(3x+5)$

80. $f(x)=\left(5x^3+3x+1\right)\left(x^2+5\right)$

81. $f(x)=(4x-3)^2$

82. $f(x)=\left(x^2-2\right)^2$

83. $f(y)=\dfrac{y^2-1}{y-1}$

84. $f(x)=\dfrac{x^3-6x^2+8x}{x^2-2x}$

85. $y=\dfrac{x-4}{5x-2}$

86. $y=\dfrac{3x-4}{2x-1}$

87. $y=\dfrac{3x+4}{2x+1}$

88. $y=\dfrac{-3x+4}{2x+1}$

89. $y=\dfrac{-3x-4}{2x-1}$

90. $y=\dfrac{2x-3}{x+1}$

91. $y=\dfrac{3x}{3x-2}$

92. $y=\dfrac{x-3}{2x+5}$

93. $y=\dfrac{5x-3}{2x+5}$

94. $y=\dfrac{6x-8}{2x-3}$

95. $f(x)=\dfrac{6x+3}{2x-3}$

96. $f(x)=\dfrac{-6x+3}{2x-3}$

97. $f(x)=\dfrac{3x-2}{2x+3}$

98. $y=\dfrac{x^2-4}{5x^2-2}$

99. $y=\dfrac{3x^2-4}{2x^2-1}$

100. $y=\dfrac{3x^2+4}{2x^2+1}$

101. $y=\dfrac{2x^2-3}{x^2+1}$

102. $y=\dfrac{3x^2}{3x^2-2}$

103. $y=\dfrac{5x^2-3}{2x^2+5}$

104. $y=\dfrac{6x^2-8}{2x^2+1}$

105. $f(x)=\dfrac{6x^2+3}{2x^2-3}$

106. $f(x)=\dfrac{-6x^2+3}{2x^2-3}$

107. $f(x)=\dfrac{3x^2-2}{2x^2+3}$

108. $f(x)=\dfrac{6x^3+3}{2x^3-3}$

109. $f(x)=\dfrac{-6x^3+3}{2x^3-3}$

110. $f(x)=\dfrac{3x^3-2}{2x^3+3}$

111. $y=\dfrac{6x^3+8}{2x^3+1}$

112. $y=\dfrac{5x^3-3}{2x^3+5}$

113. $y=\dfrac{x^3}{3x^3-2}$

114. $y=\dfrac{2x^3-3}{2x^3+1}$

115. $y=\dfrac{2x^4-3}{2x^4+1}$

116. $f(x)=\dfrac{6x^4+3}{2x^4-3}$

117. $f(x)=\dfrac{-6x^4+3}{2x^4-3}$

118. $f(x)=\dfrac{3x^4-2}{2x^4+3}$

119. $y=\dfrac{x^2-4x+1}{5x^2-2x-1}$

120. $y=\dfrac{3x^2-4x+2}{2x^2+x-1}$

121. $y=\dfrac{3x^2+x-4}{2x^2+1}$

122. $y=\dfrac{2x^2-3}{x^2+5x+1}$

123. $y=\dfrac{3x^2}{3x^2+6x-8}$

124. $y=\dfrac{x^2+2x}{2x^2+x-5}$

125. $y=\dfrac{x^2+5x+1}{x^2}$

126. $y=\dfrac{x^2-3x+1}{x^2-8x+5}$

127. $y=\dfrac{x^2-2x+3}{4x^2+5x-6}$

128. $y=\dfrac{4x^2+5x-6}{x^2-2x+3}$

129. $y=\dfrac{5x-6}{x^2-2x+3}$

130. $y=\dfrac{x^2-2x+3}{5x-6}$

131. $y=\dfrac{x^2-2}{5x-3}$

132. $y=\dfrac{2}{x^2-6x+5}$

133. $f(x)=\left(\dfrac{x^2-4}{x-1}\right)\left(\dfrac{x^2-1}{x+2}\right)$

134. $f(x)=\dfrac{x-a}{\sqrt{x}-\sqrt{a}};\quad a>0$

135. $y=\dfrac{x}{x+1}$

136. $y=\dfrac{2x^2}{3x+1}$

137. $f(x)=\dfrac{x^3-4x^2+x}{x-2}$

138. $g(x)=\dfrac{x(3-x)}{2x^2}$

139. $g(x)=\dfrac{(x-1)(2x^2-1)}{x^3-1}$

140. $f(x)=\dfrac{x+4}{x^2+x+1}$

(141 – 160) Find the ***first*** and ***second*** derivatives.

141. $y=-x^3+3$

142. $y=3x^7-7x^3+21x^2$

143. $y=6x^2-10x-\dfrac{1}{x}$

144. $f(x)=\dfrac{1}{2}x^4+\pi x^3-7x+1$

145. $f(x)=3x^4-6x^3+\dfrac{1}{2}x^2+5$

146. $f(x)=3x^5-4x^3-2x^2+3x+1$

147. $f(x)=x^7+5x^5-6x^2+2x-3$

148. $f(x)=4x^2+2x-3$

149. $f(x)=4ex^2+2x-\sqrt{3}$

150. $f(x)=4x^3+x^2-5x+2$

151. $f(x)=ax+b$

152. $y=3x^4-6x^3+\dfrac{x^2}{8}+5$

153. $y=(2x-3)(1-5x)$

154. $y=2x^2(1+3x)$

155. $y=3x(2+6x)$

156. $y=\dfrac{x^2+5x-1}{x^2}$

157. $y=\dfrac{x^2-5x+2}{x}$

158. $y=\dfrac{x^2+3x+1}{2x^2}$

159. $y=\dfrac{x^2+3}{2x^2}$

160. $y=3x^5-6x^{3/2}+x^{1/2}+\dfrac{5}{x}$

$(\mathbf{161-170})$ Find the derivatives.

161. $f(x)=3x^4-6x^3+\frac{x^2}{8}+5,\quad f^{(4)}(x)$

162. $f(x)=3x^4-6x^3+\frac{x^2}{8}+5,\quad f^{(5)}(x)$

163. $f(x)=2x^6+4x^4-x+2,\quad f^{(6)}(x)$

164. $f(x)=4x^5+4x^4+x^2-2,\quad f^{(5)}(x)$

165. $f(x)=4x^5+4x^4+x^2-2,\quad f^{(6)}(x)$

166. $f(x)=4x^4-2x^3+x+2,\quad f^{(4)}(x)$

167. $f(x)=3x^4-5x^3+2,\quad f^{(4)}(x)$

168. $f(x)=3x^4-5x^3+2,\quad f^{(6)}(x)$

169. $f(x)=2x^6-3x^3+x+2,\quad f^{(6)}(x)$

170. $f(x)=2x^5-2x^2+x-1,\quad f^{(5)}(x)$

171. Find y', y'', y''': $\quad y=(x-3)\sqrt{x+2}$

172. For what value(s) of x is the line tangent to the curve $y=x\sqrt{6-x}$ horizontal? Vertical?

173. Find an equation of the tangent line to the graph of $y=\frac{x^2-4}{2x+5}$ when $x=0$

174. Find an equation for the line perpendicular to the tangent to the curve $y=x^3-4x+1$ at the point $(2,\ 1)$.

175. If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P=\frac{nRT}{V-nb}-\frac{an^2}{V^2}$$

In which a, b, n, and R are constants. Find $\frac{dP}{dV}$.

176. Show that if $(a,\ f(a))$ is any point on the graph of $f(x)=x^2$, then the slope of the tangent line at that point is $m=2a$

177. Show that if $(a,\ f(a))$ is any point on the graph of $f(x)=bx^2+cx+d$, then the slope of the tangent line at that point is $m=2ab+c$

178. Let $f(x) = x^2$

a) Show that $\dfrac{f(x) - f(y)}{x - y} = f'\left(\dfrac{x+y}{2}\right)$, for all $x \neq y$

b) Is this property true for $f(x) = ax^2$, where a is a nonzero real number?

c) Give a geometrical interpretation of this property.

d) Is this property true for $f(x) = ax^3$?

Section 2.3 –Derivatives of Trigonometric Functions

In physics, the harmonic motion is associated with an object with a mass oscillating. A simple harmonic motion can be determined by using sine or cosine functions.

2.3−1 Derivative of the *Sine* Function

To determine the derivative of the ***sine*** function. We have to let $f(x)=\sin x$, then:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}
\end{aligned}
$$

$$\cos h = 1-2\sin^2\left(\frac{h}{2}\right)$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{\cosh-1}{h} &= \lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h}\\
&= \lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h} \qquad \textit{Let } \theta=\frac{h}{2}\\
&= -\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta}
\end{aligned}
$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \sin\theta$$

$$= -(1)(0)$$

$$= 0$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$

$$= \sin x \bullet (0) + \cos x \bullet (1)$$

$$= \cos x$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 1

Find the derivative of $y = x^2 - \sin x$

Solution

$$y' = 2x - (\sin x)'$$

$$= 2x - \cos x$$

Example 2

Find the derivative of $y = x^2 \sin x$

Solution

$$y' = 2x\sin x + x^2 \cos x \qquad (uv)' = u'v + uv'$$

Example 3

Find the derivative of $y = \dfrac{\sin x}{x}$

Solution

$$y' = \frac{x\cos x - \sin x \cdot (1)}{x^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

2.3–2 Derivative of the *Cosine* Function

Similar to sine, we are going to apply the derivative definition, different quotient, to find the derivative of the ***cosine*** function.

If $f(x) = \cos x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \cos x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{0} - \sin x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1}$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$= \cos x \bullet (0) - \sin x \bullet (1)$$

$$= -\sin x$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example 4

Find the derivative of $y = 5x + \cos x$

Solution

$$y' = 5 - \sin x$$

Example 5

Find the derivative of $y = \sin x \cos x$

Solution

$$y' = (\sin x)' \cos x + (\cos x)' \sin x \qquad (uv)' = u'v + uv'$$

$$= (\cos x)\cos x + (-\sin x)\sin x$$

$$= \cos^2 x - \sin^2 x$$

Example 6

Find the derivative of $y = \dfrac{\cos x}{1 - \sin x}$

Solution

$$y' = \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1 - \sin x}{(1-\sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

2.3–3 Derivatives of the Other Trigonometric Functions

To determine the derivative of the ***tangent*** function. In the following example, we are going to determine the derivative of the tangent by using simple derivative rules.

Example 7

Find $\dfrac{d}{dx}(\tan x)$

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$ ***Quotient Rule***

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$ $\frac{1}{\cos x} = \sec x$

$$= \sec^2 x$$

$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

Example 8

Find y'' *if* $y = \sec x$

Solution

$$y' = \sec x \tan x$$

$$y'' = (\sec x)' \tan x + \sec x (\tan x)'$$

$$= (\sec x \tan x)\tan x + \sec x\left(\sec^2 x\right)$$

$$= \sec x \tan^2 x + \sec^3 x$$

Example 9

Find an equation of the line tangent to the function $f(x) = \cot x$ @ $x = \frac{\pi}{4}$

Solution

$$f\left(x = \frac{\pi}{4}\right) = \cot\frac{\pi}{4}$$

$$= 1$$

The slope of the tangent line is found by:

$$m = f'(x)\Big|_{x=\frac{\pi}{4}}$$

$$= \frac{d}{dx}(\cot x)\Big|_{x=\frac{\pi}{4}}$$

$$= -\csc^2 x\Big|_{x=\frac{\pi}{4}}$$

$$= -\csc^2 \frac{\pi}{4}$$

$$= -\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)^2$$

$$= -\left(\sqrt{2}\right)^2$$

$$= -2$$

$$y = -2\left(x - \frac{\pi}{4}\right) + 1 \qquad y = m\left(x - x_1\right) + y_1$$

$$= -2x + \frac{\pi}{2} + 1$$

Example 10

Find the derivative of $f(x) = \sec x + x^2 \cot x$

Solution

$$f'(x) = \sec x \tan x + 2x \cot x - x^2 \csc^2 x$$

Example 11

Find the derivative of $f(x) = \csc x + \frac{\tan x}{x} + x^4 \cos x$

Solution

$$f'(x) = -\csc x \cot x + \frac{x\sec^2 x - \tan x}{x^2} + 4x^3 \cos x - x^4 \sin x \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad (uv)' = u'v + uv'$$

Exercises *Section* 2.3 –Derivatives of Trigonometric Functions

$(\mathbf{1-40})$ Find the derivative of :

1. $y=-10x+3\cos x$

2. $y=\csc x-4\sqrt{x}+7$

3. $y=x^2\cos x$

4. $y=\csc x\cot x$

5. $y=(\sin x+\cos x)\sec x$

6. $y=(\sec x+\tan x)(\sec x-\tan x)$

7. $y=\dfrac{\cos x}{x}+\dfrac{x}{\cos x}$

8. $y=x^2\cos x-2x\sin x-2\cos x$

9. $y=(2-x)\tan^2 x$

10. $y=t^2-\sec t+1$

11. $y=\dfrac{1+\csc t}{1-\csc t}$

12. $r=\theta\sin\theta+\cos\theta$

13. $p=\dfrac{\sin q+\cos q}{\cos q}$

14. $p=\dfrac{3q+\tan q}{q\sec q}$

15. $f(x)=\dfrac{\sin x+2x}{x}$

16. $f(x)=\dfrac{\sin x}{x^2}$

17. $f(x)=x^3\cos x$

18. $f(x)=\dfrac{1}{x}-12\sec x$

19. $f(\theta)=5\theta\sec\theta+\theta\tan\theta$

20. $y=\sec\pi x$

21. $y=\cos 5x$

22. $y=\cos(4-3x)$

23. $f(x)=\sin(4-3x)$

24. $f(\theta)=\dfrac{\sin a\theta}{\cos b\theta}$

25. $f(\theta)=\sin 2\theta-\cos 2\theta$

26. $f(\theta)=\tan\theta-\cot\theta$

27. $\dfrac{d}{dx}\left(5x^2\sin x\right)$

28. $\dfrac{d}{dx}\left(2x(\sin x)\sqrt{3x-1}\right)$

29. $f(x)=4x^3\sin x$

30. $f(x)=4x^3-\sec x+2$

31. $f(x)=3\tan x+\dfrac{1}{x}$

32. $f(x)=x^4\cot x$

33. $y=x-x^5\cos x$

34. $y=\sin x\,\tan x$

35. $y=\cot x\,\tan x$

36. $y=(1-\cos x)(x-\sin x)$

37. $f(\theta)=\dfrac{\tan\theta}{1-\sec\theta}$

38. $f(\theta)=\dfrac{1+\tan\theta}{1-\cot\theta}$

39. $f(\theta)=\dfrac{1-\cot\theta}{1+\cot\theta}$

40. $f(\theta)=\sec\theta+\theta^2\cos\theta$

$(\mathbf{41-46})$ Find the ***second*** derivative of the given functions.

41. $f(x) = x^5 \sin x$

42. $f(x) = x^2 \cos x + \sin x$

43. $f(x) = \sec x + \csc x$

44. $f(x) = 5\sin x$

45. $f(x) = 3\cos x + x$

46. $f(x) = 3\sec x - x^9$

47. Use the definition of derivative to prove: $\frac{d}{dx}(\tan x) = \sec^2 x$

48. Use the definition of derivative to prove: $\frac{d}{dx}(\sec x) = \sec x \tan x$

49. Use the definition of derivative to prove: $\frac{d}{dx}(\cot x)' = -\csc^2 x$

50. Use the definition of derivative to prove: $\frac{d}{dx}(\csc x) = -\csc x \cot x$

51. Find $y^{(4)}$ *if* $y = 9\cos x$

52. Find $y', y'', y''':$ $y = (x-3)\sqrt{x+2}$

53. Find $\frac{d^{999}}{dx^{999}}(\cos x)$

54. Find $\lim\limits_{x \to -\frac{\pi}{6}} \sqrt{1+\cos(\pi \csc x)}$

$(\mathbf{55-66})$ Find an equation of the line tangent to the function.

55. $f(x) = \tan x$ @ $x = \frac{\pi}{4}$

56. $f(x) = \cot x$ @ $x = \frac{3\pi}{4}$

57. $f(x) = \sec x$ @ $x = 0$

58. $f(x) = \csc x$ @ $x = \frac{\pi}{2}$

59. $f(x) = \cos x$ @ $x = \frac{\pi}{6}$

60. $f(x) = \sin x$ @ $x = \frac{\pi}{3}$

61. $f(x) = x - 2\sin x$ @ $x = \frac{\pi}{2}$

62. $f(x) = x + 2\cos x$ @ $x = \frac{\pi}{2}$

63. $f(x) = x - \sin x$ @ $x = \pi$

64. $f(x) = 1 + \cos x$ @ $x = \frac{3\pi}{2}$

65. $f(x) = x^2 - \tan x$ @ $x = 0$

66. $f(x) = -\sin x$ @ $x = 0$

$(\mathbf{67-70})$ Find an equation or equations of the line (*s*) tangent to the function $(0 < x < 2\pi)$

67. $f(x) = \tan x$ *parallel* $y = 2x$

68. $f(x) = \cot x$ *parallel* $y = -2x$

69. $f(x) = \cos x$ *perpendicular* $y = 2x$

70. $f(x) = -\sin x$ *perpendicular* $y = -2x$

71. Assume that a particle's position on the *x*-axis is given by

$$x = 3\cos t + 4\sin t$$

a) Find the particle's position when $t = 0, \quad t = \frac{\pi}{2}, \quad and \quad t = \pi$

b) Find the particle's velocity when $t = 0, \quad t = \frac{\pi}{2}, \quad and \quad t = \pi$

72. A weight is attached to a spring and reaches its equilibrium position $(x = 0)$. It is then set in motion resulting in a displacement of $x = 10\cos t$

Where *x* is measured in centimeters and *t* is measured in seconds.

a) Find the spring's displacement when $t = 0, \quad t = \frac{\pi}{3}, \quad and \quad t = \frac{3\pi}{4}$

b) Find the spring's velocity when $t = 0, \quad t = \frac{\pi}{3}, \quad and \quad t = \frac{3\pi}{4}$

73. After a diver jumps off a diving board, the edge of the board oscillates with position given by $d(t) = -4\cos t$, where *d* in inches and *t* in seconds.

a) Sketch one period of the position function $d(t), \quad t \geq 0$.

b) Find the line tangent of the position when $t = \pi$.

c) Find the velocity (first derivative), then sketch one period of the velocity for $t \geq 0$.

d) Determine the time when the velocity intersect *t*-axis.

e) Find the acceleration (second derivative), then sketch one period of the acceleration for $t \geq 0$.

f) Determine the time when the acceleration intersect *t*-axis.

Section 2.4 – Chain Rule

2.4–1 Derivative of a Composite Function

$$y = f(g(x)) = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$$

C: y turns B: u turns A: x turns

Example 1

Find the derivative of $y = \left(3x^2 + 1\right)^2$

Solution

$$u = 3x^2 + 1 \quad \Rightarrow \quad (u)' = 6x$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x$$

$$= 2\left(3x^2 + 1\right) \cdot 6x$$

$$= 36x^3 + 12x$$

Calculating from the expand formula: $y = \left(3x^2 + 1\right)^2 = 9x^4 + 6x^2 + 1$

$$y' = 36x^3 + 12x$$

2.4–2 Intuitive "*Proof*" of the Chain Rule

Let Δu be the change in u when x changes by Δx, so that:

$$\Delta u = g(x+\Delta x) - g(x)$$

Let Δy be the change in y when u changes by Δu, so that:

$$\Delta y = f(u+\Delta u) - f(u)$$

If $\Delta u \neq 0 \Rightarrow \quad \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

Example 2

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos\left(t^2+1\right)$.

Find the velocity of the object as a function of t.

Solution

Let: $u = t^2 + 1 \quad \Rightarrow \quad u' = 2t$

$x = \cos(u) \;\Rightarrow\; x' = -\sin(u)$

By the Chain Rule:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\
&= -\sin(u) \cdot 2t \\
&= -2t\sin\left(t^2+1\right)
\end{aligned}$$

2.4–3 The General *Power* Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\,u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(u^n\right)$$

$$= n\,u^{n\text{-}1}u'$$

Example 3

Find the derivative of $\frac{d}{dx}\left(5x^3 - x^4\right)^7$

Solution

$$\frac{d}{dx}\overbrace{\left(5x^3 - x^4\right)^7}^{nu^n} = \overbrace{7\left(5x^3 - x^4\right)^6}^{nu^{n-1}}\ \overbrace{\left(15x^2 - 4x^3\right)}^{u'}$$

Example 4

Find the derivative of $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$

Solution

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -3(3x-2)^{-2}$$

$$= -\frac{3}{(3x-2)^2}$$

Example 5

Find the derivative of $\frac{d}{dx}\left(\sin^5 x\right)$

Solution

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x(\sin x)'$$

$$= 5\sin^4 x\cos x$$

Example 6

Find the derivative of $g(t)=\tan(5-\sin 2t)$

Solution

$$g'(t)=\sec^2(5-\sin 2t)\cdot(5-\sin 2t)' \qquad u=5-\sin 2t \quad (\tan u)'=\sec^2 u\cdot(u')$$

$$=\sec^2(5-\sin 2t)\cdot\left(0-(\cos 2t)(2t)'\right)$$

$$=\sec^2(5-\sin 2t)\cdot(-2\cos 2t)$$

$$=-2(\cos 2t)\sec^2(5-\sin 2t)$$

Example 7

Show that the slope of every line tangent to the curve $y=\dfrac{1}{(1-2x)^3}$ is positive.

Solution

$$y=(1-2x)^{-3}$$

$$y'=-3(1-2x)^{-4}(-2)$$

$$=\frac{6}{(1-2x)^4}$$

At any point except $\left(x\neq\frac{1}{2}\right)$, the slope is $\dfrac{6}{(1-2x)^4}$ which is positive.

Example 8

Find the derivative of $f(x)=\sec\left(4x^3+2x\right)$

Solution

$$f'(x)=\left(4x^3+2x\right)'\sec\left(4x^3+2x\right)\tan\left(4x^3+2x\right) \qquad (\sec U)'=U'\sec U\tan U$$

$$=\left(12x^2+2\right)\sec\left(4x^3+2x\right)\tan\left(4x^3+2x\right)$$

Example 9

Find the derivative of $f(x)=\cos\left(x^2-5\right)$

Solution

$$f'(x)=-2x\sin\left(x^2-5\right)$$

$$(\cos U)'=-U'\sin U$$

Formula:

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m=mn(ad-bc)x^{n-1}\frac{\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

For the proof, see *Appendix* ***A***.

Example 10

Find the derivative of $f(x)=\left(\frac{2x+1}{3x-2}\right)^5$

Solution

$$\left(\frac{2x+1}{3x-2}\right)'=\frac{-4-3}{(3x-2)^2}$$

$$=\frac{-7}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$f'(x)=5\,\frac{-7}{(3x-2)^2}\left(\frac{2x+1}{3x-2}\right)^4$$

$$=-\frac{35(2x+1)^4}{(3x-2)^6}$$

Example 11

Find the derivative of $f(x)=\left(\frac{x^3+1}{2x^3-3}\right)^4$

Solution

$$f'(x)=(3)(4)(-3-2)x^2\frac{\left(x^3+1\right)^3}{\left(2x^3-3\right)^4}$$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m=mn(ad-bc)x^{n-1}\frac{\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

$$=-\frac{60x^2\left(x^3+1\right)^3}{\left(2x^3-3\right)^4}$$

Formula $\left(U^mV^nW^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$

Proof

$$\left(U^mV^nW^p\right)'=\left(U^m\right)'V^nW^p+U^m\left(V^n\right)'W^p+U^mV^n\left(W^p\right)'$$

$$=mU^{m-1}U'V^nW^p+nU^mV^{n-1}V'W^p+pU^mV^nW^{p-1}W'$$

factor $U^{m-1}V^{n-1}W^{p-1}$

$$=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

Example 12

Find the derivative of $f(x)=\left(3x^2+x-2\right)^5(4x+1)^4$

Solution

$$f'(x)=\overbrace{\left(3x^2+x-2\right)^4}^{U^{m-1}}\overbrace{(4x+1)^3}^{V^{n-1}}\overbrace{\left(5(6x+1)(4x+1)+4(4)\left(3x^2+x-2\right)\right)}^{(m\ U'\ V\ +\ n\ V'\ U)}$$

$$=\left(3x^2+x-2\right)^4(4x+1)^3\left(120x^2+50x+5+48x^2+16x-32\right)$$

$$=\left(3x^2+x-2\right)^4(4x+1)^3\left(168x^2+66x-27\right)$$

Example 13

Find the derivative of $f(x)=(3x-2)^5\left(x^2-3\right)^4\left(2x^3+5\right)^{-6}$

Solution

$$\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$\begin{aligned} f'(x) = (3x-2)^4\left(x^2-3\right)^3\left(2x^3+5\right)^{-7} \Big(& 5(3)\left(x^2-3\right)\left(2x^3+5\right) \\ & +4(2x)(3x-2)\left(2x^3+5\right) \\ & -6\left(6x^2\right)(3x-2)\left(x^2-3\right) \Big) \end{aligned}$$

$$\begin{aligned} f'(x) = (3x-2)^4\left(x^2-3\right)^3\left(2x^3+5\right)^{-7} \Big(& \left(15x^2-45\right)\left(2x^3+5\right) \\ & +\left(24x^2-16x\right)\left(2x^3+5\right) \\ & -\left(108x^3-72x^2\right)\left(x^2-3\right) \Big) \end{aligned}$$

x^5	x^4	x^3	x^2	x^1	x^0
30		–90	75		–225
48	–32		120	–80	
–108	72	324	–216		

$$f'(x) = \frac{(3x-2)^4\left(x^2-3\right)^3}{\left(2x^3+5\right)^7}\left(-30x^5+40x^4+234x^3-21x^2-80x-225\right)$$

Formulas

$$\left(\cos^n U\right)' = -nU'\ \sin U\ \cos^{n-1}U \qquad \left(\sin^n U\right)' = nU'\ \cos U\ \sin^{n-1}U$$

$$\left(\tan^n U\right)' = nU'\ \sec^2 U\ \tan^{n-1}U \qquad \left(\sec^n U\right)' = nU'\ \tan U\ \sec^n U$$

$$\left(\cos^n U\right)' = -nU'\ \csc^2 U\ \cot^{n-1}U \qquad \left(\csc^n U\right)' = -nU'\ \cot U\ \csc^n U$$

Example 14

Find the derivative of $f(x) = \cos^4\left(2x^3 + 5\right)$

Solution

$$f'(x) = -\,4\left(6x^2\right)\ \sin\left(2x^3+5\right)\ \cos^3\left(2x^3+5\right) \qquad \left(\cos^n U\right)' = -U'\ \sin U\ \cos^{n-1} U$$

$$= -\,24x^2\ \sin\left(2x^3+5\right)\ \cos^3\left(2x^3+5\right)$$

Example 15

Find the derivative of $f(x) = \sec^5\left(3x^2 + 4x\right)$

Solution

$$f(x) = 5(6x+4)\ \tan\left(3x^2+4x\right)\ \sec^5\left(3x^2+4x\right) \qquad \left(\sec^n U\right)' = U'\ \tan U\ \sec^n U$$

Exercises *Section* 2.4 – Chain Rule

$(\mathbf{1-108})$ Find the derivative of:

1. $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

2. $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

3. $y=\left(x^3+1\right)^2$

4. $y=\left(x^2+3x\right)^4$

5. $y=\dfrac{4}{2x+1}$

6. $y=\dfrac{2}{(x-1)^3}$

7. $y=x^2\sqrt{x^2+1}$

8. $y=\left(\dfrac{x+1}{x-5}\right)^2$

9. $s(t)=\sqrt{2t^2+5t+2}$

10. $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

11. $y=t^2\sqrt{t-2}$

12. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$

13. $y=4x(3x+5)^5$

14. $y=\left(3x^2-5x\right)^{1/2}$

15. $D_x\left(x^2+5x\right)^8$

16. $y=\dfrac{(3x+2)^7}{x-1}$

17. $y=\left(\dfrac{x^2}{8}+x-\dfrac{1}{x}\right)^4$

18. $y=\sqrt{3x^2-4x+6}$

19. $y=\cot\left(\pi-\dfrac{1}{x}\right)$

20. $y=5\cos^{-4}x$

21. $y=\sin\left(\dfrac{3\pi t}{2}\right)+\cos\left(\dfrac{3\pi t}{2}\right)$

22. $r=6(\sec\theta-\tan\theta)^{3/2}$

23. $g(x)=\dfrac{\tan 3x}{(x+7)^4}$

24. $f(\theta)=\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

25. $y=\sin^2(\pi t-2)$

26. $y=(t\tan t)^{10}$

27. $y=\cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$

28. $y=4\sin\left(\sqrt{1+\sqrt{t}}\right)$

29. $y=\tan^2\left(\sin^3 x\right)$

30. $f(x)=\left(\left(x^2+3\right)^5+x\right)^2$

31. $y=\left(\dfrac{3x-1}{x^2+3}\right)^2$

32. $y=\cos\sqrt{\sin(\tan\pi x)}$

33. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$

34. $y=\cos(1-2x)^2$

35. $f(x)=(4x-3)^2$

36. $f(x)=\dfrac{x}{\sqrt[3]{x^2+4}}$

37. $f(x)=\left(\dfrac{x^2}{x^3+2}\right)^2$

38. $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$

39. $f(\theta)=4\tan\left(\theta^2+3\theta+2\right)$

40. $f(\theta)=\tan(\sin\theta)$

41. $y=5x+\sin^3 x+\sin x^3$

42. $y=\csc^5 3x$

43. $f(\theta)=\cos^4\left(7\theta^2+1\right)$

44. $f(x)=\sin^5\left(x^3\right)$

45. $f(x)=\cos^4\left(x^2+2\right)$

46. $f(\theta)=\dfrac{1}{\sin^2\theta}$

47. $f(\theta)=(\tan\theta+\sin\theta)^{-3}$

48. $y=x^2\cos^4 x$

49. $y=\sin\left(\cos^4 x\right)$

50. $y=\cos(\sin 3x)$

51. $y=\sqrt{3+\sec\pi x^2}$

52. $y=\cot^3(3x+1)$

53. $y=\csc^2(2x-3)$

54. $y=\cos^4\pi x$

55. $y=\sin^5 x$

56. $y=-\dfrac{6}{\sin^3 x}$

57. $y=2x\sqrt{x^2-2x+2}$

58. $\dfrac{d}{du}\left(\dfrac{4u^2+u}{8u+1}\right)^3$

59. $y=\frac{1}{2}x^2\sqrt{16-x^2}$

60. $f(x)=\left(2x^3-2x+1\right)^5$

61. $f(x)=\left(3x^3+x^2-2\right)^4$

62. $f(x)=\left(2x^3-x^2+6x-1\right)^3$

63. $f(x)=\left(3x^4+x^2+1\right)^{5/2}$

64. $f(x)=\left(x^2-2x+1\right)^{3/2}$

65. $f(x)=\left(x^3+2x+3\right)^{-1/3}$

66. $y=\left(\dfrac{x-3}{2x+5}\right)^4$

67. $y=\left(\dfrac{5x-3}{2x+5}\right)^5$

68. $y=\left(\dfrac{6x-8}{2x-3}\right)^6$

69. $f(x)=\dfrac{1}{(2x-5)^5}$

70. $f(x)=\dfrac{1}{(3x-2)^2}$

71. $f(x)=\dfrac{1}{(x+2)^2}$

72. $f(x)=\dfrac{1}{\left(2x^3-3\right)^5}$

73. $f(x)=\dfrac{1}{\left(3x^4+2\right)^3}$

74. $f(x)=\dfrac{1}{\left(x^3+3\right)^8}$

75. $f(x)=\dfrac{1}{\left(3x^2+1\right)^3}$

76. $f(x)=\left(\dfrac{6x+3}{2x-3}\right)^4$

77. $f(x)=\left(\dfrac{-6x+3}{2x-3}\right)^5$

78. $f(x)=\left(\dfrac{3x-2}{2x+3}\right)^6$

79. $f(x)=\left(\dfrac{6x^2+3}{2x^2-3}\right)^7$

80. $f(x)=\left(\dfrac{-6x^2+3}{2x^2-3}\right)^6$

81. $f(x)=\left(\dfrac{3x^2-2}{2x^2+3}\right)^5$

82. $y=\left(\dfrac{3x^2-4}{2x^2-1}\right)^3$

83. $y=\left(\dfrac{3x^2+4}{2x^2+1}\right)^{-3}$

84. $y=\left(\dfrac{2x^2-3}{x^2+1}\right)^{1/3}$

85. $y=\sqrt{\dfrac{2x^3-3}{2x^3+1}}$

86. $y=\left(\dfrac{2x^4-3}{2x^4+1}\right)^5$

87. $y=\left(\dfrac{2x^3-3}{x^3+1}\right)^6$

88. $y=\left(\dfrac{3x^3-2}{2x^3+3}\right)^6$

89. $y=\left(\dfrac{x^2-4x+1}{5x^2-2x-1}\right)^3$

90. $y=\left(\dfrac{3x^2-4x+2}{2x^2+x-1}\right)^{2/3}$

91. $f(x)=\left(\dfrac{3t^2-1}{3t^2+1}\right)^{-3}$

92. $f(x)=\left(\dfrac{x}{3x^2+2x+1}\right)^{1/3}$

93. $f(x)=\left(x^3+x+5\right)^4\left(2x^2+3\right)^5$

94. $f(x)=\left(x^2+2x-3\right)^5(2x+3)^6$

95. $f(x)=\left(2x^2-4x+3\right)^4(3x-5)^5$

96. $f(x)=\left(x^2+2x-3\right)^4\left(x^2+3x+5\right)^6$

97. $f(x)=\left(2x^3-5x\right)^3\left(x^2+2x+1\right)^4(2x-3)^5$

98. $f(x)=\left(x^4+3x\right)^4\left(x^3+2x\right)^5(2x-3)^6$

99. $f(x)=\left(x^2-3\right)^4\left(x^3+1\right)^5(5x-5)^6$

100. $f(x)=\left(2x^3-3\right)^4\left(x^4-1\right)^5\left(x^2+1\right)^{-4}$

101. $f(x)=\dfrac{\left(x^2-6x\right)^5}{\left(3x^2+5x-2\right)^4}$

102. $f(x)=\dfrac{\left(2x^2+3x1\right)^4}{\left(x^2+5x-6\right)^5}$

103. $f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

104. $f(x)=\dfrac{x^2+3}{(2x-1)^3(3x+1)^4}$

105. $f(x)=\dfrac{\left(x^2-3x\right)^3\left(x^2+3x-3\right)^4}{\left(x^2-3x+2\right)^2}$

106. $f(x)=\dfrac{(x+2)^3(x-3)^5}{(5x-6)^3}$

107. $f(x)=\dfrac{\sin^4 x}{\left(x^2+1\right)^6}$

108. $f(x)=\dfrac{\left(3x^2+2x-5\right)^5}{\cos^6(5x)}$

$(\mathbf{109-122})$ Find the ***second*** derivative.

109. $y=\dfrac{x^2+3}{(x-1)^3+(x+1)^3}$

110. $y=\left(1+\dfrac{1}{x}\right)^3$

111. $y=9\tan\left(\dfrac{x}{3}\right)$

112. $y=\tan x+\cot x$

113. $y=\sec x+\cot x$

114. $y=\csc x+\tan x$

115. $y=4\sin^2(3x)$

116. $f(x)=2\cos^2\left(x^3\right)$

117. $f(x)=\sqrt{3x+2}$

118. $f(x)=\sqrt[3]{x^2-2x}$

119. $f(x)=\dfrac{3x-1}{x+2}$

120. $f(x)=(3x-1)^3(x+2)^4$

121. $f(x)=(x+2)^4\sin^3\pi x$

122. $f(x)=\left(x^3+1\right)^4\tan^2 3x$

123. Find the tangent line to the graph of $y=\sqrt[3]{(x+4)^2}$ when $x=4$.

$(\mathbf{124-125})$ Evaluate the limit.

124. $\displaystyle\lim_{h\to 0}\frac{\sin^2\left(\frac{\pi}{4}+h\right)-\frac{1}{2}}{h}$

125. $\displaystyle\lim_{x\to 5}\frac{\tan\left(\pi\sqrt{3x-11}\right)}{x-5}$

Section 2.5 – Derivatives of Logarithmic, Exponential, & Hyperbolic Functions

In mathematics, the ***logarithm*** is the inverse function of ***exponential***.

Logarithmic functions $(\ln x)$, can rescale large quantities.

Exponential functions $\left(e^x\right)$ play an important role in modeling population growth, the decay of radioactive materials, and most engineering problems.

2.5–1 Derivative of Logarithmic Functions

Derivative of $y = \ln x$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}} \qquad x \neq 0$$

The chain rule extends: $\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}}$ $\quad u > 0$

Example 1

Find $\quad \frac{d}{dx}\ln 2x$

Solution

$$\frac{d}{dx}\ln 2x = \frac{(2x)'}{2x}$$

$$= \frac{2}{2x}$$

$$= \frac{1}{x}$$

Example 2

Find the derivative of $\ln\left(x^2+3\right)$

Solution

$$\frac{d}{dx}\ln\left(x^2+3\right) = \frac{2x}{x^2+3}$$

Example 3

Find the derivative of $f(x) = \ln\left(x^3 + 3x + 4\right)$

Solution

$$f'(x) = \frac{3x^2 + 3}{x^3 + 3x + 4}$$

2.5–2 Properties of the logarithm

Product Rule	$\ln bx = \ln b + \ln x$
Quotient Rule	$\ln \frac{b}{x} = \ln b - \ln x$
Reciprocal Rule	$\ln \frac{1}{x} = -\ln x$
Power Rule	$\ln x^r = r \ln x$

Example 4

Find $\frac{dy}{dx}$ if $y = \frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}$, $x > 1$

Solution

$$\ln y = \ln \frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}$$

$$= \ln\left(x^2+1\right)(x+3)^{1/2} - \ln(x-1) \qquad \textit{Quotient Rule}$$

$$= \ln\left(x^2+1\right) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \textit{Product Rule}$$

$$= \ln\left(x^2+1\right) + \frac{1}{2}\ln(x+3) - \ln(x-1) \qquad \textit{Power Rule}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x+3} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

Example 5

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{3x+1}\right)$

Solution

$$f(x) = \ln\left(\frac{x^2 \sin x}{3x+1}\right)$$

$$= 2\ln x + \ln(\sin x) - \ln(3x+1)$$

$$f'(x) = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{3}{3x+1}$$

$$= \frac{2}{x} + \cot x - \frac{3}{3x+1}$$

2.5–3 The Derivative of e^x

The natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\ln\left(e^x\right) = x \qquad \textit{Inverse relationship}$$

$$\frac{d}{dx}\ln\left(e^x\right) = 1 \qquad \textit{Differentiate both sides.}$$

$$\frac{1}{e^x}\frac{d}{dx}\left(e^x\right) = 1 \qquad \frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}e^x = e^x$$

If u is any differentiable function of x, then

$$\boxed{\frac{d}{dx}e^u = e^u\frac{du}{dx}} \qquad \boxed{\left(e^u\right)' = u'e^u}$$

Example 6

Find the derivative of $\frac{d}{dx}\left(5e^x\right)$

Solution

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x$$

$$= 5e^x$$

Example 7

Find the derivative of $\frac{d}{dx}\left(e^{\sin x}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x)$$

$$= e^{\sin x}\cdot\cos x$$

Example 8

Find the derivative of $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}}\frac{d}{dx}\left(\sqrt{3x+1}\right)$$

$$= e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{-1/2}\cdot 3$$

$$= \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

Example 9

Find the derivative of $f(x) = e^{x^3} + \ln\left(x^2+1\right)$

Solution

$$f'(x) = 3x^2e^{x^3} + \frac{2x}{x^2+1}$$

Example 10

Find the derivative of $f(x) = e^{x^3\ln x}$

Solution

$$f'(x) = \left(x^3\ln x\right)' e^{x^3\ln x}$$

$$= \left(3x^2\ln x + x^2\right)e^{x^3\ln x}$$

2.5–4 *Definition*

For any numbers $a > 0$ and x, the ***exponential function with base a*** is

$$a^x = e^{x \ln a}$$

When $a = e$, the function gives $a^x = e^{x \ln a} = e^{x \ln e} = e^x$

2.5–5 Power Rule – *Definition*

For any $x > 0$ and for any real number n, $\quad x^n = e^{n \ln x}$

Example 11

Evaluate the derivative $f(x) = x^{2x}$

Solution

$$\begin{aligned}
\frac{d}{dx}\left(x^{2x}\right) &= \frac{d}{dx}\left(e^{2x \ln x}\right) \\
&= e^{2x \ln x}\left(2x \ln x\right)' \\
&= 2e^{2x \ln x}\left(\ln x + 1\right) \\
&= 2x^{2x}\left(\ln x + 1\right)
\end{aligned}$$

2.5–6 General Power Rule for Derivatives

For any $x > 0$ and for any real number n, $\frac{d}{dx}x^n = nx^{n-1}$

Proof

$$\begin{aligned}\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x}\\ &= e^{n\ln x}\frac{d}{dx}(n\ln x)\\ &= x^n \cdot \frac{n}{x}\\ &= nx^{n-1} \quad \checkmark\end{aligned}$$

Example 12

Differentiate $f(x) = x^x, \quad x > 0$

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(e^{x\ln x}\right)\\ &= e^{x\ln x}\frac{d}{dx}(x\ln x)\\ &= e^{x\ln x}\frac{d}{dx}(x\ln x)\\ &= e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right)\\ &= x^x(\ln x + 1) \quad x > 0\end{aligned}$$

2.5–7 *Theorem* – The Number *e* as a Limit

The number e can be calculated as the limit $e = \lim_{x\to 0}(1+x)^{1/x}$

Proof

If $f(x) = \ln x \quad \to f'(x) = \frac{1}{x} \quad so \quad f'(1) = 1$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{x\to 0} \frac{f(1+x)-f(1)}{x}$$

$$= \lim_{x\to 0} \frac{\ln(1+x)-\ln(1)}{x}$$

$$= \lim_{x\to 0} \left[\frac{1}{x}\ln(1+x)\right]$$

$$= \lim_{x\to 0} \ln(1+x)^{1/x}$$

$$= \ln\left[\lim_{x\to 0} (1+x)^{1/x}\right] \qquad f'(1)=1$$

$$= 1$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

2.5–8 *Definition*

If $a > 0,\ \neq 1$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\boxed{\frac{d}{dx}a^u = a^u \ \ln a \ \frac{du}{dx}}$$

$$\left(a^u\right)' = u'\ a^u \ \ln a$$

Example 13

Evaluate $\frac{d}{dx}\left(3^x\right)$

Solution

$$\frac{d}{dx}\left(3^x\right) = 3^x \ln 3$$

Example 14

Evaluate $\frac{d}{dx}\left(3^{-x}\right)$

Solution

$$\frac{d}{dx}3^{-x} = 3^{-x}\ln 3\frac{d}{dx}(-x)$$

$$= -3^{-x}\ln 3$$

***Example* 15**

Evaluate $\frac{d}{dx}\left(3^{\sin x}\right)$

Solution

$$\frac{d}{dx}3^{\sin x} = 3^{\sin x}\ \ln 3\ \frac{d}{dx}(\sin x)$$

$$= 3^{\sin x}(\cos x)\ \ln 3$$

2.5–9 Logarithms with base *a*

For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x

Inverse Equations for $\log_a x$ and a^x

$$a^{\log_a x} = x \quad (\forall x > 0)$$

$$\log_a\left(a^x\right) = x \quad (all\ x)$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Derivative $\frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}$

***Example* 16**

Evaluate $\frac{d}{dx}\log_{10}(3x+1)$

Solution

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{(3x+1)\cdot\ln 10}\frac{d}{dx}(3x+1)$$

$$= \frac{3}{(3x+1)\cdot\ln 10}$$

2.5–9 Exponential *Models*: Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0 e^{kt}$.

The ***initial value*** of y at $t = 0$ is $y(0) = y_0$ and the ***rate constant*** $k > 0$ determines the rate of the growth.

Exponential growth is characterized by a constant ***relative growth rate*** and is given by the quantity

$$\frac{1}{y}\frac{dy}{dt}$$

Example 17

Suppose the population of the town of Pine is given by $P(t) = 1{,}500 + 125t$, while the population of the town of Spruce is given by $S(t) = 1{,}500e^{0.1t}$, where $t \geq 0$ is measured in years. Find the growth rate and the relative growth rate of each town.

Solution

$$\frac{dP}{dt} = 125$$

$$\frac{dS}{dt} = 150e^{0.1t}$$

The relative growth rate of Pine is

$$\frac{1}{P}\frac{dP}{dt} = \frac{125}{1{,}500 + 125t},$$

which decreases in time.

The relative growth rate of Spruce is

$$\frac{1}{S}\frac{dS}{dt} = \frac{150e^{0.1t}}{1{,}500e^{0.1t}}$$

$$= \frac{1}{10}$$ Contant for all times

The linear population function has a constant absolute growth rate and the exponential population function has a constant relative growth rate.

Hyperbolic Functions

2.5–11 Relationship Between Trigonometric and Hyperbolic Functions

The trigonometric functions are based on relationships involving a circle, also known as ***circular*** functions. Specifically, $\cos t$ and $\sin t$ are equal to *x*- and *y*-coordinates, respectively, of the point $P(x, y)$ on the unit circle that corresponds to an angle of *t* radians.

Observe that *t* is twice the area of the circular sector.

The ***hyperbolic cosine*** and ***hyperbolic sine*** are defined in analogous fashion using the hyperbola $x^2 - y^2 = 1$ instead the circle $x^2 + y^2 = 1$.

***t* is twice the area of hyperbolic sector**

Consider the region bouunded by the x-axis, the right branch of the unit hyperbola $x^2 - y^2 = 1$, and a line segment from the origin to a point $P(x, y)$ on the hyperbola; let *t* equal twice the area of this region.

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}

2.5–12 Definitions, Identities, and Graphs of the Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \textit{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

We pronounce sinh x as "cinch x", rhyming with "pinch x"

cosh x as "kosh x", rhyming with "gosh x"

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

Example 18

Derive identity $\sinh 2x = 2\sinh x \cosh x$

Solution

$$2\sinh x \cosh x = 2\left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= \left(\frac{e^{2x} - e^{-2x}}{2}\right)$$

$$= \sinh 2x \quad \checkmark$$

Example 19

Use the fundamental identity $\cosh^2 x - \sinh^2 x = 1$ to prove that $1 - \tanh^2 x = \operatorname{sech}^2 x$

Solution

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \quad \checkmark$$

Circular Functions: $\cosh^2 u - \sinh^2 u = 1$

Identities	*Derivatives*
$\cosh^2 u - \sinh^2 u = 1$	$\frac{d}{dx}(\sinh u) = u' \cosh u$
$\sinh 2x = 2\sinh x \cosh x$	$\frac{d}{dx}(\cosh u) = u' \sinh u$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\frac{d}{dx}(\tanh u) = u' \operatorname{sech}^2 u$
$\cosh^2 x = \frac{\cosh 2x + 1}{2}$	$\frac{d}{dx}(\coth u) = -u' \operatorname{csch}^2 u$
$\sinh^2 x = \frac{\cosh 2x - 1}{2}$	$\frac{d}{dx}(\operatorname{sech} u) = -u' \operatorname{sech} u \ \tanh u$
$\tanh^2 x = 1 - \operatorname{sech}^2 x$	$\frac{d}{dx}(\operatorname{csch} u) = -u' \operatorname{csch} u \ \coth u$
$\coth^2 x = 1 + \operatorname{csch}^2 x$	

Example 20

Evaluate $\frac{d}{dx}(\sinh x) = \cosh x$

Solution

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Example 21

Evaluate $\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right)$

Solution

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right)$$

$$= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2 \sqrt{1+t^2}$$

Example 22

Evaluate $\frac{d}{dx}(\cosh x)^2$

Solution

$$\frac{d}{dx}(\cosh x)^2 = 2\cosh x \ \frac{d}{dx}(\cosh x)$$

$$= 2\cosh x \sinh x$$

$$= \sinh 2x$$

Example 23

Evaluate $\frac{d^2}{dx^2}(\operatorname{sech} 3x)$

Solution

$$\frac{d^2}{dx^2}(\operatorname{sech} 3x) = \frac{d}{dx}(-3 \operatorname{sech} 3x \tanh 3x)$$

$$= 9 \operatorname{sech} 3x \tanh^2 3x - 9 \operatorname{sech}^3 3x$$

$$= 9 \operatorname{sech} 3x \left(\tanh^2 3x - \operatorname{sech}^2 3x\right)$$

Exercises *Section* 2.5 – Derivatives of Logarithmic, Exponential, & Hyperbolic Functions

$(\mathbf{1-155})$ Find the derivative.

1. $y=\ln\sqrt{x+5}$

2. $y=(3x+7)\ln(2x-1)$

3. $f(x)=\ln\sqrt[3]{x+1}$

4. $f(x)=\ln\left(x^2\sqrt{x^2+1}\right)$

5. $y=\ln\dfrac{x^2}{x^2+1}$

6. $y=\ln\left(\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

7. $y=\ln(x^2+1)$

8. $f(x)=\ln\left(x^2-4\right)$

9. $y=2\ln\left(x^2-3x+4\right)$

10. $f(x)=3\ln\left(1+x^2\right)$

11. $f(x)=(1+\ln x)^3$

12. $f(x)=(x-2\ln x)^4$

13. $f(x)=x^2\ln x$

14. $f(x)=-\dfrac{\ln x}{x^2}$

15. $y=\ln\left(t^2\right)$

16. $y=\ln(2\theta+2)$

17. $y=(\ln x)^3$

18. $y=x(\ln x)^2$

19. $y=\dfrac{x^4}{4}\ln x-\dfrac{x^4}{16}$

20. $y=\dfrac{1+\ln t}{t}$

21. $f(x)=\dfrac{\ln x}{1+x}$

22. $f(x)=\dfrac{2x}{1+\ln x}$

23. $f(x)=x^3\ln x$

24. $f(x)=6x^4\ln x$

25. $f(x)=\ln x^8$

26. $f(x)=5x-\ln x^5$

27. $f(x)=\ln x^{10}+2\ln x$

28. $f(x)=\dfrac{\ln x}{2x+5}$

29. $y=-2\ln x+x^2-4$

30. $y=\ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$

31. $y=\ln(\ln(\ln x))$

32. $y=\ln(\sec(\ln x))$

33. $y=\ln\left(\dfrac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right)$

34. $y=\ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

35. $f(x)=e^{3x}$

36. $f(x)=e^{-2x^3}$

37. $f(x)=4e^{x^2}$

38. $f(x)=x^2e^x$

39. $f(x)=2x^3e^x$

40. $f(x)=\dfrac{3e^x}{1+e^x}$

41. $f(x)=5e^x+3x+1$

42. $f(x)=\dfrac{e^x+e^{-x}}{2}$

43. $f(x)=\dfrac{e^x}{x^2}$

44. $f(x)=x^2e^x-e^x$

45. $f(x)=(1+2x)e^{4x}$

46. $y=x^2e^{5x}$

47. $y=x^2e^{-2x}$

48. $f(x)=\dfrac{e^x}{x^2+1}$

49. $f(x)=\dfrac{1-e^x}{1+e^x}$

50. $y=\dfrac{e^x+e^{-x}}{x}$

51. $y=\sqrt{e^{2x^2}+e^{-2x^2}}$

52. $y=\dfrac{x}{e^{2x}}$

53. $y=3e^{5x^3+1}$

54. $f(x)=\left(x^2-2x+2\right)e^x$

55. $f(\theta)=e^\theta(\sin\theta+\cos\theta)$

56. $f(\theta)=\ln\left(3\theta e^{-\theta}\right)$

57. $f(\theta)=\theta^3e^{-2\theta}\cos 5\theta$

58. $f(\theta)=\ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

59. $f(t)=e^{(\cos t+\ln t)}$

60. $y=e^{\sin t}\left(\ln t^2+1\right)$

61. $y=e^{x^2}\ln x$

62. $f(x)=e^x+x-\ln x$

63. $y=\ln x+2e^x-3x^2$

64. $f(x)=\ln x^2+4e^x$

65. $y=\ln\dfrac{1+e^x}{1-e^x}$

66. $y=\dfrac{\ln x}{e^{2x}}$

67. $f(x)=e^{2x}\ln\left(xe^x+1\right)$

68. $f(x)=\dfrac{xe^x}{\ln\left(x^2+1\right)}$

69. $f(x)=xe^{-10x}$

70. $f(x)=x\ln^2 x$

71. $f(x)=e^{-x}\ln x$

72. $f(x)=2^{x^2-x}$

73. $y=\ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

74. $f(x)=e^{\left(4\sqrt{x}+x^2\right)}$

75. $f(t)=\ln\left(3te^{-t}\right)$

76. $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

77. $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

78. $f(x)=(2x)^{4x}$

79. $f(x)=2^{x^2}$

80. $h(y)=y^{\sin y}$

81. $f(x)=x^{\pi}$

82. $h(t)=(\sin t)^{\sqrt{t}}$

83. $p(x)=x^{-\ln x}$

84. $f(x)=x^{2x}$

85. $f(x)=x^{\tan x}$

86. $f(x)=x^e+e^x$

87. $f(x)=x^{x^{10}}$

88. $f(x)=\left(1+\dfrac{4}{x}\right)^x$

89. $f(x)=\cos\left(x^{2\sin x}\right)$

90. $f(x)=\ln(\ln x)$

91. $f(x)=\ln\left(\cos^2 x\right)$

92. $f(x)=\dfrac{\ln x}{\ln x+1}$

93. $f(x)=\dfrac{\ln x}{x}$

94. $f(x)=\dfrac{\tan^{10} x}{(5x+3)^6}$

95. $y=\dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$

96. $f(x)=\dfrac{x^8\cos^3 x}{\sqrt{x-1}}$

97. $f(x)=(\sin x)^{\tan x}$

98. $f(x)=\left(1+\dfrac{1}{x}\right)^{2x}$

99. $f(x)=\dfrac{e^{-x}}{x}$

100. $f(x)=x^{\pi}\cdot\pi^x$

101. $f(x)=\ln\left(4x^4+x\right)$

102. $f(x)=x^3\ln\left(6x^2\right)$

103. $f(x)=x\ln x$

104. $f(x)=x\ln x-x$

105. $y=\frac{1}{2}\sinh(2x+1)$

106. $y=2\sqrt{t}\tanh\sqrt{t}$

107. $y=\ln(\cosh z)$

108. $y=\operatorname{csch}\theta(1-\ln\operatorname{csch}\theta)$

109. $y=\ln\sinh v-\frac{1}{2}\coth^2 v$

110. $y=\left(x^2+1\right)\operatorname{sech}(\ln x)$

111. $y=\left(4x^2-1\right)\operatorname{csch}(\ln 2x)$

112. $y=-\sinh^3 4x$

113. $y=\sqrt{\coth 3x}$

114. $y=\dfrac{x}{\operatorname{csch}x}$

115. $y=\tanh^2 x$

116. $y=\ln\left(\operatorname{sech}2x\right)$

117. $y=x^2\cosh^2 3x$

118. $y=6\sinh\frac{x}{3}$

119. $y=\ln(\sinh x)$

120. $y=x^2\tanh x$

121. $y=x^2\tanh\frac{1}{x}$

122. $y=x\operatorname{sech}x$

123. $f(x)=\sinh\left(x^2-3\right)$

124. $y=x\sinh x-\cosh x$

125. $f(x)=\ln(\sinh x)$

126. $f(x)=\ln\left(\tanh\frac{x}{2}\right)$

127. $f(x)=\tanh(2x-3)$

128. $f(x)=\cosh(2x+1)$

129. $f(x)=\sinh(3x+2)$

130. $f(x)=\tanh(4x-1)$

131. $f(x)=3x^3\tanh\left(x^4\right)$

132. $f(x)=\sinh(\ln x)$

133. $f(x)=\tanh^2(x)\cosh x$

134. $f(x)=\dfrac{\sinh x}{1+\cosh x}$

135. $f(x)=\tanh\sqrt{x^2+1}$

136. $f(x)=\dfrac{a^x}{\ln a}$

137. $f(x)=\ln\sqrt{4x-5}$

138. $f(x)=\ln(\sec x)$

139. $f(x)=\ln(\tan x)$

140. $f(x)=\ln\left(5x^3+2\right)^5$

141. $f(x)=\ln(\ln x)$

142. $y=\ln\left(x+\sqrt{3x^2+1}\right)$

143. $f(x)=\sin(\ln x)$

144. $f(x)=xe^{-x}$

145. $f(x)=\ln\left(e^{-x}\right)$

146. $y=t^{1-e}$

147. $y=2^{\sin 3t}$

148. $y=\log_3(1+\theta\ln 3)$

149. $y=\log_{25}e^x-\log_5\sqrt{x}$

150. $y=\log_3 r\cdot\log_9 r$

151. $y=\log_7\left(\dfrac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

152. $y=3\log_8\left(\log_2 t\right)$

153. $y=t\ \log_3\left(e^{(\sin t)(\ln 3)}\right)$

154. $f(x)=\log_3(x+8)$

155. $f(x)=2^{x^2-x}$

(156 – 171) Use logarithmic differentiation to find the derivative of:

156. $y=\sqrt{x(x+1)}$

157. $y=\sqrt{\left(x^2+1\right)(x-1)^2}$

158. $y=\sqrt{\dfrac{1}{t(t+1)}}$

159. $y=\dfrac{\theta+5}{\theta\cos\theta}$

160. $y=\sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}$

161. $y=(x+1)^x$

162. $y=x^2+x^{2x}$

163. $y=(\sin x)^x$

164. $y=x^{\sin x}$

165. $y=(\ln x)^{\ln x}$

166. $y=(3x)^x$

167. $y=x^{e^x}$

168. $y=\dfrac{x\sqrt{2x+1}}{e^x\sin^3 x}$

169. $y=\left(x^2-1\right)^{\ln x}$

170. $y=x^{\tan x}$

171. $y=\dfrac{x\ \sin x}{\left(1+e^x\right)^3\cos x}$

$(\mathbf{172-174})$ Find the derivative of:

172. $\dfrac{d^{99}}{dx^{99}}\left(e^x\right)$

173. $\dfrac{d^5}{dx^5}(\cosh x)$

174. $\dfrac{d^6}{dx^6}(\sinh x)$

$(\mathbf{175-190})$ Find the second derivative of:

175. $y = e^{2x}$

176. $y = 3\,e^{5x^3+1}$

177. $y = \left(x^2-3\right)e^x$

178. $y = x^2\,e^{-2x^2}$

179. $y = e^{x^2}$

180. $y = x^2\ln x - \frac{1}{2}x^2$

181. $y = x\ln x - x$

182. $y = e^x \sin x$

183. $y = e^{-x}\cos x$

184. $y = e^{3x}\ln x$

185. $y = \ln(3x+2)$

186. $y = \ln\left(3x^2+x-2\right)$

187. $y = \ln|\sin x|$

188. $y = \ln(\sec x)$

189. $y = \dfrac{e^x+e^{-x}}{2}$

190. $y = \dfrac{e^x}{e^x+e^{-x}}$

191. Rewrite the expression $\cosh 3x - \sinh 3x$ in terms of exponentials and simplify the results as much as you can.

192. Rewrite the expression $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$ in terms of exponentials and simplify the results as much as you can.

193. Prove the identities.

a) $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

b) $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$

194. Show that $\dfrac{d}{dx}\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \dfrac{1}{2}e^{x/2}$

195. Find the equation of the tangent line to $f(x) = e^x$ at the point (0, 1).

196. Find the equation of the tangent line to $f(x) = e^x$ at the point (1, e).

197. Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4).

198. Find the equation of the tangent line to $y = 4xe^{-x} + 5$ at $x = 1$.

199. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450\left(1 - e - 0.0022t\right)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 *months*.

200. A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 hours?

201. A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after t *hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

202. The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln\left(t + 2\right)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in 2013 $(t = 13)$.

Section 2.6 – Implicit Differentiation

In most mathematics, we can express implicit equation as the value of the dependent variable directly (explicitly) as a function of the independent variable. The majority of differentiation problems in first-year calculus involve functions y written explicitly as functions of *x*. In section, we are going to introduce the Implicit differentiation which is nothing more than the chain rule for derivatives.

2.6−1 *Definition*

A relation $F(x,y)=0$ is said to define the function $y=f(x)$ implicitly if, for *x* in the domain of f $\rightarrow F(x,f(x))=0$

Examples: $x^3+y^3-9xy=0,\quad x^2+y^2=25$

2.6−2 Implicitly Defined Functions

It is always assumed that the given equation determines *y* implicitly as a differentiable function of *x* so that $\frac{dy}{dx}$ exists.

2.6–3 Chain rule *vs*. Implicit differentiation $\frac{dy}{dx} = y'$

Chain Rule	*Implicit*
$\left(U^n\right)' = nU'\,U^{n-1}$	$\left(y^n\right)' = n\,y^{n-1}y'$
$(\sin u)' = u'\cos u$	$(\sin y)' = y'\cos y$
$\left(e^u\right)' = e^u\,u'$	$\left(e^y\right)' = e^y\,y'$
$(\ln u)' = \frac{u'}{u}$	$(\ln y)' = \frac{y'}{y}$

Example 1

Find $\frac{dy}{dx}$ if $y^2 = x$

Solution

$$y^2 = x$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

Example 2

Find the slope of the circle $x^2 + y^2 = 25$ at the point (3, –4).

Solution

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope at (3, –4) is:

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4}$$

$$= \frac{3}{4}$$

2.6–4 Implicit Differentiation

- ✓ Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
- ✓ Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.
- ✓ Or use y' instead of $\frac{dy}{dx}$.

Example 3

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Solution

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} = 2x + y\cos(xy) + x\cos(xy)\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy) \qquad \text{Let } y' = \frac{dy}{dx}$$

$$2yy' = 2x + \cos(xy)(xy)'$$

$$2yy' = 2x + \cos(xy)(y + xy')$$

$$2yy' = 2x + y\cos(xy) + x\cos(xy)y'$$

$$2yy' - x\cos(xy)y' = 2x + y\cos(xy)$$

$$(2y - x\cos xy)y' = 2x + y\cos xy$$

$$\boxed{y' = \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

Example 4

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

Solution

Let $y' = \dfrac{dy}{dx}$

$$\frac{d}{dy}\left(2x^3 - 3y^2\right) = \frac{d}{dy}(8)$$

$$6x^2 - 6yy' = 0$$

$$6x^2 = 6yy'$$

$$y' = \frac{6x^2}{6y}$$

$$= \frac{x^2}{y}$$

$$y'' = \left(\frac{x^2}{y}\right)' \qquad \begin{array}{ll} u = x^2 & v = y \\ u' = 2x & v' = y' \end{array}$$

$$y'' = \frac{2xy - x^2 y'}{y^2}$$

$$= \frac{2xy - x^2 \frac{x^2}{y}}{y^2}$$

$$= \frac{\frac{2xy^2 - x^4}{y}}{y^2}$$

$$= \frac{2xy^2 - x^4}{y^3}$$

2.6–5 Normal Lines

The ***normal*** line is the line ***perpendicular*** to the ***tangent*** of the profile curve at the point of entry.

$$m_{normal} \bullet m_{\text{tangent}} = -1$$

Example 5

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

Solution

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(9xy) = 0$$

$$3x^2 + 3y^2 y' - 9\left(y + xy'\right) = 0$$

$$3x^2 + 3y^2 y' - 9y - 9xy' = 0$$

$$3\left(y^2 - 3x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

The slope: $y'\Big|_{(2,4)} = \dfrac{3(4) - (2)^2}{(4)^2 - 3(2)}$

$$= \frac{8}{10}$$

$$= \frac{4}{5}$$

The tangent at (2, 4) is the line passes thru (2, 4) with slope $\frac{4}{5}$

$$y = \frac{4}{5}(x - 2) + 4 \qquad y = m(x - x_1) + y_1$$

$$y = \frac{4}{5}x - \frac{8}{5} + 4$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

The Normal to the curve at (2, 4) is the line perpendicular to the tangent thru (2, 4) with slope

$$m_1 \cdot m_2 = -1$$

$$\left(\frac{4}{5}\right) \cdot m_2 = -1$$

$$m_2 = -\frac{5}{4}$$

$$y = -\frac{5}{4}(x - 2) + 4 \qquad y = m(x - x_1) + y_1$$

$$y = -\frac{5}{4}x + \frac{5}{2} + 4$$

$$y = -\frac{5}{4}x + \frac{13}{2}$$

Example 6

Find $\frac{dy}{dx}$: $x^3 \sin y + y = 3x - 2$

Solution

$$\frac{d}{dx}\left(x^3 \sin y + y\right) = \frac{d}{dx}(3x-2)$$

$$3x^2 \sin y + x^3(\cos y)y' + y' = 3$$

$$\left(x^3 \cos y + 1\right)y' = 3 - 3x^2 \sin y$$

$$\boxed{\frac{dy}{dx} = \frac{3 - 3x^2 \sin y}{x^3 \cos y + 1}}$$

Exercises *Section* 2.6 – Implicit Differentiation

$(\mathbf{1-60})$ Find $\frac{dy}{dx}$.

1. $y^2 + x^2 - 2y - 4x = 4$

2. $x^2y^2 - 2x = 3$

3. $x + \sqrt{x}\sqrt{y} = y^2$

4. $x^2y + xy^2 = 6$

5. $x^3 - xy + y^3 = 1$

6. $y^2 = \frac{x-1}{x+1}$

7. $(3xy+7)^2 = 6y$

8. $xy = \cot(xy)$

9. $x + \tan(xy) = 0$

10. $x\cos(2x+3y) = y\sin x$

11. $y = \frac{e^y}{1+\sin x}$

12. $\sin x \cos(y-1) = \frac{1}{2}x^2 - 2y^2 - 1 = 0$

13. $y\sqrt{x^2+y^2} = 15$

14. $4x^5 + \tan y = x^2 + 5y$

15. $x^2 - y^2 = 16$

16. $3x^2 + 6y^2 = 12$

17. $3x^3 + 6xy^2 = 4x^3$

18. $xy - \cos(xy) = 3$

19. $x^2y^2 = y - 5$

20. $y\sqrt{x+3} = xy + 9$

21. $y\sin(xy) = y^2 + 1$

22. $(xy)^2 + 2x = y^2$

23. $x^3y + xy^3 = 8$

24. $x^4y - xy^3 = 2$

25. $e^{2x}\cos y = 0$

26. $xy + e^{xy} = x$

27. $\ln(1+xy) = x^2$

28. $\ln(x+y) + x^2 = y$

29. $(x+y)e^{xy} = y^2 + 3x$

30. $e^{-y}\left(x^2+y^2\right) = 2$

31. $2x^2 + 3xy + 4y^2 - 5x + 2y = 0$

32. $\sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x = 0$

33. $4 = \frac{1}{x^2+y^2-1}$

34. $\cos^2\left(3x - y^2\right) = x + 2$

35. $5 = e^{xy}\ln y$

36. $x\cos y - y\sin x = xy$

37. $x^2y + \cos y + y\sin x = 1$

38. $y\left(3x^2 - y^2\right) = e$

39. $\ln\left(x^2+y^2\right) = 2$

40.

41. $x^3 + 3xy^2 - y^5 = 0$

42. $2\sin xy = 1$

43. $ye^{xy} - 2 = 0$

44. $\sqrt{x^2 + 2xy + y^4} = 3$

45. $y\ln\left(x^2+y^2+4\right) = 3$

46. $y\ln\left(x^2+y^2\right) = 4$

47. $2x^2 + 3xy - 3y^4 = 2$

48. $xe^y + ye^x + 2\ln x - 2 - 3\ln 2 = 0$

49. $\frac{1}{2}\ln\left(x^2+y^2\right)+\tan\frac{y}{x}=2$

50. $x^3+y^3=1$

51. $x^2y^3=2x-y$

52. $x^2y^3-x^3y^2=8$

53. $x^2+xy=y^3$

54. $x^3y+xy^5=2$

55. $\frac{x-y}{x+y}=1+\frac{x^2}{y}$

56. $x\sqrt{x+y}=8-xy$

57. $\frac{x}{y}+\left(\frac{y}{x}\right)^3=1$

58. $x+2y+1=\frac{y^2}{x-1}$

59. $x\sin\left(xy-y^2\right)=x^2+1$

60. $\cos\left(\frac{\pi y}{x}\right)=\frac{x^2}{y}$

(61 – 64) Find $\frac{dr}{d\theta}$.

61. $r-2\sqrt{\theta}=\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}$

62. $\sin(r\theta)=\frac{1}{2}$

63. $r\theta^2=\cos^2\theta$

64. $\frac{1}{2}r^2\theta=r\cos\theta+\sin\theta$

(65 – 80) Find $\frac{d^2y}{dx^2}$.

65. $x^{2/3}+y^{2/3}=1$

66. $2\sqrt{y}=x-y$

67. $x^2+y^2=16$

68. $x^3-y^3=5$

69. $x^2+6xy-2y^2=5$

70. $x^4+y^4-16xy=3$

71. $\frac{1}{3}x^3-\frac{1}{3}y^3+2xy=3$

72. $y\left(3x^2-y^2\right)=1$

73. $ye^{x^2-y}=1$

74. $x^2+4y^2=2$

75. $x^3-2y^2+y^3=x$

76. $x^3-2xy+y^3=1$

77. $\ln\left(x^2-y^2\right)=4$

78. $\ln\left(x^2+y^2\right)=x$

79. $e^{xy}=x$

80. $e^{xy}=y$

81. If $x^3+y^3=16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

82. Find dy/dx: $x^2-xy+y^2=4$ and evaluate the derivative at the given point $(0,-2)$.

83. Find the slope of the curve $\left(x^2+y^2\right)^2=(x-y)^2$ at the point (–2, 1) and (–2, –1).

84. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1).

$(\mathbf{85-90})$ Find an equation of the line tangent to the following curves at the given point.

85. $y = 3x^3 + \sin x$; $(0,\ 0)$

86. $y = \dfrac{4x}{x^2+3}$; $(3,\ 1)$

87. $y + \sqrt{xy} = 6$; $(1,\ 4)$

88. $x^2y + y^3 = 75$; $(4,\ 3)$

89. $x^3 + y^3 = 9xy$; $(2,\ 4)$

90. $(x+1)^2(y-1)^2 = 1$; $\left(1,\ \frac{1}{2}\right)$

91. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point $(2,\ 3)$.

92. Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point $(-1,\ 0)$.

93. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y - \sin y = 0$ at the point $(0,\ \pi)$.

94. Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation.

$$xy^2 + y = x^2 + 17$$

Suppose further that when *x* = 2 and *y* = 3, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

95. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

Section 2.7 – Derivatives of Inverse Functions

2.7–1 Defining the inverses

The six basic trigonometric functions are not ***one-to-one***.

Sine	***Cosine***	***Tangent***
$y = \sin x$ *Domain*: $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ *Range*: $[-1,\ 1]$	$y = \cos x$ *Domain*: $[0,\ \pi]$ *Range*: $[-1,\ 1]$	$y = \tan x$ *Domain*: $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ *Range*: $(-\infty,\ \infty)$
Inverse		
$y = \sin^{-1} x$ *Domain*: $-1 \le x \le 1$ *Range*: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$y = \sin^{-1} x$ *Domain*: $-1 \le x \le 1$ *Range*: $0 \le y \le \pi$	$y = \tan^{-1} x$ *Domain*: $-\infty < x < \infty$ *Range*: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Cosecant	***Secant***	***Cotangent***
$y = \csc x$ *Domain*: $\left[-\frac{\pi}{2},\ 0\right)\cup\left(0,\ \frac{\pi}{2}\right]$ *Range*: $(-\infty,\ -1]\cup[1,\ \infty)$	$y = \sec x$ *Domain*: $\left[0,\ \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\ \pi\right]$ *Range*: $(-\infty,\ -1]\cup[1,\ \infty)$	$y = \cot x$ *Domain*: $(0,\ \pi)$ *Range*: $(-\infty,\ \infty)$
Inverse		
$y = \csc^{-1} x$ *Domain*: $x \le -1$ *or* $x \ge 1$ *Range*: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $(y \ne 0)$	$y = \sec^{-1} x$ *Domain*: $x \le -1$ *or* $x \ge 1$ *Range*: $0 \le y \le \pi$ $\left(y \ne \frac{\pi}{2}\right)$	$y = \cot^{-1} x$ *Domain*: $-\infty < x < \infty$ *Range*: $0 < y < \pi$

Since these restricted functions are ***now*** one-to-one, they have inverses, which we denoted by

$$y = \sin^{-1} x \quad or \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad or \quad y = \arccos x$$

$$y = \tan^{-1} x \quad or \quad y = \arctan x$$

$$y = \cot^{-1} x \quad or \quad y = \operatorname{arc}\cot x$$

$$y = \sec^{-1} x \quad or \quad y = \operatorname{arc}\sec x$$

$$y = \csc^{-1} x \quad or \quad y = \operatorname{arc}\csc x$$

2.7–2 *Definitions*

- ✓ $y = \sin^{-1} x$ is the number in $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ for which $\sin y = x$.
- ✓ $y = \cos^{-1} x$ is the number in $[0,\ \pi]$ for which $\cos y = x$.
- ✓ $y = \tan^{-1} x$ is the number in $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$ for which $\tan y = x$.
- ✓ $y = \cot^{-1} x$ is the number in $(0,\ \pi)$ for which $\cot y = x$.

Example 1

Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

2.7–3 *Inverse Function* – Inverse Cofunction Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \qquad \cot^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x \qquad \sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \qquad \sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$$

2.7–4 Derivatives of *Inverse* Trigonometric

Derivative of $y = \sin^{-1} x$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$y = \sin^{-1} x \quad \Rightarrow \quad f = \sin y$$

$$\left(f^{-1}\right)'(x) = \frac{1}{\cos\left(\sin^{-1} x\right)} \qquad \cos\alpha = \sqrt{1-\sin^2\alpha}$$

$$= \frac{1}{\sqrt{1-\sin^2\left(\sin^{-1} x\right)}}$$

$$= \frac{1}{\sqrt{1-\left(\sin\left(\sin^{-1} x\right)\right)^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Or we can use implicitly to find the inverse derivative of the sine function.

$$y = \sin^{-1} u \quad \Rightarrow \quad u = \sin y$$

$$\frac{d}{dx}(u) = \frac{d}{dx}(\sin y)$$

$$u' = y' \cos y$$

$$y' = \frac{u'}{\cos y} \qquad \cos^2 y + \sin^2 y = 1$$

$$= \frac{u'}{\sqrt{1-\sin^2 y}} \qquad u = \sin y$$

$$= \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$$

Derivative of $y = \cos^{-1} x$

$$y = \cos^{-1} u \quad \Rightarrow \quad u = \cos y$$

$$\frac{d}{dx}(u) = \frac{d}{dx}(\cos y)$$

$$u' = -y' \sin y$$

$$y' = -\frac{u'}{\sin y} \qquad \cos^2 y + \sin^2 y = 1$$

$$= -\frac{u'}{\sqrt{1-\cos^2 y}} \qquad u = \sin y$$

$$= -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} u\right) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\sin^{-1} u\right)' = \frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\cos^{-1} u\right)' = -\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$	$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$
$\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\cot^{-1} u\right)' = -\frac{u'}{1+u^2}$
$\frac{d}{dx}\sec^{-1} u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\sec^{-1} u\right)' = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}\csc^{-1} u = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\csc^{-1} u\right)' = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

Example 2

Find the derivative of $\frac{d}{dx}\left(\sin^{-1} x^2\right)$

Solution

$$\frac{d}{dx}\left(\sin^{-1}x^2\right)=\frac{1}{\sqrt{1-\left(x^2\right)^2}}\cdot\frac{d}{dx}\left(x^2\right)$$

$$=\frac{2x}{\sqrt{1-x^4}}$$

Example 3

Find the derivative of $\frac{d}{dx}\left(\sec^{-1}5x^4\right)$

Solution

$$\frac{d}{dx}\left(\sec^{-1}5x^4\right)=\frac{\left(5x^4\right)'}{5x^4\sqrt{\left(5x^4\right)^2-1}} \qquad 5x^4-1>0$$

$$=\frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$=\frac{4}{x\sqrt{25x^8-1}}$$

Example 4

Find the derivative of $\frac{d}{dx}\left(\arccos x^3\right)$

Solution

$$\frac{d}{dx}\left(\arccos x^3\right)=-\frac{1}{\sqrt{1-\left(x^3\right)^2}}\cdot\left(x^3\right)'$$

$$=-\frac{3x^2}{\sqrt{1-x^6}}$$

Example 5

Find the derivative of $f(x)=x^2\sin^{-1}x^3$

Solution

$$f'(x) = 2x\sin^{-1}x^3 + x^2\frac{3x^2}{\sqrt{1-x^6}}$$

$$= 2x\sin^{-1}x^3 + \frac{3x^4}{\sqrt{1-x^6}}$$

***Example* 6**

Find the derivative of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$

Solution

$$f'(x) = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{x^2}{x^2+1}\left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2+1}$$

2.7–5 Inverse Hyperbolic Trigonometric Functions

$y = \cosh^{-1}x \quad \Leftrightarrow \quad x = \cosh y$

for $x \geq 1$ *and* $0 \leq y < \infty$

$y = \sinh^{-1}x \quad \Leftrightarrow \quad x = \sinh y$

$-\infty < x < \infty$ *and* $-\infty < y < \infty$

$y = \tanh^{-1} x \quad \Leftrightarrow \quad x = \tanh y$

$-1 < x < 1 \quad and \quad -\infty < y < \infty$

$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y$

$0 < x \le 1 \quad and \quad 0 \le y < \infty$

$y = \coth^{-1} x \quad \Leftrightarrow \quad x = \coth y$

$|x| > 1 \quad and \quad y \neq 0$

$y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad x = \operatorname{csch} y$

$x \neq 0 \quad and \quad y \neq 0$

Example:

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)}$$

$$= \frac{1}{\frac{1}{x}}$$

$$= x$$

Identities	*Derivatives*
$\operatorname{sech}^{-1}x = \cosh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\sinh^{-1}u\right) = \frac{u'}{\sqrt{1+u^2}}$
$\operatorname{csch}^{-1}x = \sinh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\cosh^{-1}u\right) = \frac{u'}{\sqrt{u^2-1}}$
$\coth^{-1}x = \tanh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\tanh^{-1}u\right) = \frac{u'}{1-u^2}$
	$\frac{d}{dx}\left(\coth^{-1}u\right) = \frac{u'}{1-u^2}$
	$\frac{d}{dx}\left(\operatorname{sech}^{-1}u\right) = -\frac{u'}{u\sqrt{1-u^2}}$
	$\frac{d}{dx}\left(\operatorname{csch}^{-1}u\right) = -\frac{u'}{\lvert u\rvert\sqrt{1+u^2}}$

Example 7

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}\left(\cosh^{-1}u\right) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

Solution

$$f = \cosh u$$
$$f' = u'\sinh u$$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$= \frac{1}{\sinh\left(\cosh^{-1}x\right)} \qquad \cosh^2 u - \sinh^2 u = 1 \Rightarrow \sinh u = \sqrt{\cosh^2 u - 1}$$

$$= \frac{1}{\sqrt{\cosh^2\left(\cosh^{-1}x\right)-1}} \qquad \cosh\left(\cosh^{-1}x\right) = x$$

$$= \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cosh^{-1}u\right)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

Example 8

Find the points at which the curves $y=\cosh x$ and $y=\frac{5}{3}$ intersect.

Solution

$$\cosh x=\frac{5}{3}$$

$$\cosh^{-1}(\cosh x)=\cosh^{-1}\left(\frac{5}{3}\right)$$

$$|x|=\ln\left(\frac{5}{3}+\sqrt{\left(\frac{5}{3}\right)^2+1}\right) \qquad \cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$$

$$=\ln\left(\frac{5}{3}+\frac{4}{3}\right)$$

$$x=\pm\ln 3$$

The points of intersection lie on the line $y=\frac{5}{3}$, are $\left(-\ln 3,\ \frac{5}{3}\right)$ *and* $\left(\ln 3,\ \frac{5}{3}\right)$.

Example 9

Find the derivative $y=\tanh^{-1}3x$

Solution

$$y'=\frac{3}{1-9x^2}$$

Example 10

Find the derivative $y=x^2\sinh^{-1}x$

Solution

$$y'=2x\sinh^{-1}x+x^2\frac{1}{\sqrt{x^2+1}}$$

Example 11

The velocity v (in *m/s*) of an idealized surface wave traveling on the ocean is modeled by the equation

$$v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi d}{\lambda}\right)}$$

Where $g = 9.8\ m/s^2$ is the acceleration due the gravity, λ is the wavelength measured in meters from crest to crest, and d is the depth of the undisturbed water, also measured in *meters*.

a) A sea kayaker observes several waves that pass beneneath her hayak, and she estimates that $\lambda = 12\ m$ and $v = 4\ m/s$. How deep is the water in which she is kayaking?

b) The deep-water equation for wave velocity is $v = \sqrt{\frac{g\lambda}{2\pi}}$, which approximates the velocity formula given above. Waves are said to be in deep water if the depth-to-wavelength ration d/λ is greater than $\frac{1}{2}$. Explain why $v = \sqrt{\frac{g\lambda}{2\pi}}$ is a good approximation when $\frac{d}{\lambda} > \frac{1}{2}$.

Solution

a) ***Given***: $\lambda = 12\ m$, $v = 4\ m/s$

$$4 = \sqrt{\frac{9.8(12)}{2\pi}\tanh\left(\frac{2\pi d}{12}\right)}$$

$$16 = \frac{117.6}{2\pi}\tanh\left(\frac{\pi d}{6}\right)$$

$$\frac{32\pi}{117.6} = \tanh\left(\frac{\pi d}{6}\right)$$

$$\frac{\pi d}{6} = \tanh^{-1}\left(\frac{32\pi}{117.6}\right)$$

$$d = \frac{6}{\pi}\tanh^{-1}\left(\frac{32\pi}{117.6}\right)$$

$$\approx 2.4\ \ m$$

Therefore, the kayaker is in water that is about 2.4 *m* deep.

b) Since $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x > 0$, then $\tanh x$ is an incrasing function whose values approaches 1 as $x \to \infty$.

Also when $\frac{d}{\lambda} = \frac{1}{2}$, $\tanh\left(\frac{2\pi d}{\lambda}\right) = \tanh \pi \approx 0.996$, which is nearly equal to 1.

These facts imply that whenever $\frac{d}{\lambda} > \frac{1}{2}$, we can replace $\tanh\left(\frac{2\pi d}{\lambda}\right)$ with 1 in the velocity formula, resulting in the deep-water velocity function $v = \sqrt{\frac{g\lambda}{2\pi}}$

Inverse Hyperbolic Functions	
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$	$(-\infty, -1)\cup(1, \infty)$
$\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0)\cup(0, \infty)$

Exercises *Section* 2.7 – Derivatives of Inverse Functions

$(\mathbf{1-76})$ Find the derivative of:

1. $y=\cos^{-1}\left(\frac{1}{x}\right)$

2. $y=\sin^{-1}\sqrt{2}\,t$

3. $y=\sec^{-1}(5s)$

4. $y=\cot^{-1}\sqrt{t-1}$

5. $y=\ln\left(\tan^{-1}x\right)$

6. $y=\tan^{-1}(\ln x)$

7. $y=\csc^{-1}\left(e^{t}\right)$

8. $y=x\sqrt{1-x^2}+\cos^{-1}x$

9. $y=\ln\left(x^2+4\right)-x\tan^{-1}\left(\frac{x}{2}\right)$

10. $f(x)=\sin^{-1}\frac{1}{x}$

11. $f(x)=\sec^{-1}\frac{1}{x}$

12. $f(x)=\cos^{-1}\left(\sqrt{x}\right)$

13. $y=\sqrt{\csc^{-1}x}$

14. $y=\left(1-\tan^{-1}x\right)^3$

15. $y=\dfrac{1}{\cot^{-1}x}$

16. $y=\csc^{-1}(-x)$

17. $y=\tan^{-1}\sqrt{9-x^2}$

18. $y=x^2\sec^{-1}\left(x^2\right)$

19. $y=x\ \sin^{-1}\left(\sqrt{x}\right)$

20. $y=\dfrac{\cos^{-1}x}{x}$

21. $y=\dfrac{\sin^{-1}\sqrt{x}}{x}$

22. $y=\tan^{-1}x+\cot^{-1}x$

23. $y=\sin^{-1}(\cos x)$

24. $y=\cos^{-1}(\sin x)$

25. $y=\sin\left(\cos^{-1}x\right)$

26. $y=\tan^{-1}(\tan x)$

27. $y=\tan^{-1}(\cot x)$

28. $y=x-\cos^{-1}(\cos x)$

29. $f(x)=\arccos(3x)$

30. $f(x)=\arcsin(2x)$

31. $f(x)=\arctan\left(e^{x}\right)$

32. $f(x)=\operatorname{arccot}(\ln x)$

33. $f(x)=\operatorname{arcsec}\left(e^{x^2}\right)$

34. $f(x)=\operatorname{arccsc}\left(e^{4x}\right)$

35. $f(x)=\dfrac{\arccos(2x)}{x+1}$

36. $f(x)=e^{3x}\arccos(5x)$

37. $y=\ln\left(t^2+2\right)-\frac{1}{2}\arctan\left(\frac{1}{2}t\right)$

38. $y=6\arcsin\frac{t}{4}-\frac{1}{2}t\ \sqrt{16-t^2}$

39. $y=\arctan t+\dfrac{t}{1+t^2}$

40. $y=\arctan x-\dfrac{1}{x^2+4}$

41. $y=36\arctan\frac{x}{6}-x\sqrt{36-x^2}$

42. $y=\sin^{-1}\left(1-x^2\right)+e^{x}\ \ln\left(x^2+1\right)$

43. $y=x\sec^{-1}\left(x^2\right)$

44. $y = x\tan^{-1}(x)$

45. $y = \cosh^{-1} 2\sqrt{x+1}$

46. $y = \left(\theta^2 + 2\theta\right)\tanh^{-1}(\theta+1)$

47. $y = (1-t)\coth^{-1}\sqrt{t}$

48. $y = \ln x + \sqrt{1-x^2}\ \operatorname{sech}^{-1} x$

49. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

50. $y = \cosh^{-1}(\sec x)$

51. $f(t) = 2\tanh^{-1}\sqrt{t}$

52. $f(x) = \sinh^{-1} x^2$

53. $f(x) = \csc h^{-1}\left(\frac{2}{x}\right)$

54. $f(x) = x\sinh^{-1} x - \sqrt{x^2+1}$

55. $f(x) = \sinh^{-1}(\tan x)$

56. $y = \sinh^{-1}\sqrt{x}$

57. $y = (1-t)\tanh^{-1} t$

58. $f(x) = \arctan(\sinh x)$

59. $f(x) = \tanh^{-1}(\cos x)$

60. $f(x) = \sinh^{-1}(\cosh x)$

61. $f(x) = \cosh^{-1}\left(x^3\right)$

62. $f(x) = e^{\sinh^{-1} x}$

63. $f(x) = \ln\left(\tanh^{-1}(x)\right)$

64. $f(x) = \left(\tanh^{-1} x\right)^2$

65. $f(x) = \left(\coth^{-1} x\right)^3$

66. $y = \frac{1}{16}\arctan\frac{x}{2} + \frac{1}{8}\frac{x}{x^2+4}$

67. $y = \frac{1}{7}\sqrt{x^2-49} - \operatorname{arcsec}\frac{x}{7}$

68. $y = x\ln\left(2x^2+1\right) - 2x + \frac{2}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\,x\right)$

69. $y = \ln\left(\frac{27}{4} + \left(x+\frac{3}{2}\right)^2\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+3}{3\sqrt{3}}\right)$

70. $y = \frac{10\sqrt{x^2-100}}{x^2} + \arctan\frac{\sqrt{x^2-100}}{10}$

71. $y = \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} + \frac{1}{4}\sqrt{x}\sqrt{1-x}$

72. $y = -\frac{1}{2}\arcsin(1-x) - \frac{1}{2}(1-x)\sqrt{2x-x^2} - \frac{1}{3}\left(\sqrt{2x-x^2}\right)^3$

73. $y = \frac{1}{8}a^4\arcsin\frac{x}{a} - \frac{1}{8}a^2x\sqrt{a^2-x^2} + \frac{1}{4}x^3\sqrt{a^2-x^2}$

74. $y = \frac{1}{2}\arcsin\left(e^x\right) + \frac{1}{2}e^x\sqrt{1-e^{2x}}$

75. $y = \ln\left(x^2+1\right) + \arctan x + \frac{x}{x^2+1} + 2$

76. $y = \left(2x^2-1\right)\arcsin x + \frac{1}{x\sqrt{1-x^2}}$

(77 – 80) Find the value of:

77. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

78. $\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

79. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

80. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

(81 – 83) Find the limit.

81. $\lim\limits_{x\to -1^+} \cos^{-1} x$

82. $\lim\limits_{x\to -\infty} \tan^{-1} x$

83. $\lim\limits_{x\to \infty} \csc^{-1} x$

(84 – 87) Find the derivative.

84. $\frac{d}{dx}\left(x\sec^{-1} x\right)\Big|_{x=\frac{2}{\sqrt{3}}}$

85. $\frac{d}{dx}\left(\cot^{-1}(\ln x)\right)\Big|_{x=e}$

86. $\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)\Big|_{x=0}$

87. $\frac{d}{dx}\left(\frac{\sin^{-1} x}{x}\right)\Big|_{x=\frac{1}{2}}$

88. Show that $\frac{d}{dx}\arctan(\tanh x) = \operatorname{sech} 2x$.

(89 – 94) Prove the derivative.

89. $\frac{d}{dx}(\arctan u) = \frac{u'}{1+u^2}$

90. $\frac{d}{dx}(\operatorname{arccsc} u) = \frac{-u'}{|u|\sqrt{u^2-1}}$

91. $\frac{d}{dx}(\operatorname{arcsec} u) = \frac{u'}{|u|\sqrt{u^2-1}}$

92. $\frac{d}{dx}(\operatorname{arccot} u) = -\frac{u'}{1+u^2}$

93. $\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{u^2+1}}$

94. $\frac{d}{dx}\left(\cosh^{-1} u\right) = \frac{u'}{\sqrt{u^2-1}}$

95. Derive the formula $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$ for all real x. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

96. Derive the formula $\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2-1}\right)$ for all real x.

97. Find the angle α

98. A person is holding a rope, the length rope is 20 feet, that is tied to a boat. As the person walks along the dock, the boat travels along a ***tractrix***, given by the equation

$$y = 20\,\mathrm{sech}^{-1}\frac{x}{20} - \sqrt{20^2 - x^2}$$

Find the distance the person must walk to bring the boat 5 *feet* from the dock.

Section 2.8 – Related Rates

2.8–1 *Definition*

The ***instantaneous rate of change*** of f with respect to x at x_0 is the derivative:

$$f'(x_0) = \lim_{x \to \infty} \frac{f(x_0 + h) - f(x)}{h}$$

Provided the limit exists.

Example 1

The area **A** of the circle is related to its diameter by the equation $A = \frac{\pi}{4} D^2$.

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}$$

When $D = 10\ m$, the area is changing with respect to the diameter at the rate of

$$\frac{dA}{dD} = \frac{\pi(10)}{2} \approx 15.71\ m^2 / m$$

2.8–2 *Motion* along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t)$$

The ***displacement*** of the object over the time interval from t to $t + \Delta t$ is:

$$\Delta s = f(t + \Delta t) - f(t)$$

And the ***average velocity*** of the object over that time interval is:

$$v_{avg} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition

Speed is the absolute value of velocity $\quad speed = |v(t)| = \left|\frac{ds}{dt}\right|$

s increasing: positive slope moving upward

s decreasing: negative slope moving downward

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time *t* is $s = f(t)$, then the body's acceleration at time *t* is:

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Jerk is the derivative of acceleration with the respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

When a ride in a car is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

Example 2

The free fall of a heavy ball bearing released from rest at time $t = 0$ *sec*.

a) How many meters does the ball fall in the first 2 *sec*?
b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

a) The metric free-fall equation is $s = 4.9t^2$.

During the first 2 *sec*:

$$s(2) = 4.9(2)^2$$
$$= 19.6\ m$$

b) At any time, the velocity is:

$$v = \frac{ds}{dt}$$
$$= \frac{d}{dt}\left(4.9t^2\right)$$
$$= 9.8t$$

At $t = 2$,

Velocity: $v = 9.8(2)$

$$= 19.6\ m/\sec$$

$Speed = |v|$ $= 19.6\ \ m/\sec$

Acceleration:

$$a(t) = v'(t)$$
$$= 9.8\ m/\sec^2$$

Example 3

A dynamic blast blows a heavy rock straight up with a launch velocity of 160 *ft/sec* (about 109 *mph*). It reaches a height of $s(t) = 160t - 16t^2$ after *t sec*.

a) How high does the rock go?
b) What are the velocity and speed of the rock when it is 256 *ft* above the ground on the way up? On the way down?
c) What is the acceleration of the rock at any time *t* during its flight (after the blast)?
d) When does the rock hit the ground again?

Solution

a) At any time *t* during the rock's motion, its velocity is

$$v(t) = 160 - 32t \qquad\qquad v(t) = \frac{ds}{dt}$$

The velocity is zero when it reaches maximum height:

$$v = 160 - 32t = 0$$

$$160 = 32t$$

$$t = \frac{160}{32}$$

$$= 5 \text{ sec}$$

The rock's height at $t = 5$ sec is

$$s(t=5) = 160(5) - 16(5)^2$$

$$= 400 \text{ ft}$$

b) $s = 160t - 16t^2 = 256$

$$-16t^2 + 160t - 256 = 0$$

$$t^2 - 10t + 16 = 0$$

$$t = 2 \text{ sec}, \quad t = 8 \text{ sec}$$

$t = 2\text{sec}$

$$v = 160 - 32(2)$$

$$= 96 \text{ ft / sec}$$

$t = 8\text{sec}$

$$v = 160 - 32(8)$$

$$= -96 \text{ ft / sec}$$

The rock's speed is 96 *ft/sec*.

Since $v(t=2)>0$, the rock is moving *upward* and s is increasing.

$v(t=8)<0$, the rock is moving *downward* and s is decreasing.

c) Acceleration at any time is: $a = v' = -32\ ft/\sec^2$

d) $s = 160t - 16t^2 = 0$

$t(160-16t)=0$

$t = 0, 10$

At t = 0, the blast occurred, and the rock was thrown upward, it took 10 *sec* to return to ground.

2.8–3 Related Rates: Applications

The problem of finding a rate of change from other known rates of change is called a ***related rates problem***.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

Example 4

Water runs into a conical tank at the rate of 9 $ft^3/\min$. The tank stands point down and has a height of 10 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

Solution

V = volume $\left(ft^3\right)$ of the water in the tank at time t (*min*)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

Given: $\begin{cases} \dfrac{dV}{dt} = 9\ \dfrac{ft^3}{min} & y = 6\,ft \\ h = 10\,ft & r = 5\,ft \end{cases}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi x^2 y$$

From the triangles:

$$\frac{x}{y} = \frac{5}{10}$$

$$x = \frac{y}{2}$$

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$$

$$= \frac{1}{3}\pi\left(\frac{y^2}{4}\right)y$$

$$= \frac{1}{12}\pi y^3$$

The derivative in function of time:

$$\frac{dV}{dt} = \frac{1}{12}\pi\left(3y^2\frac{dy}{dt}\right)$$

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{(9)(4)}{36\pi} = \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi}\ ft/min$$

$$\approx 0.3183\ ft/min$$

The water level is rising at about 0.32 ***ft/min***.

2.8–4 Related Rates Problem Strategy

1. Draw a picture and name the variables and constants. (Use *t* for time).
2. Write down the given information (numerical).
3. Write down what you are asked to find.
4. Write an equation that relates the variables.
5. Differentiate with respect to *t*.
6. Evaluate.

Example 5

A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?

Solution

θ = the angle in *rad*.
y = the height in *feet* of the balloon

Given: $\frac{d\theta}{dt} = 0.14\ rad/\min \quad when \quad \theta = \frac{\pi}{4}$

$distance = 500\ ft$

$$\tan\theta = \frac{y}{500}$$

$$y = 500\tan\theta$$

$$\frac{dy}{dt} = 500\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$= 500\left(\sec^2\frac{\pi}{4}\right)(0.14)$$

$$= 140$$

The balloon is rising at the rate of 140 *ft/min*.

Example 6

A police cruiser, approaching a right-angle intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 *mi* north of the intersection and the car is 0.8 *mi* to the east, the police determined with radar that the distance between them and the car is increasing at 20 *mph*. If the cruiser is moving at 60 *mph* at the instant of measurement.

what is the speed of the car?

Solution

x = position of car at time t.
y = position of cruiser at time t.
s = distance between car and cruiser at time t.

Given: $\begin{cases} x = 0.8\ mi & y = 0.6\ mi \\ \dfrac{ds}{dt} = 20\ mph & \dfrac{dy}{dt} = -60\ mph \end{cases}$

Using the Pythagorean Theorem to get the distance:

$$s^2 = x^2 + y^2$$

$$s = \sqrt{x^2 + y^2}$$

$$= \sqrt{0.8^2 + 0.6^2}$$

$$= 1$$

$$s^2 = x^2 + y^2$$

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$s\frac{ds}{dt} - y\frac{dy}{dt} = x\frac{dx}{dt}$$

$$x\frac{dx}{dt} = s\frac{ds}{dt} - y\frac{dy}{dt}$$

$$0.8\frac{dx}{dt} = (1)(20) - (0.6)(-60)$$

$$\frac{dx}{dt} = \frac{20 + 36}{0.8}$$

$$= 70\ \textbf{mph}$$

The car's speed is 70 *mph*.

Example 7

A particle moves clockwise at a constant rate along a circle of radius 10 *feet* centered at the origin. The particles initial position is (0, 10) on the *y*-axis and its final destination is the point (10, 0) on the *x*-axis. Once the particle is in motion, the tangent line at *P* intersects the *x*-axis at a point *Q* (which moves over time).

If it takes the particle 30 *sec* to travel from start to finish, how fast is the point *Q* moving along the *x*-axis when it is 20 *feet* from the center of the circle?

Solution

Since the particle travels 30 *sec* from start to finish of angle $90° = \frac{\pi}{2}$, the particle is traveling along the circle at a constant rate of:

$$\frac{\frac{\pi}{2}\ rad}{30\ \sec} \cdot \frac{60\ \sec}{1\ \min} = \pi\ rad\ /\ \min$$

That implies $\frac{d\theta}{dt} = -\pi$

$$x = 20\ \ ft \quad and \quad \frac{d\theta}{dt} = -\pi\ \ rad\ /\ \min$$

$$\cos\theta = \frac{10}{x}$$

$$x = \frac{10}{\cos\theta}$$

$$= 10\sec\theta$$

$$\frac{dx}{dt} = \frac{d}{dt}(10\sec\theta)$$

$$= 10\ \sec\theta\ \tan\theta\ \frac{d\theta}{dt}$$

$$= 10\ \sec\theta\ \tan\theta\ (-\pi)$$

$$= -10\pi\ \sec\theta\ \tan\theta$$

$$20 = 10\sec\theta$$

$$\sec\theta = 2$$

$$\tan\theta = \sqrt{\sec^2\theta - 1}$$

$$= \sqrt{4-1}$$

$$= \sqrt{3}$$

$$\frac{dx}{dt} = -10\pi\ \sec\theta\ \tan\theta$$

$$= -10\pi\ (2)\left(\sqrt{3}\right)$$

$$= -20\pi\sqrt{3} \ \ ft / \min$$

The point Q is moving towards the origin at the speed of $20\sqrt{3}\pi \approx 108.8 \ ft / \min$.

Example 8

A jet airliner is flying at a constant altitude of 12,000 *feet* above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise ***ccw***) at the rate of $\frac{2}{3} \ deg / sec$ in order to keep the aircraft within its direct line of sight?

Solution

From the triangle:

$$\tan\theta = \frac{12,000}{x}$$

$$x = \frac{12,000}{\tan\theta}$$

$$= \frac{12,000}{5,280}\cot\theta \quad mi$$

$$\frac{dx}{dt} = -\frac{12,000}{5,280}\csc^2\theta\,\frac{d\theta}{dt}$$

$$= -\frac{12,000}{5,280}\csc^2\left(\frac{\pi}{6}\right)\cdot\left(\frac{2}{3}\right)\frac{deg}{sec}\ \frac{\pi}{180°}\ \frac{3600 \ sec}{1 \ hr}$$

$$\approx -380 \ \ mi / hr$$

The negative appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 ***mi/hr*** when first detected by the radar.

Example 9

A rope is running through a pulley at P and bearing a weight W at one end. The other end is held 5 *feet* above the ground in the hand M of a worker. Suppose the pulley is 25 *feet* above ground, the rope is 45 *feet* long, and the worker is walking rapidly away from the vertical line PW at the rate of 6 *ft/sec*. How fast is the weight being raised when the worker's hand is 21 *feet* away from PW?

Solution

Given: $\quad when \;\; x = 21 \quad \rightarrow \quad \dfrac{dx}{dt} = 6$

$$45 = z + 20 - h$$

$$z = 45 - 20 + h$$

$$= 25 + h$$

$$z^2 = 20^2 + x^2$$

$$(25+h)^2 = 20^2 + x^2$$

$$\frac{d}{dt}\left((25+h)^2\right) = \frac{d}{dt}\left(20^2 + x^2\right)$$

$$2(25+h)\frac{dh}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dh}{dt} = \frac{x}{25+h}\frac{dx}{dt}$$

$when \;\; x = 21$

$$(25+h)^2 = 20^2 + x^2$$

$$= 20^2 + 21^2$$

$$= 29^2$$

$$25 + h = 29$$

$$\frac{dh}{dt} = \frac{21}{29}(6)$$

$$\approx 4.3 \; ft \,/\, \text{sec}$$

As the rate 4.3 ***ft/sec*** at which the weight is being raised when x = 21 *feet*.

Exercises *Section* 2.8 – Related Rates

1. If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

2. If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

3. The position $s(t) = t^2 - 3t + 2, \quad 0 \le t \le 2$ of a body moving on a coordinate line, with *s* in meters and *t* in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

4. The position $s(t) = \frac{25}{t+5}, \quad -4 \le t \le 0$ of a body moving on a coordinate line, with *s* in meters and *t* in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

5. At time *t*, the position of a body moving along the *s*-axis is $s = t^3 - 6t^2 + 9t$ ***m***.

a) Find the body's acceleration each time the velocity is zero.
b) Find the body's speed each time the acceleration is zero.
c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

6. A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/sec* (about 86 *km/h*) reaches a height of $s(t) = 24t - 0.8t^2$ *m* in *t sec*.

a) Find the rock's velocity and acceleration at time *t*. (The acceleration in this case is the acceleration of gravity on the moon.)
b) How long does it take the rock to reach its highest point?
c) How high does the rock go?
d) How long does it take the rock to reach half its maximum height?
e) How long is the rock aloft?

7. Had Galileo dropped a cannonball from the Tower of Pisa, 179 *feet* above the ground, the ball's height above the ground *t sec* into the fall would have been $s = 179 - 16t^2$.

a) What would have been the ball's velocity, speed, and acceleration at time *t*?
b) About how long would it have taken the ball to hit the ground?
c) What would have been the ball's velocity at the moment of impact?

8. A toy rocket fired straight up into the air has height $s(t) = 160t - 16t^2$ *feet* after t seconds.

a) What is the rocket's initial velocity (when $t = 0$)?
b) What is the acceleration when $t = 3$?
c) At what time will the rocket hit the ground?
d) At what velocity will the rocket be traveling just as it smashes into the ground?

9. A helicopter is rising straight up in the air. Its distance from the ground t seconds after takeoff is $s(t) = t^2 + t$ *feet*

a) How long will it take for the helicopter to rise 20 *feet* ?
b) Find the velocity and the acceleration of the helicopter when it is 20 *feet* above the ground.

10. The position of a particle moving on a line is given by $s(t) = 2t^3 - 21t^2 + 60t, \quad t \geq 0$, where t is measured in *seconds* and s in *feet*.

a) What is the velocity after 3 *seconds* and after 6 *seconds*?
b) When the particle moving in the positive direction?
c) Find the total distance traveled by the particle during the first 7 *seconds*.

11. A small probe is launched vertically from the ground. After it reaches its high point, a parachute deploys and the probe descends to Earth. The height of the probe the ground is

$$s(t) = \frac{300t - 50t^2}{t^3 + 2} \quad for \quad 0 \leq t \leq 6$$

a) Graph the height function and describe the motion of the probe.
b) Find the velocity of the probe.
c) Graph the velocity function and determine the approximate time at which the velocity is a maximum.

12. Suppose the cost of producing x lawn mowers is $C(x) = -0.02x^2 + 400x + 5000$

a) Determine the average and marginal costs for $x = 3000$ lawn mowers.
b) Interpret the meaning of your results in part (*a*)

13. Suppose a company produces fly rods. Assume $C(x) = -0.0001x^3 + 0.05x^2 + 60x + 800$ represents the cost of making x fly rods.

a) Determine the average and marginal costs for $x = 400$ fly rods.
b) Interpret the meaning of your results in part (*a*)

14. Suppose $p(t) = -1.7t^3 + 72t^2 + 7200t + 80{,}000$ is the population of a city t years after 1950.

a) Determine the average rate of growth of the city from 1950 to 2000.
b) What was the rate of growth of the city in 1990?

15. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per *hour*, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

16. A cube's surface area increases at the rate of 72 $in^2/\sec$. At what rate is the cube's volume changing when the edge length is $x = 3$ *in*?

17. The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if $\boldsymbol{r}$ is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if $\boldsymbol{h}$ is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither $\boldsymbol{r}$ nor $\boldsymbol{h}$ is constant?

18. The voltage $\boldsymbol{V}$ (volts), current $\boldsymbol{I}$ (amperes), and resistance $\boldsymbol{R}$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 *volt/sec* while I is decreasing at the rate of $\frac{1}{3}\,amp/\sec$. Let t denote time in seconds.

a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

19. Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

20. A 13-*ft* ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *ft* from the house, the base is moving at the rate of 5 *ft/sec*.

a) How fast is the top of the ladder sliding down the wall then?

b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?

21. A 13-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 *ft/s*. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 *ft* from the wall?

22. A 12-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 *ft/s*. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

23. A swimming pool is 50 *m* long and 20 *m* wide. Its length decreases linearly along the length from 3 *m* to 1 *m*. It is initially empty and is filed at a rate of 1 m^3 / min .

a) How fast is the water level rising 250 *min* after the filling begins?

b) How long will it take to fill the pool?

24. An inverted conical water tank with a height of 12 *feet* and a radius of 6 *feet* is drained through a hole in the vertex at a rate of 2 ft^3 / sec. What is the rate of change of the water depth when the water depth is 3 *feet*?

25. Water runs into a conical tank at the rate of 6 ft^3/min. The tank stands point down and has a height of 20 *feet* and a base radius of 8 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

26. Water runs into a conical tank at the rate of 5 ft^3/min. The tank stands point down and has a height of 10 *feet* and a base radius of 4 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

27. Water runs into a conical tank at the rate of 5 ft^3/min. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?

28. Water runs into a conical tank at the rate of $4\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 5 *feet* deep?

29. Water runs into a conical tank at the rate of $2\ ft^3/min$. The tank stands point down and has a height of 10 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?

30. Water runs into a conical tank at the rate of $3\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

31. Water runs into a conical tank at the rate of $6\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 12 *feet* deep?

32. A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of $3\ m^3/\min$.

(Hint: The volume of a cap of thickness *h* sliced from a sphere of radius *r* is $\frac{\pi h^2(3r-h)}{3}$).

a) How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?

b) What is the rate of change of the surface area of the water when the water is 5 *m* deep?

33. A fisherman hooks a trout and reels in his line at 4 *in/sec*. Assume the trip of the fishing rod is 12 *ft* above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal speed of the fish when it is 20 *ft* from the fisherman.

34. Water is flowing at the rate of $6\ m^3/\min$ from a reservoir shaped like a hemispherical bowl of radius 13 *m*. Answer the following questions, given that the volume of water in a hemispherical bowl of radius *R* is $V=\frac{\pi}{3}y^2(3R-y)$ when the water is y meters deep.

a) At what rate the water level changing when the water is 8 *m* deep?

b) What is the radius *r* of the water's surface when the water is *y m* deep?

c) At what rate is the radius *r* changing when the water is 8 *m* deep?

35. A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3/min$. How fast is the balloon's radius increasing at the instant the radius is 5 *feet*? How fast the surface area increasing?

36. An observer stands 300 *feet* from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 *ft/sec*. what is the rate of change of the angle of elevation of the balloon when it is 400 *feet* from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

37. A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *feet* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

a) How fast is the boat approaching the dock when 10 *feet* of rope are out?
b) At what rate is the angle θ changing at this instant?

38. A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?

39. A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of $0.2\ rad/\text{min}$. How fast is the balloon rising at that moment?

40. A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of $0.2\ rad/\text{min}$. How fast is the balloon rising at that moment?

41. A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *feet* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 *sec* later?

42. The figure shows a boat 1 *km* offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $\frac{d\theta}{dt} = -0.6\ rad/\sec$.

a) How fast is the light moving along the shore when it reaches point *A*?
b) How many revolutions per minute is 0.3 *rad/sec*?

43. You are videotaping a race from a stand 132 *feet* from the track, following a car that is moving at 180 *mi/h* (264 *ft/sec*).
How fast will your camera angle θ be changing when the car is right in front of you?
A half second later?

44. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3 / min.

a) How fast is the level in the pot rising when the coffee in the cone is 5 *in*. deep?

b) How fast is the level in the cone falling then?

45. The coordinates of a particle in the metric *xy*–plane are differentiable functions of time *t* with $\frac{dx}{dt} = -1\ m/\sec$ and $\frac{dy}{dt} = -5\ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

46. A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its *x*-coordinate (measure in meters) increases at a steady 10 *m/sec*. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3m$?

47. To find the height of a lamppost, you stand a 6 *feet* pole 20 *feet* from the lamp and measure the length *a* of its shadow, finding it to be 15 *feet*, give or take an inch. Calculate the height of the lamppost using the value of $a = 15$ and estimate the possible error in the result.

48. A light shine from the top of a pole 50 *feet* high. A ball is dropped from the same height from a point 30 *feet* away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later? (Assume the ball falls a distance $s = 16t^2$ *foot* in t sec.)

49. A lamppost stands 75 *feet* tall. An angle θ is formed when wires of various lengths of x *feet* are attached from the ground to the top of the pole.

Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 100 *feet* is attached.

50. A building that is 225 *feet* tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building on the tip of the shadow.

Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when 272 *feet*

51. On a morning of a day when the sun will pass directly overhead, the shadow of an 80–*feet* building on level ground is 60 *feet* long.
At the moment, in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27° / *min*. At what rate is the shadow decreasing?

52. A spherical iron ball 8 *in*. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3 / min , how fast is the thickness of the ice decreasing when it is 2 *in*. thick? How fast is the outer surface area of ice decreasing?

53. A baseball diamond is a square 90 *feet* on a side. A player runs from first base to second at a rate of 16 *ft/sec*.

a) At what rate is the player's distance from third base changing when the player is 30 *ft* from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 *ft/sec*. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

54. Runners stand at first and second base in a baseball game. At the moment a ball is hit the runner at first base runs to second base at 18 *ft/s*; simultaneously the runner on second runs to third base at 20 *ft/s*. How fast is the distance between the runners changing 1 *sec* after the ball is hit?

(***Hint***: The distance between consecutive bases I 90 *feet* and the bases lie at the corners of a square.)

55. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius *r* of the outer ripple is increasing at a constant rate of 1 *foot* per *second*.

When the radius is 4 *feet*, at what rate is the total area ***A*** of the disturbed water changing?

56. Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of $2 \ ft/\sec$. How fast is the area of the spill increasing when the radius of the spill is 60 *ft*?

57. The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find $\frac{dy}{dt}$ when $x = 1$, given $\frac{dx}{dt} = 2$ when $x = 1$.

58. Air is being pumped into a spherical balloon at a rate of 4.5 $ft^3 / \min$.

Find the rate of change of the radius when the radius is 2 *feet*.

59. An Airplane is flying on a flight path that will take it directly over a radar tracking station. The distance s is decreasing at a rate of 400 *mph* when $s = 10$ *mi*. What is the speed of the plane?

60. Find the rate of change in the angle of elevation of the camera at 10 *seconds* after lift-off.

61. In the engine, a 7–inch connecting rod is fastened to a crank of radius 3 *inches*, the crankshaft rotates counterclockwise at a constant rate of 200 *revolutions per minute*.

Find the velocity of the piston when $\theta = \frac{\pi}{3}$.

62. A swimming pool is 12 *meters* long, 6 *meters* wide, 1 *meter* deep at the shallow end, and 3 *meters* deep at the deep end. Water is being pumped into the pool at $\frac{1}{4}\ m^3/\text{min}$, and there is 1 *meter* of water at the deep end.

a) What percent of the pool is filled?
b) At what rate is the water level rising?

63. A trough is 12 *feet* long and 3 *feet* across the top. Its ends are isosceles triangles with altitudes of 3 *feet*.

a) Water is being pumped into the trough at $2\ ft^3/\text{min}$. How fast is the water level rising when the depth h is 1 *foot*?
b) The water is rising at a rate of $\frac{3}{8}\ in/\text{min}$ when $h = 2$. Determine the rate at which water is being pumped into the trough.

64. A ladder 25 *feet* long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 *feet* per *second*.

a) How fast is the top of the ladder moving down the wall when its base is 7 *feet*, 15 *feet*, and 24 *feet* from the wall?
b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 *feet* from the wall.
c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 *feet* from the wall.

65. A construction worker pulls a five-*meter* plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of $0.15\ m/\sec$. How fast is the end of the plank sliding along the ground when it is 2.5 *m* from the wall of the building?

66. A winch at the top of a 12-*meter* building pulls a pipe of the same length to a vertical position. The winch pulls in rope at a rate of $-0.2\ m/\sec$. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$

67. A boat is pulled into a dock by means of a winch 12 *feet* above the deck of the boat.

a) The winch pulls in rope at a rate of 4 *feet* per *second*. Determine the speed of the boat when there is 13 *feet* of rope out. What happens to the speed of the boat as it gets closer to the dock?

b) Suppose the boat is moving at a constant rate of 4 *feet* per *second*. Determine the speed at which the winch pulls in rope when there is a total of 13 *feet* of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

68. An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 225 *miles* from the point moving at 450 *mph* . The other plane is 300 *miles* from the point moving at 600 *mph* .

a) At what rate is the distance between the planes decreasing?

b) How much time does the air traffic controller have to get one of the planes on a different flight path?

69. Two airplanes are flying in the air at the same height; airplane *A* is flying at 250 *mph* and airplane *B* is flying north at 300 *mph* . They are both heading to the same airport, located 30 *miles* east of airplane *A* ands 40 *miles* north of airplane *B*, at what rate is the distance between the airplanes changing?

70. Two cars are driving along parallel freeways that are 6 miles apart, one heading east and the other heading west. Each car drives a constant 65 *mph* Find the rate at which the distance between the cars is changing when they are 15 miles apart, heading toward each other.

71. Two people are riding bikes to a store at the same distance, and $\frac{1}{2}$ mile away from each other. The store for the first bike is east with 12 mph speed and north for the second biker with speed of 16 mph. Find the rate of change at which the distance between the bikes between the bikes is $\frac{1}{4}$ mile.

72. An airplane is flying at an altitude of 5 *miles* and passes directly over a radar antenna. When the plane is 10 *miles* away $(s=10)$, the radar detects that the distance s is changing at a rate of 240 mph. What is the speed of the plane?

73. A man 6 *feet* tall walks at a rate of 5 *feet* per *second* away from a light that is 16 *feet* above the ground.

a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?
b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

74. A man 6 *feet* tall walks at a rate of 5 *feet* per *second toward* a light that is 20 *feet* above the ground.

a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?

b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

75. A man 6 *feet* tall walks at a rate of 5 ft/sec toward a streetlight that is 30 *feet* high. The man's 3-*foot*-tall child follows at the same speed, but 10 *feet* behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

a) Suppose the man is 90 *feet* from the streetlight. Show that the man's shadow extends beyond the child's shadow.

b) Suppose the man is 60 *feet* from the streetlight. Show that the child's shadow extends beyond the man's shadow.

c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

d) Determine how fast the tip of the man's shadow is moving as a function of x, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

76. A ball is dropped from a height of 20 *m*, 12 *m* away from the top of a 20-*meter* lamppost. The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 *second* after the ball is released?

77. The endpoints of a movable rod of length 1 *meter* have coordinates $(x,\ 0)$ and $(0,\ y)$. The position of the end on the *x-axis* is:

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6} \qquad \text{where } t \text{ is the time in } \textit{seconds}.$$

a) Find the time of one complete cycle of the rod.
b) What is the lowest point reached by the end of the rod on the *y-axis*?
c) Find the speed of the *y-axis* endpoint when the *x-axis* endpoints is $\left(\frac{1}{4},\ 0\right)$

78. Cars on a certain roadway travel on a circular arc of radius *r*. in order not to rely on friction alone to overcome to centrifugal force, the road is banked at an angle of magnitude θ from the horizontal. The banking angle must satisfy the equation $rg\tan\theta = v^2$, where *v* is the velocity of the cars and $g = 32\ ft/\sec^2$ is the acceleration due to gravity. Find the relationship between the related rates $\frac{dv}{dt}$ and $\frac{d\theta}{dt}$.

79. A fish is reeled in at a rate of 1 $ft/\sec$ from a point 10 *feet* above the water. At what rate is the angle θ between the line and the water changing when there is a total of 25 *feet* of line from the end of the rod to the water?

80. An airplane flies at an altitude of 5 *miles* toward a point directly over an observer. The speed of the plane is 600 mph. Find the rates at which the angle of elevation θ is changing when the angle is

a) $\theta = 30°$ *b)* $\theta = 60°$ *c)* $\theta = 75°$

81. A patrol car is parked 50 *feet* from a long warehouse. The revolving light on top of the car turns at a rate of 30 *revolutions* per *minute*. How fast is the light beam moving along the wall when the beam makes angles of

a) $\theta = 30°$ *b)* $\theta = 60°$ *c)* $\theta = 70°$

With the perpendicular line from the light to the wall?

82. A wheel of radius 30 *cm* revolves at a rate of 10 *revolutions* per *second*. A dot is painted at a point *P* on the rim of the wheel.

a) Find $\frac{dx}{dt}$ as a function of θ.

b) Graph the function.

c) When is the absolute value of the rate of change of *x* greatest?

d) When is it least?

e) Find $\frac{dx}{dt}$ when $\theta = 30°$ and $\theta = 60°$

83. A security camera is centered 50 *feet* above a 100-*foot* hallway. It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $\left|\frac{dx}{dt}\right| = 2 \ ft/\sec$

84. A rotating beacon is located 1 *km* off a straight shoreline. The beacon rotates at a rate of $3 \ rev/\min$. How fast (in km/hr) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ *km* down the shoreline?

85. A sandbag is dropped from a balloon at a height of 60 *m* when the angle of elevation to the sun is 30°. The position of the sandbag is $s(t) = 60 - 4.9t^2$

Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at height of 35 *m*.

86. The distance between the head of a piston and the end of a cylindrical chamber is given by $x(t) = \frac{8t}{t+1}$ *cm*, for $t \geq 0$ (measured in seconds). The radius of the cylinder is 4 *cm*.

a) Find the volume of the chamber, for $t \geq 0$.

b) Find the rate of change of the volume $V'(t)$ for $t \geq 0$.

c) Graph the derivative of the volume function. On what intervals is the volume increasing? Decreasing?

87. Two boats leave a dock at the same time. One boat travels south at 30 *mi/hr* and the other travels east at 40 *mi/hr*. after half an hour, how fast is the distance between the boats increasing?

88. A spherical balloon is inflated at a rate of 10 $cm^3 / \min$. At what rate is the diameter of the balloon increasing when the balloon has a diameter of 5 *cm*.

89. A rope is attached to the bottom of a hot-air balloon that is floating above the flat field. If the angle of the rope to the ground remains 65° and the rope is pulled in at 5 *ft/s*, how quickly is the elevation of the balloon changing?

90. Water flows into a conical tank at a rate of $2\ ft^3 / \min$. If the radius of the top of the tank is 4 *feet* and the height is 6 *feet*, determine how quickly the water level is rising when the water is 2 *feet* deep in the tank.

91. A jet flies horizontally 500 *feet* directly above a spectator at an air show at 450 *mi/hr*. How quickly is the angle of elevation (between the ground and the line from the spectator to the jet) changing 2 seconds later?

92. A man who's eyelevel is 6 *feet* above the ground walks toward a billboard at a rate of 2 *ft/s*. The bottom of the billboard is 10 *feet* above the ground, and it is 15 *feet* high. The man's viewing angle is the angle formed by the lines between the man's eyes and the top and bottom of the billboard. At what rate is the viewing angle changing when the man is 30 *feet* from the billboard?

93. A trough is shaped like a half cylinder with length 5 *m* and radius 1 *m*. The trough is full of water when a valve is opened, and water flows out of the bottom of the trough at a rate of $1.5\ m^3 / hr$.

(***Hint***: Area of the sector = $\frac{1}{2}r^2\theta$, *r* is the radius of a sector of the circle subtended by an angle of θ)

a) How fast is the water level changing when the water level is 0.5 *m* from the bottom of the through?

b) What is the rate of change of the surface area of the water when the water is 0.5 *m* deep?

94. A conical tank with an upper radius of 4 *m* and a height of 5 *m* drains into a cylindrical tank with a radius of 4 *m* and a height of 5 *m*. If the water level in the conical tank drops at a rate of 0.5 *m/min*, at what rate does the water in the cylindrical tank rise when the water level in the conical tank is 3 *m*? 1 *m*?

95. Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min). The smaller pool has a radius of 5 *m*, and the water level rises at a rate of 0.5 *m/min*. The larger pool has a radius of 8 *m*. How fast is the water level rising in the larger pool?

96. An observer is 20 *m* above the ground floor of a large hotel atrium looking at a glass enclosed elevator shaft that is 20 *m* horizontally from the observer. The angle of elevation of the elevator is the angle that the observer's line of sight makes with the horizontal (it may be positive or negative).

a) If the elevator rises at a rate of 5 *m/s*, what is the rate of change of the angle of the angle of elevation when the elevator is 10 *m* above the ground?

b) When the elevator is 40 *m* above the ground?

97. A camera is set up at the starting line of a drag race 50 *ft*. from a dragster at the starting line (camera 1). Two seconds after the start race, the dragster has traveled 100 *ft*. and the camera is turning at 0.75 *rad/s* while filming the dragster.

a) What is the speed of the dragster at this point?

b) A second camera (camera 2) filming the dragster is located on the starting line 100 *ft*. away from the dragster at the start of the race. How fast is this camera turning 2 *sec* after the start of the race?

98. A port and a radar station are 2 *mi* apart on a straight shore running east and west. A ship leaves the port at noon traveling northeast at a rate of 15 *mi/hr*. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the shore and the line between the radar station and the ship at 12:30 PM?

99. A ship leaves port traveling southwest at a rate of 12 *mi/hr*. At noon, the ship reaches its closest approach to a radar station, which is on the shore 1.5 *mi* from the port. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the radar station and the ship at 1:30 PM?

100. A lighthouse stands 500 *m* off a straight shore, the focused beam of its light revolving four times each minute. *P* is the point on shore closest to the lighthouse and *Q* is a point on the shore 200 *m* from *P*.

a) What is the speed of the beam along the shore when it strikes the point *Q*?
b) Describe how the speed of the beam along the shore varies with the distance between *P* and *Q*. (neglect the height of the lighthouse)

101. A boat leaves a port traveling due east at 12 *mi/hr*. At the same time, another boat leaves the same port traveling northeast at 15 *mi/hr*. The angle θ of the line between the boats is measured relative to due north. What is the rate of change of this angle 30 *min*. after the boats leave the port? 2 *hr*. after the boats leave the port?

102. An observer stands 20 *m* from the bottom of a 10-*m* tall Ferris wheel on a line that is perpendicular to the face of the Ferris wheel. The wheel revolves at a rate of π *rad/min* and the observer's line of sight with a specific seat on the wheel makes an angle θ with the ground. 40 *seconds* after that seat leaves the lowest point on the wheel, what is the rate of change of θ?
Assume the observer's eyes are level with the bottom of the wheel.

103. The bottom of a large theater screen is 3 *ft*. above your eye level and the top of the screen is 10 *ft*. above your eye level. Assume you walk away from the screen (perpendicular to the screen) at a rate of 3 *ft/s* while looking at the screen. What is the rate of change of the viewing angle $\boldsymbol{\theta}$ when you are 30 *ft*. from the wall on which the screen hangs, assuming the floor is flat?

104. A revolving searchlight, 100 *m* from the nearest point on the center line of a straight highway, casts a horizontal beam along a highway. The beam leaves the spotlight at an angle of $\frac{\pi}{16}$ *rad* and revolves at a rate $\frac{\pi}{16}$ *rad/s*. Let $\boldsymbol{w}$ be the width of the beam as it sweeps along the highway and $\boldsymbol{\theta}$ be the angle that the center of the beam makes with the perpendicular to the highway. What is the rate of change of $\boldsymbol{w}$ when $\theta = \frac{\pi}{3}$? *Neglect the height of the lighthouse.*

105. A piston is seated at the top of a cylindrical chamber with radius 5 *cm* when it starts moving into the chamber at a constant speed of 3 *cm/sec*. What is the rate of change of the volume of the cylinder when the piston is 2 *cm* from the base of the chamber?

106. You are watching a bird is located 40 *ft* above your head. The bird is flying horizontally at a rate 10 ft/s. How fast does the angle of elevation change when the horizontal distance between you and the bird is 80 *ft*?

107. A paricle is moving along the curvewhose equation is given by: $\dfrac{xy^3}{1+y^2}=\dfrac{8}{5}$

Assume that the *x*-coordinate is increasing at the rate of 6 *units* / sec wheen the particle position is at the point $(1,\ 2)$.

a) At what rate is the *y*-coordinate of the point changing at that instant?
b) Is the paritcle rising or falling at that instant?

108. An aircraft is flying at a constant altitude with a constant speed of 600 *mi/hr*. an antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point *P*.

At the instant the aircraft is 2 *miles* from the impact point *P* the missile is 4 *miles* from *P* and flying at 1,200 *mi/hr*. At that instant, how rapidly is the distance between missile and aircraft decreasing?

109. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant *w* and the horizontal tension at its lowest point is a vector of length *H*. If we choose a coordinate system for the plane of the cable in which the *x*-axis is horizontal, the force of gravity is straight down, the positive *y*-axis points straight up, and the lowest point of the cable lies at the point $y=\dfrac{H}{w}$ on the *y*-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine

$$y=\frac{H}{w}\cosh\left(\frac{H}{w}x\right)$$

Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

a) Let $P(x,y)$ denote an arbitrary point on the cable. The next accompanying displays the

tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\left(\frac{w}{H}x\right)$$

b) Using the result in part (a) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.

c) The length of arc AP is $s = \frac{1}{a}\sinh ax$, where $a = \frac{w}{H}$. Show that the coordinates of P may be expressed in terms of s as $x = \frac{1}{a}\sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}$

Summary

Derivative Definition: $f'(x) = \lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

$(uv)' = u'v + v'u$ $\quad \left(\dfrac{u}{v}\right)' = \dfrac{u'v - v'u}{v^2}$ $\quad (U^n)' = nU^{n-1}U'$ $\quad \left(\dfrac{1}{U^n}\right)' = -\dfrac{n \cdot U'}{U^{n+1}}$

$\left(\dfrac{1}{U}\right)' = -\dfrac{U'}{U^2}$ $\quad \left(\dfrac{1}{\sqrt{U}}\right)' = -\dfrac{U'}{2U^{3/2}}$ $\quad (\sqrt{U})' = \dfrac{U'}{2\sqrt{U}}$

$$\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$\left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2} \qquad \frac{d}{dx}\left(\frac{ax^n + b}{cx^n + d}\right)^m = mn(ad - bc)x^{n-1}\frac{\left(ax^n + b\right)^{m-1}}{\left(cx^n + d\right)^{m+1}}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{\left(dx^2 + ex + f\right)^2}$$

$\left(\sin^n u\right)' = nu'\cos u \sin^{n-1} u$	$\left(\cos^n u\right)' = -nu' \sin u \cos^{n-1} u$
$\left(\tan^n u\right)' = nu' \sec^2 u \tan^{n-1} u$	$\left(\sec^n u\right)' = nu' \tan u \sec^n u$
$\left(\cot^n u\right)' = -nu' \csc^2 u \cot^{n-1} u$	$\left(\csc^n u\right)' = -nu' \cot u \csc^n u$
$(\arcsin u)' = \dfrac{u'}{\sqrt{1 - u^2}}$	$(\arccos u)' = \dfrac{-u'}{\sqrt{1 - u^2}}$
$(\arctan u)' = \dfrac{u'}{1 + u^2}$	$(\operatorname{arcsec} u)' = \dfrac{u'}{\lvert u \rvert \sqrt{u^2 - 1}}$
$(\operatorname{arccot} u)' = \dfrac{-u'}{1 + u^2}$	$(\operatorname{arccsc} u)' = \dfrac{-u'}{\lvert u \rvert \sqrt{u^2 - 1}}$

$(\operatorname{arc}\csc u)' = \dfrac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$	$(\cosh u)' = u'\sinh u$
$(\coth u)' = -u'\left(\operatorname{csch}^2 u\right)$	$(\operatorname{csch} u)' = -u'\coth u\ \operatorname{csch} u$
$(\tanh u)' = u'\left(1-\tanh^2 u\right) = u'\left(\operatorname{sech}^2 u\right)$	$(\operatorname{sech} u)' = -u'\tanh u\ \operatorname{sech} u$
$\left(\sinh^{-1} u\right)' = \dfrac{u'}{\sqrt{u^2+1}}$	$\left(\cosh^{-1} u\right)' = \dfrac{u'}{\sqrt{u^2-1}}$
$\left(\tanh^{-1} u\right)' = \dfrac{u'}{1-u^2}$	$\left(\operatorname{sech}^{-1} u\right)' = -\dfrac{u'}{u\sqrt{1-u^2}}$
$\left(\coth^{-1} u\right)' = \dfrac{u'}{1-u^2}$	$\left(\operatorname{csch}^{-1} u\right)' = -\dfrac{u'}{\lvert u\rvert\sqrt{1+u^2}}$
$\left(e^u\right)' = u'e^u$	$(\ln u)' = \dfrac{u'}{u}$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Identities	
$\cosh^2 u - \sinh^2 u = 1$	
$\sinh 2x = 2\sinh x\cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$	$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
$\tanh^2 x = 1 - \operatorname{sech}^2 x$	$\coth^2 x = 1 + \operatorname{csch}^2 x$

Exercises *Lecture 2* – Review

$(\mathbf{1-80})$ Find the ***first*** derivative.

1. $f(x)=50\pi$

2. $f(x)=t^2+\pi$

3. $g(t)=t^2+\pi$

4. $f(x)=2x-5$

5. $f(x)=x^2+\pi x-5$

6. $f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2-5x+3$

7. $f(x)=x^6-5x^4$

8. $f(x)=2x^2+3x$

9. $g(t)=t^{-3}-3t^{-1/2}+5t^2$

10. $f(x)=2x^{-3/2}+3x^{5/3}$

11. $f(x)=3x^{3/2}+5x^{-5/3}$

12. $f(x)=2\sqrt{x}-\frac{1}{x}+\frac{4}{\sqrt{x}}$

13. $f(x)=(x+2)(x+3)$

14. $f(x)=\left(x^2+2\right)\left(x^2-2\right)$

15. $f(x)=\left(\sqrt{x}+3\right)\left(\sqrt{x}-3\right)$

16. $f(x)=\sqrt{x}\left(\sqrt{x}+5\right)$

17. $g(t)=\frac{t^2-3t+5}{t^2}$

18. $f(x)=\frac{2x-3}{2x+3}$

19. $f(x)=\frac{x-5}{3x+4}$

20. $f(x)=\frac{3x+4}{2x-5}$

21. $f(x)=\frac{x^2+2x+5}{3x^2-x+4}$

22. $f(x)=\frac{2x^2+5x-6}{x^2-3x-1}$

23. $f(x)=\frac{1}{\sqrt{5-4x-x^2}}$

24. $f(x)=\left(x^2+3x+4\right)^6$

25. $f(x)=\left(\frac{1}{3}x^3+4x-1\right)^7$

26. $f(x)=\left(x^2+4\right)^3\left(x^2+x-5\right)^4$

27. $f(x)=\frac{\left(x^2+2x-3\right)^6\left(2x^2+3x-5\right)^4}{\left(x^3+2x^2\right)^5}$

28. $f(x)=\cos 3x-\sin 5x$

29. $f(x)=\cos^3 x-\sin^5 x$

30. $f(x)=\tan 3x-\sec 2x$

31. $f(x)=\tan^2 x-\csc^3 x$

32. $f(x)=x^2\tan 3x$

33. $f(x)=\cot \pi x-\csc\frac{\pi}{2}x$

34. $f(x)=\frac{1-\tan x}{1+\tan x}$

35. $f(x)=\frac{\sin^3\left(\sqrt{5}x\right)}{x^2}$

36. $f(\theta)=3\theta\tan\theta\cot\theta$

37. $f(\theta)=\frac{\sin\theta}{\theta}$

38. $f(x)=e^{3x}+x^2$

39. $f(x)=e^{3x^2}$

40. $f(x)=e^{2x}\ln x$

41. $f(x)=e^{3x}\cos x$

42. $f(x)=e^{x^3}\left(x^2+x\right)$

43. $f(x)=e^{x}\left(2x^2-5x\right)$

44. $f(x)=\left(x^2e^{x}\right)^3$

45. $f(x)=e^{2x}\sec x$

46. $f(x)=\left(e^{2x}+1\right)\sin x$

47. $f(x)=(\sin x+\cos x)\,e^{x}$

48. $f(x)=e^{-5x^2}$

49. $f(x)=\dfrac{\ln x}{e^{x}}$

50. $f(x)=\ln\left(x^2+1\right)$

51. $f(x)=\ln\left(x^2+1\right)^3$

52. $f(x)=\ln\left(\dfrac{ax+b}{cx+d}\right)^3$

53. $f(x)=\sqrt{1+\ln^2 x}$

54. $f(x)=\ln\left(\dfrac{x^3(x+1)^4}{(2x-1)^2}\right)$

55. $y=\ln\sqrt{\dfrac{x+1}{x-1}}$

56. $y=(\ln x)^{\tan x}$

57. $y=\left(x^3+2x\right)^{\ln x}$

58. $y=x^{\tan x}$

59. $y=\left(x^2+2\right)^{\ln x}$

60. $y=\ln(\tan x)$

61. $y=\ln(\sin x)$

62. $y=\ln(\ln x)$

63. $f(x)=\arccos(\pi x)$

64. $f(x)=\arcsin\left(2x^2\right)$

65. $f(x)=\sin^{-1}\left(\dfrac{x+2}{3}\right)$

66. $f(x)=\arctan(2\pi x)$

67. $f(x)=\tan^{-1}\left(\sqrt{2}\,x\right)$

68. $f(x)=\sinh x^2$

69. $f(x)=\cosh(\sin 2x)$

70. $f(x)=x^3\tanh 4x$

71. $f(x)=\dfrac{\sinh 2x}{\ln x}$

72. $f(x)=e^{-2x}\cosh(2x)$

73. $f(x)=\sinh(\cos\pi x)$

74. $f(x)=\operatorname{sech}\left(e^{2x}\right)$

75. $f(x)=\cosh^{-1}(\cosh^{-1}x)$

76. $f(x)=\ln\left(\cosh^{-1}(x)\right)$

77. $f(x)=\sinh^{-1}(\cos 2x)$

78. $f(x)=\tanh^{-1}(\ln x)$

79. $f(x)=\left(\coth^{-1}x\right)^2$

80. $f(x)=\sinh^{-1}(\tanh x)$

(81 – 86) Find the ***second*** derivative.

81. $f(x)=x^4+5x^3+x^2-1$

82. $f(x)=3x^3+2x^2+\sqrt{5}\,x+\pi$

83. $f(x)=\sin 2x+\cos 3x$

84. $f(x)=\sec 3x+\tan 3x$

85. $f(x) = \sinh\left(x^3\right)$

86. $f(x) = e^{5x} + \ln x$

(87 – 92) Find the derivatives.

87. $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$, $f^{(4)}(x)$

88. $f(x) = x^4 - 4x^3 + 2x^2 + 5x - 3$, $f^{(5)}(x)$

89. $f(x) = 2x^6 - 3x^4 + 2x^2 + x + 1$, $f^{(6)}(x)$

90. $f(x) = 2x^6 - 3x^4 + 2x^2 + x + 1$, $f^{(7)}(x)$

91. $f(x) = e^x$, $f^{(12)}(x)$

92. $f(x) = e^{2x}$, $f^{(10)}(x)$

(93 – 98) Find $\frac{dy}{dx}$ by implicit differentiation.

93. $x^3y^3 + y^2 = x + y$

94. $x^2y + xy^2 = x + 2$

95. $x^3 - 3xy^2 - y^5 = 0$

96. $y \ln\left(x^2 + y^2 + 4\right) = 3$

97. $y e^{xy} - 2 = 0$

98. $3\sin(xy) = 1$

(99 – 104) Find $\frac{d^2y}{dx^2}$ by implicit differentiation.

99. $4x^2 + 2y^2 = 5$

100. $5y^2 + \sin y = x^2$

101. $x^3 + y^3 = 3xy$

102. $3x^2 - 4y^2 = 4$

103. $x\cos y = y$

104. $x^3y^3 - 5 = 0$

105. An inverted conical water tank with a height of 15 *feet* and a radius of 5 *feet* is drained through a hole in the vertex at a rate of $3\ ft^3/min$. What is the ***rate of change*** of the water depth when the water depth is 3 *feet*?

106. A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{6}$, the angle is increasing at the rate of 0.25 *rad/min*. ***How fast*** is the balloon rising at that moment?

107. A man 5 *feet* tall walks at a rate of 5 *feet* per *second* away from a light that is 20 *feet* above the ground.

a) When he is 15 *feet* from the base of the light, at what rate is the tip of his shadow moving?

b) When he is 15 *feet* from the base of the light, at what rate is the length of his shadow changing?

108. A helicopter starting on the ground is rising directly into the air at a rate of 25 *ft/sec*. You are running on the ground starting directly under the helicopter at a rate of 10 *ft/sec*. Find the rate of change of the distance between the helicopter and yourself after 5 *sec*.

109. A camera located 3,000 *ft* away from a rocket launch pad. If the rocket is rising vertically at 880 ft/s when it is 4,000 *ft* up, how fast must the camera elevation angle change at the instant to keep the rocket in sight?

110. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 *ft/s*. How rapidly is the area enclosed by the ripple increasing at the end of 10 *sec*?

111. Suppose that it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce *x* radiators when 8 to 30 radiators are produced and that $R(x) = x^3 - 3x^2 + 12x$ gives the dollar revenue from selling *x* radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to

produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

112. A baseball diamond is a square whose sides are 90 *ft* long. Suppose that a player running from second base to third base has a speed of $30\ ft/s$ at the instant when he is 20 *ft* from third base. At what rate is the player's distance from home plate changing at that instant?

Practice *Exam*

Find the ***first*** derivative.

1. $f(t) = 4\sqrt{t} - \frac{1}{4}t^4 + t + 1 + \frac{1}{t}$

2. $f(x) = 3x^4 - 3x^3 + 6x^2 - x + 5$

3. $f(x) = \dfrac{1}{x+2}$

4. $f(x) = \dfrac{x^2 + 4x^{1/2}}{x^2}$

5. $y = \ln\left(\sec^2\theta\right)$

6. $y = t^2 - \sec t + 1$

7. $y = x^2\cos x - 2x\sin x - 2\cos x$

8. $y = \left(x^2 + 3x\right)^4$

9. $f(x) = e^{-x}\ln x$

10. $y = \cos^3(6x)$

11. $y = \dfrac{x^2 - 1}{x^2 + 1}$

12. $f(x) = \dfrac{4x - 2}{3x + 5}$

13. $f(x) = \dfrac{1}{\sqrt{5 - 4x - x^2}}$

14. $f(x) = e^{2x^2}$

15. $y = x\ln^2 x$

16. $y = \sin^{-1}\left(x^2\right)$

17. $y = \arctan(3x)$

18. $y = \cosh(4x)$

19. $y = \sinh(6x)$

20. $f(x) = \left(x^2 + 4\right)^3\left(x^2 + x - 5\right)^4$

21. Find the ***fourth*** derivative $\quad f(x) = 3x^4 - 3x^3 + 6x^2 - x + 5$

22. Find the ***fifth*** derivative $\quad f(x) = x^4 - 4x^3 + 2x^2 + 5x - 3$

23. Find $\dfrac{dy}{dx}$ by implicit differentiation $\quad x^3y^3 + y^2 = x + y$

24. An inverted conical water tank with a height of 18 *feet* and a radius of 6 *feet* is drained through a hole in the vertex at a rate of $6\ ft^3/min$. What is the ***rate of change*** of the water depth when the water depth is 3 *feet*? *Volume of a cone*: $V = \frac{1}{3}\pi r^2 h$

25. A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 *rad/min*.

How fast is the balloon rising at that moment?

Lecture Three

Applications of Derivatives

Section 3.1 – Maxima and Minima

3.1−1 Definition

Let f be a function with Domain D. Then f has an ***absolute maximum*** value on D at a point c if

$$f(x) \le f(c) \quad \textit{for all } x \textit{ in } D$$

And an ***absolute maximum*** value on D at a point c if

$$f(x) \ge f(c) \quad \textit{for all } x \textit{ in } D$$

Maximum and minimum values are called ***extreme values*** of the function f .

Absolute maxima or minima are also referred to as ***global*** maxima or minima.

Function rule	*Domain D*	*Absolute extrema on D*
$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$	Absolute minimum of 0 at $x = 0$. Absolute minimum of 4 at $x = 2$.
$y = x^2$	$(0, 2]$	No absolute minimum. Absolute minimum of 4 at $x = 2$.

$y = x^2$	(0, 2)	No absolute extrema.

3.1–2 *Theorem* – The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

Maximum & Minimum at interior points

Maximum & Minimum at endpoints

Maximum at interior point
& Minimum at endpoint

Minimum at interior point
& Maximum at endpoint

3.1–3 *Definitions*

A function f has a ***local maximum*** (***LMAX***) value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying is some open interval containing c.

A function f has a ***local minimum*** (***LMIN***) value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying is some open interval containing c.

An absolute maximum is also a local maximum. Being the ***largest*** value overall, it is also the largest value in its immediate neighborhood.

3.1–4 Finding Extrema

Theorem – The First Derivative Theorem for Local Extreme Values

If f has a ***local minimum*** or ***local maximum*** value at a point c of its domain D, and f' is defined at c, then

$$f'(c) = 0$$

Proof

For $f'(c) = 0$ at a local extremum, we need to show that $f'(c)$ can't be positive or negative at c.

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c > 0} \right\} \le 0$$

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c < 0} \right\} \ge 0$$

$$\Rightarrow \quad f'(c) = 0$$

3.1–5 *Definition*

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

3.1–6 How to find the Absolute Extrema of a continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

3.1–7 Critical Points (*CP*) or *Critical Numbers*

The critical points for a function f are those numbers c in the domain of f for which $f'(c)=0$ or $f'(c)$ doesn't exist. A critical point is a point whose x-coordinate is the critical point c, and whose y-coordinate is $f(c)$.

If $f'(x)=0$ undefined

Example 1

Find the critical point(s) of $f(x)=x^2$

Solution

$f'(x)=2x=0$

$\rightarrow$ $\underline{x=0}$ is a Critical Number (CN)

$f(0)=0$

Critical Point (CP): $\underline{(0,\ 0)}$

Example 2

Find the absolute maximum and minimum values of $f(x)=x^2$ on [–2, 1].

Solution

$f'(x)=2x=0$

$x=0$ (CN)

x	$f(x)$
–2	4
0	0
1	1

The function has an absolute maximum value of 4 at x = –2 and an absolute minimum value of 0 at $x=0$.

Example 3

Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on [–2, 1].

Solution

$g'(t) = 8 - 4t^3 = 0$

$t^3 = 2$

$t = \sqrt[3]{2} \ > 1$

t	$g(t)$
–2	–32
1	7

The function has an absolute maximum value of 7 at $x = 1$ and an absolute minimum value of –32 at $x = -2$.

Example 4

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on [–2, 3].

Solution

$f'(x) = \frac{2}{3}x^{-1/3}$

$= \dfrac{2}{3\sqrt[3]{x}} = 0 \quad \Rightarrow \quad$ *Undefined*

CN: $x = 0$

$f(0) = 0$

Critical point (*CP*): $(0,\ 0)$

x	$f(x)$
–2	$\sqrt[3]{4}$
0	0
3	$\sqrt[3]{9}$

Absolute ***MIN***: $(0,\ 0)$ Absolute ***MAX***: $\left(3,\ \sqrt[3]{9}\right)$

Example 5

Find the absolute maximum and minimum values of $f(x)=\sqrt{a^2-x^2}$ on $[-a,\ a]$.

Solution

Domain: $f(x)$: $a^2-x^2\geq 0$

$-a\leq x\leq a$

$$f'(x)=-\frac{x}{\sqrt{a^2-x^2}}=0 \qquad \left(\sqrt{U}\right)'=\frac{U'}{2\sqrt{U}}$$

CN: $\underline{x=0,\ \pm a}$

x	$f(x)$
$-a$	0
0	a
a	0

Absolute ***MIN***: $(\pm a,\ 0)$ Absolute ***MAX***: $(0,\ a)$

3.1–8 First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f.

1. If f' changes from negative to positive at c, then f has a local minimum (***LMIN***).
   Or Relative minimum (***RMIN***)
2. If f' changes from positive to negative at c, then f has a local maximum (***LMAX***).
   Or Relative maximum (***RMAX***)
3. If f' doesn't change sign at c, then f has no local extremum at c.

Example 6

Find the extreme value(s) of $f(x) = x^3 - \frac{5}{2}x^2 + 2x + 2$.

Solution

$$f(x) = x^3 - \frac{5}{2}x^2 + 2x + 2$$

$$f'(x) = 3x^2 - 5x + 2 = 0$$

CN : $x = 1,\ \frac{2}{3}$

x	$f(x)$
$\frac{2}{3}$	$\frac{8}{27} - \frac{10}{9} + \frac{4}{3} + 2 = \frac{68}{27}$
1	$\frac{5}{2}$

LMIN: $\left(1,\ \frac{5}{2}\right)$ ***LMAX***: $\left(\frac{2}{3},\ \frac{68}{27}\right)$

Example 7

Find the extreme value(s) of $f(x) = x^2 + 4x + 3$.

Solution

$$f(x) = x^2 + 4x + 3$$

$$f'(x) = 2x + 4 = 0$$

CN : $x = -2$

x	$f(x)$
-2	-1

-2	0
$-$	$f'(0) = +$

∴ The point $(-2,\ -1)$ can called ***MIN. Point*** *or* ***Abs. Min point***. Since it the lowest point.

Example 8

Find the extreme value(s) of $f(x)=\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{5}{2}x^2+6x$.

Solution

$$f(x)=\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{5}{2}x^2+6x$$

$$f'(x)=x^3-2x^2-5x+6=0$$

$x_1=1$

1	1	−2	−5	6
		1	−1	−6
	1	−1	−6	0

$$x^2-x-6=0$$

CN: $x_{1,2,3}=-2,\ 1,\ 3$

x	$f(x)$
−2	$4+\frac{16}{3}-10-12=-\frac{38}{3}$
1	$\frac{1}{4}-\frac{2}{3}-\frac{5}{2}+6=\frac{37}{12}$
3	$\frac{81}{4}-18-\frac{45}{2}+18=-\frac{9}{4}$

	−2	0	1	3
	−	$f'>0$	−	+
	↘	↗	↘	↗

LMIN: $\left(-2,\ -\frac{38}{3}\right)$ $\left(3,\ -\frac{9}{4}\right)$ ***LMAX***: $\left(1,\ \frac{19}{6}\right)$

Example 9

Find the extreme value(s) of $f(x) = x - \sin x \qquad x \in [0,\ 4\pi]$.

Solution

$f'(x) = 1 - \cos x = 0$

$\cos x = 1$

$CN: \quad x_{1,2,3} = 0,\ 2\pi,\ 4\pi$

x	$f(x)$
0	0
2π	2π
4π	4π

Abs. Min: $(0,\ 0)$ ***Abs. Max***: $(4\pi,\ 4\pi)$

Example 10

Find the extreme value(s) of $f(x) = e^{3x}$.

Solution

$f'(x) = 3e^{3x} \neq 0$

$\therefore$ The function has no extreme point.

Exercises *Section* 3.1 – Maxima and Minima

(1 – 10) Determine all critical points of each function

1. $y = x^2 - 6x + 7$

2. $y = x^2 - 5x + 4$

3. $g(x) = (x-1)^2(x-3)^2$

4. $f(x) = 3x^3 + 2x^2 + x - 1$

5. $f(x) = \tan x$

6. $f(x) = \cosh x$

7. $f(x) = e^{2x}$

8. $f(x) = \ln(x-1)$

9. $f(x) = \dfrac{x^2}{x-2}$

10. $g(x) = x^2 - 32\sqrt{x}$

(11 – 18) Find the absolute maximum and minimum values of each function. Then graph the function. Identify the points on the graph where the absolute extrema occur and include their coordinates.

11. $f(x) = \frac{2}{3}x - 5$ $\quad -2 \le x \le 3$

12. $f(x) = x^2 - 1$ $\quad -1 \le x \le 2$

13. $f(x) = -\dfrac{1}{x^2}$ $\quad 0.5 \le x \le 2$

14. $f(x) = x^2 - 3x + 2$ $\quad 1 \le x \le 3$

15. $f(x) = x^2 - 3x^{2/3}$ $\quad 0 \le x \le 2$

16. $f(x) = \sqrt{4 - x^2}$ $\quad -2 \le x \le 1$

17. $f(\theta) = \sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

18. $g(x) = \sec x$ $\quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

(19 – 45) Find the absolute maximum and minimum values of each function (if they exist).

19. $f(x) = x^{4/3}$, $\quad -1 \le x \le 8$

20. $f(\theta) = \theta^{3/5}$, $\quad -32 \le \theta \le 1$

21. $f(x) = 2^x \sin x$ $\quad [-2,\ 6]$

22. $f(x) = \sec x$ $\quad \left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

23. $f(x) = x^3 e^{-x}$ $\quad [-1,\ 5]$

24. $f(x) = x\ln\left(\frac{x}{5}\right)$ $\quad \left[\frac{1}{10},\ 5\right]$

25. $f(x) = x^{8/3} - 16x^{2/3}$ $\quad [-1,\ 8]$

26. $f(x) = x^2 - 8x + 10$ $\quad [0,\ 7]$

27. $f(x) = 2(3 - x)$, $\quad [-1,\ 2]$

28. $f(x) = x^3 - 3x^2$, $\quad [0,\ 4]$

29. $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 4$, $\quad [-2,\ 5]$

30. $f(x) = \dfrac{1}{x+2}$, $\quad [-4,\ 1]$

31. $f(x) = \left(x^2 + 4\right)^{2/3}$, $\quad [-2,\ 2]$

32. $f(x) = \sin 2x + 3$ *on* $\quad [-\pi,\ \pi]$

33. $f(x) = 2x^3 - 3x^2 - 36x + 12$ $\quad (-\infty,\ \infty)$

34. $f(x) = 4x^{1/2} - x^{5/2}$ $\quad [0,\ 4]$

35. $f(x) = 2x\ln x + 10$ $\quad (0,\ 4)$

36. $f(x) = x\sin^{-1} x$ $\quad [-1,\ 1]$

37. $f(\theta) = \sin\theta + \cos\theta$ $\quad [0,\ 2\pi]$

38. $f(\theta) = \theta + \cos\theta$ $\quad [0,\ 2\pi]$

39. $f(x)=\dfrac{x}{x+1}$ $[0,\ 10]$

40. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$ $[-2,\ 2]$

41. $f(x)=\dfrac{2x^2-3x-1}{x-2}$ $[-2,\ 2)$

42. $f(x)=\dfrac{x^2-3}{x^2+4}$ $[-2,\ 2]$

43. $f(x)=x^4-2x^3+x^2$ $[-1,\ 1]$

44. $f(x)=-x^4+12x^2-27$ $[-3,\ 3]$

45. $f(x)=2x^3+11x^2-7x-6$ $[-4,\ 1]$

(46 – 70) Find the extreme values (absolute and local) of the function and where they occur

46. $f(x)=2x^2+5x+3$

47. $f(x)=3x^2-4x+2$

48. $f(x)=x^3-x-1$

49. $f(x)=x^3-4x^2+2$

50. $y=x^3-2x+4$

51. $f(x)=2x^4-4x^2+1$

52. $f(x)=x^4+6x^3-18x^2$

53. $y=\sqrt{x^2-1}$

54. $y=\sqrt{16-x^2}$

55. $y=x\sqrt{4-x^2}$

56. $y=\dfrac{1}{\sqrt[3]{1-x^2}}$

57. $y=x^2\sqrt{3-x}$]

58. $y=\dfrac{x+1}{x^2+2x+2}$

59. $y=\dfrac{3x-1}{x^2-x-6}$

60. $y=x^{2/3}(x+2)$

61. $f(x)=\dfrac{e^x+e^{-x}}{2}$

62. $f(x)=\frac{1}{8}x^3-\frac{1}{2}x$ $[-1,\ 3]$

63. $f(x)=\frac{1}{x}-\ln x$

64. $f(x)=\sin x\cos x$ $[0,\ 2\pi]$

65. $f(x)=3\cos x-x$

66. $f(x)=\sin^2 x$

67. $f(x)=x-\tan^{-1}x$

68. $f(x)=(\cosh x)^2$

69. $f(x)=2x^3-3x^2-36x+12$ $(-\infty,\ \infty)$

70. $f(x)=2x\ln x+10$ $(0,\ 4)$

71. Let $f(x)=(x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

72. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y=30\left(e^{x/60}+e^{-x/60}\right)$ $-30\le x\le 30$ models the shape of the telephone wire strung between two poles that are 60 *feet.* apart (x & y are measured in

feet.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

73. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 *feet* long and starts 3 *feet* from the wall you are sitting next to.

a) Show that your viewing angle is $\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$. If you are *x feet* from the front wall.

b) Find *x* so that α is as large as possible

Section 3.2 – Graphing Functions

3.2–1 Increasing and Decreasing Functions

Corollary

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

When the function f increases, the first derivative is positive and above the $x-axis$ from the left to right.

When the function f decreases, the first derivative is negative and below the $x-axis$ from the left to right.

When the function f at its extreme points, the first derivative intersect $x-axis$, where the sign is neither positive or negative.

Example 1

Find the open intervals on which the function $f(x) = x^3 - 12x - 5$ is increasing or decreasing

Solution

$$f'(x) = 3x^2 - 12$$

$$3\left(x^2 - 4\right) = 0$$

CN : $\underline{x = \pm 2}$

$-\infty$	-2	2	∞
$f'(-3) > 0$	$f'(0) < 0$	$f'(3) > 0$	
Increasing	***Decreasing***	***Increasing***	

Increasing: $(-\infty,\ -2)\quad(2,\ \infty)$

Decreasing: $(-2,\ 2)$

Example 2

Find the open intervals on which the function $f(x) = x^{1/3}(x-4)$ is increasing or decreasing

Solution

$$f(x) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}\left(x^{1/3} - x^{-2/3}\right)\frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{4}{3}\frac{x-1}{x^{2/3}} = 0$$

$$\Rightarrow \begin{cases} x = 1 \\ x \neq 0 \end{cases} \quad (CN)$$

$-\infty$	0	1	∞
$f'(-1) < 0$	$f'(0.5) < 0$	$f'(2) > 0$	
Decreasing	***Decreasing***	***Increasing***	

Increasing: $(1,\ \infty)$ ***Decreasing***: $(-\infty,\ 1)$

Example 3

Find extreme points and the open intervals on which the function is increasing or decreasing

$$f(x) = x\sqrt{x+2}$$

Solution

Domain: $x \geq -2$

$$f'(x) = \frac{x+2+\frac{1}{2}x}{\sqrt{x+2}}$$

$$= \frac{1}{2}\frac{3x+4}{\sqrt{x+2}} = 0$$

CN : $x = -\frac{4}{3}$

$$f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}}$$

-2	$-\frac{4}{3}$	0
$-$		$+$

Increasing: $\left(-\frac{4}{3}, \infty\right)$ ***Decreasing***: $\left(-2, -\frac{4}{3}\right)$

LMIN: $\left(-\frac{4}{3}, -\frac{4}{3}\sqrt{\frac{2}{3}}\right)$

3.2–2 Concavity

Definition

Let f be differentiable on an open interval I. The graph of f is

1. **Concave *upward*** on I if f' is increasing on the interval.
2. **Concave *downward*** on I if f' is decreasing on the interval.

3.2–3 Test for Concavity

Let f be function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is ***concave up*** on I.
2. If $f''(x) < 0$ for all x in I, then f is ***concave down*** on I.

   i. Locate the x values @ which $f''(x) = 0$ or undefined
   ii. Use these test x-value to determine the test intervals
   iii. Test the sign of $f''(x)$ in each interval

Example 4

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = x^4 - 8x^3 + 18x^2$$

Solution

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36 = 0$$

$$12 - 48 + 36 = 0$$

$x = 1 \quad x = 3$ | $(pt.\ infl.)$

$-\infty$	1	3	∞
$f''(0) > 0$	$f''(2) < 0$	$f''(4) > 0$	
upward	***downward***	***upward***	

f is concave upward on $(-\infty,\ 1)$ *and* $(3,\ \infty)$

f is concave downward on $(1,\ 3)$

Example 5

Determine the concavity of $y = 3 + \sin x$ *on* $[0,\ 2\pi]$

Solution

$$y' = \cos x$$

$$y'' = -\sin x = 0$$

$pt.\ Infl.:\ x = 0,\ \pi,\ 2\pi$

0	π	2π
$f''\left(\frac{\pi}{2}\right) < 0$		$f''\left(\frac{3\pi}{2}\right) > 0$
downward		***upward***

The graph y is ***concave down*** on $(0,\ \pi)$

The graph y is ***concave up*** on $(\pi,\ 2\pi)$

3.2–4 Points of Inflection

Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a ***point of inflection***.

At a point of inflection $(c,\ f(c))$, either $f''(c) = 0$ *or* $f''(c)$ fails to exist.

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$f'(0) = 0$$

Point of inflection: $(0,\ 0)$

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$f'(0) = 0$$

$(0,\ 0)$ is not a point of inflection

Example 6

A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0$$

Find the velocity and acceleration and describe the motion of the particle.

Solution

The velocity is: $v(t) = s'(t) = 6t^2 - 28t + 22 = 0$

$$\Rightarrow \quad \boxed{t = 1,\ \frac{11}{3}}$$

The acceleration is: $a(t) = v'(t) = 12t - 28 = 0$

$$\boxed{t = \frac{7}{3}}$$

0	1	$\frac{7}{3}$	$\frac{11}{3}$
$f'(.5) > 0$	$f'(2) < 0$		$f'(4) > 0$
Increasing right	***Decreasing left***		***Increasing right***
$f''(1) < 0$			$f''(4) > 0$
Concave down			***Concave up***

The particle starts moving to the right while slowing down, and then reverses by moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $\left[0,\ \frac{7}{3}\right)$. The acceleration then changes direction changes direction at $t = \frac{7}{3}$ but the particles continues moving leftward, while slowing down under the rightward acceleration. At $t = \frac{11}{3}$ the particle reverses direction again; moving to the right in the same direction as the acceleration.

3.2–5 Second Derivative Test for local Extrema

Let $f'(c)=0$ and let f'' exist ($\exists$)

1. If $f'(c)=0$ *and* $f''(c)>0 \Rightarrow f$ is a local Minimum at $x=c$
2. If $f'(c)=0$ *and* $f''(c)<0 \Rightarrow f$ is a local Maximum at $x=c$
3. If $f'(c)=0$ *and* $f''(c)=0 \Rightarrow$ Test fails $\rightarrow$ use f' to determine *Max*, *Min*.

$f'=0 \quad f''>0$

Local Min.

$f'=0 \quad f''<0$

Local Max.

Example 7

Sketch a graph of the function $f(x)=x^4-4x^3+10$ using the following steps

a) Identify where the extrema of f occur

b) Find the intervals on which f is increasing and decreasing

c) Find where the graph of f is concave up and down

d) Sketch the general shape of the graph for f

e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution

$$f'(x)=4x^3-12x^2$$
$$=4x^2(x-3)=0$$

$CN: \quad x=0,\ 0,\ 3$

$-\infty$	0		3		∞
$f'(-1)<0$	‖	$f'(1)<0$		$f'(4)>0$	
decreasing	‖	*decreasing*		*increasing*	

a) *At* $x=3$

$$y=3^4-4(3)^3+10$$

$= -17$

A ***local minimum*** at $(3,\ -17)$

b) f is ***decreasing***: $(-\infty,\ 0]\cup[0,\ 3)$

f is ***increasing***: $(3,\ \infty)$

c) $f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$$\Rightarrow \begin{cases} x = 0 & \rightarrow\ f(0) = 10 \\ x = 2 & \rightarrow\ f(2) = -6 \end{cases}$$

$-\infty$	0	2	∞
$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$	
Concave up	***Concave down***	***Concave up***	

f is ***concave up***: $(-\infty,\ 0)\cup(2,\ \infty)$

f is ***concave down***: $(0,\ 2)$

d) $f(x=0) = 0^4 - 4(0)^3 + 10$

$= 10$

Example 8

Sketch the graph of $f(x)=\dfrac{(x+1)^2}{1+x^2}$

Solution

Domain of f is $(-\infty,\ \infty)$ ***Horizontal Asymptotes*** $y=1$

$$f(x)=\frac{x^2+2x+1}{x^2+1}$$

$$\begin{matrix}1 & 2 & 1\\ 1 & 0 & 1\end{matrix}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$$f'(x)=\frac{-2x^2+2}{\left(1+x^2\right)^2}=0$$

$$-2x^2+2=0$$

$$x^2=1$$

$$\underline{x=\pm 1}\ \ (CN)$$

$$f'(x)=2\left(1-x^2\right)\left(1+x^2\right)^{-2}$$

$$f''(x)=2\left(1+x^2\right)^{-3}\left[-2x\left(1+x^2\right)-2(2x)\left(1-x^2\right)\right]$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=2\frac{-2x-2x^3-4x+4x^3}{\left(1+x^2\right)^3}$$

$$=2\frac{2x^3-6x}{\left(1+x^2\right)^3}$$

$$=\frac{4x\left(x^2-3\right)}{\left(1+x^2\right)^3}=0$$

$\rightarrow$ $\underline{x=0\quad x=\pm\sqrt{3}}$ ***Point of inflections***

$-\infty$	$-\sqrt{3}$	-1	**0**	1	$\sqrt{3}$	∞
	$-$		$+$		$-$	
	Decreasing		***Increasing***		***Decreasing***	
$-$		$+$		$-$		$+$
Concave down		***Concave up***		***Concave down***		***Concave up***

RMAX: $(1,\ 2)$

Decreasing: $(-\infty,\ -1)\cup(1,\ \infty)$

RMIN: $(-1,\ 0)$

Increasing: $(-1,\ 1)$

Concave down: $\left(-\infty,\ -\sqrt{3}\right)\cup\left(0,\ \sqrt{3}\right)$

Concave up: $\left(-\sqrt{3},\ 0\right)\cup\left(\sqrt{3},\ \infty\right)$

3.2–6 Summary

$f(x)$ has ***C***ritical ***N***umber (*s*) by forcing $f'(x)=0$, where f' intersects the $x-axis$

Critical ***P***oint (*s*) $(x,\ y)$ called extreme point (*s*).

$f'(x)>0$ $\quad f$ *is increasing*
$f'(x)<0$ $\quad f$ *is decreasing*

First derivative f' above $x-axis$ (positive): $f'(x)>0$

f' below $x-axis$ (negative): $f'(x)<0$

$f(x)$ changes direction from increasing to decreasing, that critical point called ***Max***imum Point. f' has positive values

$f(x)$ changes direction from decreasing to increasing. That critical point called ***Min***imum Point. f' has negative values

$f(x)$ has ***P***oint of ***I***nflection (*s*) by forcing $f''(x)=0$

$f'(x)$ is increasing, $f''>0$ has positive values, then $f(x)$ *concave* ***upward***.

$f'(x)$ is decreasing, $f''>0$ has positive values, then $f(x)$ *concave* ***downward***.

Exercises *Section* 3.2 – Graphing Functions

(1 – 15) Find the critical numbers and the open intervals on which the function is increasing or decreasing.

1. $f(x) = 3x^2 - 7x + 4$

2. $f(x) = x^3 + 3x^2 - 9x + 4$

3. $f(x) = (x-1)^{2/3}$

4. $f(x) = \sqrt{3 - x^2}$

5. $f(x) = \dfrac{x}{x^2 + 4}$

6. $f(x) = \dfrac{x}{x^2 + 1}$

7. $f(x) = x\sqrt{x+1}$

8. $f(x) = x^3 - 12x$

9. $f(x) = x^{2/3}$

10. $f(x) = \cos x \quad (0,\ 2\pi)$

11. $f(x) = 2 - |1 - x|$; $[0,\ 2]$

12. $f(x) = \dfrac{x^2}{x^2 + 1}$

13. $f(x) = -\dfrac{1}{x^2 + 1}$

14. $f(x) = -\frac{1}{3}x^3 + x^2$

15. $f(x) = x^3 - 3x^2 + 3$

(16 – 22) Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

16. $g(t) = -t^2 - 3t + 3$

17. $h(x) = 2x^3 - 18x$

18. $f(\theta) = 3\theta^2 - 4\theta^3$

19. $g(x) = x^4 - 4x^3 + 4x^2$

20. $f(x) = x - 6\sqrt{x-1}$

21. $f(x) = \dfrac{x^3}{3x^2 + 1}$

22. $f(x) = x^{1/3}(x+8)$

(23 – 45) Find all relative Extrema as well as where the function is increasing and decreasing

23. $f(x) = 2x^3 - 6x + 1$

24. $f(x) = 6x^{2/3} - 4x$

25. $f(x) = x^4 - 4x^3$

26. $f(x) = 3x^{2/3} - 2x$

27. $y = \sqrt{4 - x^2}$

28. $f(x) = x\sqrt{x+1}$

29. $f(x) = \dfrac{x}{x^2 + 1}$

30. $f(x) = x^4 - 8x^2 + 9$

31. $f(x) = 8x^3 - 24x - 47$

32. $f(x) = 8x^2 - 7x - 12$

33. $f(x) = x(x-11)^5$

34. $f(x) = (x+1)^2(x-2)^3$

35. $f(x) = 9 + 6x\ln x$

36. $f(x) = x^3 + \dfrac{1}{x^3} + 3$

37. $y = \cos 2x \quad 0 \le x \le 2\pi$

38. $y = 3x + 2\cos 3x \quad 0 \le x \le 2\pi$

39. $y = \cos x + \sin x \quad 0 \le x \le 2\pi$

40. $y = \sin^{-1} x$

41. $y = x^{2/3} + (x+2)^{1/3}$

42. $f(x) = x\sqrt{2 - x^2}$

43. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

44. $f(x) = \dfrac{x^2 - 4}{x^2 - 3}$

45. $f(x) = \dfrac{40 - 10x}{5 - x}$

(46 – 49) Find the local extrema of each function on the given interval, and say where they occur

46. $f(x) = \sin 2x \quad 0 \le x \le \pi$

47. $f(x) = \sqrt{3}\cos x + \sin x \quad 0 \le x \le 2\pi$

48. $f(x) = \frac{x}{2} - 2\sin\frac{x}{2} \quad 0 \le x \le 2\pi$

49. $f(x) = \sec^2 x - 2\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(50 – 61) Find the points of inflection.

50. $f(x) = x^2 - 3x - 18$

51. $f(x) = x^3 - 4x^2 + 4x - 16$

52. $f(x) = x^3 - 9x^2 + 24x - 18$

53. $f(x) = x^4 - x^3 - 9x^2 + 3x + 18$

54. $f(x) = \sqrt{4 - x^2}$

55. $f(x) = \dfrac{x^2 - 1}{2x + 1}$

56. $f(x) = \sin 2x \quad x \in [0,\ 4\pi]$

57. $f(x) = e^{3x}$

58. $f(x) = 6x^{1/3} + 2x$

59. $f(x) = x^3 - 9x^2 + 24x - 18$

60. $f(x) = x^3 - 3x^2 + 3$

61. $f(x) = \dfrac{1}{e^{3x}}$

(62 – 80) Determine the intervals on which the graph of the function is concave upward or concave downward.

62. $f(x) = x^2 + 2x - 3$

63. $f(x) = -2x^2 - 5x - 3$

64. $f(x) = x^3 + x^2 - 6x - 8$

65. $f(x) = 2x^3 + x^2 - 25x + 12$

66. $f(x) = 2x^3 + x^2 + 1$

67. $f(x) = -4x^3 - 8x^2 + 32$

68. $f(x) = \dfrac{x^2 - 1}{2x + 1}$

69. $f(x) = \dfrac{12}{x^2 + 4}$

70. $f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$

71. $f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

72. $f(x) = \sqrt{2}\cos x + \sqrt{2}\sin x \quad 0 \le x \le 2\pi$

73. $f(x) = e^{2x}$

74. $f(x) = -e^{-5x}$

75. $f(x) = \ln x$

76. $f(x) = 2x + 4\sin x \quad 0 \le x \le 2\pi$

77. $f(x) = 13x + \sin 13x \quad 0 \le x \le \frac{2\pi}{13}$

78. $f(x) = 2xe^{-x^2}$

79. $f(x) = \dfrac{x^2}{x^2 - 4}$

80. $f(x) = x^{1/3}(x-6)$

81. Does $f(x) = 2x^5 - 10x^4 + 20x^3 + x + 1$ have any inflection points? If so, identify them.

82. Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

83. Find the extrema using the second derivative test $f(x) = \dfrac{4}{x^2+1}$

84. Discuss the concavity of the graph of f and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

85. Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

(86 – 134) Find the critical point(s). Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values. Determine the point of inflection. Find the open intervals on which the function is concave upward and downward. ***Sketch*** the graph.

86. $y = x^3 - 3x + 3$

87. $f(x) = 2x^3 + 11x^2 - 7x - 6$

88. $f(x) = x^3 + 2x^2 - 5x - 6$

89. $f(x) = x^3 + 8x^2 + 11x - 20$

90. $f(x) = 8x^3 + 18x^2 + 45x + 27$

91. $f(x) = x^4 - 4x^3 + 5$

92. $y = -x^4 + 6x^2 - 4$

93. $f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$

94. $f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$

95. $f(x) = x^4 - 5x^2 - 2x$

96. $f(x) = 4x^5 - 8x^4 - x + 2$

97. $f(x) = \frac{1}{2}x^4 - 3x^2 + 4x + 1$

98. $y = x\left(\frac{x}{2} - 5\right)^4$

99. $y = x + \sin x \quad 0 \le x \le 2\pi$

100. $y = \cos x + \sqrt{3}\sin x \quad 0 \le x \le 2\pi$

101. $y = \dfrac{x}{\sqrt{x^2+1}}$

102. $y = x^2 + \dfrac{2}{x}$

103. $y = \dfrac{x^2 - 3}{x - 2}$

104. $y = \dfrac{5}{x^4 + 5}$

105. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

106. $y = \dfrac{x^4 + 1}{x^2}$

107. $y = \dfrac{x^2 - 4}{x^2 - 2}$

108. $y = -\dfrac{x^2 - x + 1}{x - 1}$

109. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

110. $y = \dfrac{4x}{x^2 + 4}$

111. $f(x) = \dfrac{x^2 + 4}{2x}$

112. $f(x) = \dfrac{3x}{x^2 + 3}$

113. $f(x) = 4\cos(\pi(x-1))$ *on* $[0,\ 2]$

114. $f(x) = \dfrac{x^2 + x}{4 - x^2}$

115. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

116. $f(x) = 2x^{3/2} - 6x^{1/2}$

117. $f(x) = \sqrt[3]{x} - \sqrt{x} + 2$

118. $f(x) = \dfrac{\cos \pi x}{1 + x^2}$ *on* $[-2,\ 2]$

119. $f(x) = \sin x + \cos x$

120. $f(x) = x + \cos x$

121. $f(x) = x - 2\sin x$

122. $f(x) = \sin^2 x + \cos x$

123. $f(x) = \cos^2 x - \sin x$

124. $f(x) = x^{2/3} + (x+2)^{1/3}$

125. $f(x) = x(x-1)e^{-x}$

126. $f(x) = x\left(x^2 - 1\right)e^{x}$

127. $f(x) = \dfrac{e^2}{2x^2 + x + 1}$

128. $f(x) = e^{2x} + e^{-x}$

129. $f(x) = \dfrac{x}{e^x}$

130. $f(x) = x^4 e^{-x}$

131. $f(x) = x^2 \ln x$

132. $f(x) = x^2 - x - \ln x$

133. $f(x) = \dfrac{x^2 - 3}{x^2 + 4}$

134. $f(x) = \dfrac{2x + 4}{x^2 + 4}$

135. $f(x) = \dfrac{1}{x^2 + 4}$

136. $f(x) = 3x\sqrt{1 - x^2}$

137. $f(x) = 4x^9 + 4x^7 + 4x$

138. $f(x) = x^5 - x^3$

139. $f(x) = \dfrac{1}{x^2} - \dfrac{11}{x}$

140. $r = \theta - 2\cos\theta$ $\quad \pi \le \theta \le 3\pi$

(141 – 142) For the given graph of the derivative $f'(x)$.

a) Determine the intervals on which f is increasing and decreasing.

b) Determine the intervals on which f is concave up and down.

141. $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$

142.

143. Find the points of inflection and concavity of $f'(x) = x^4 - 16x^3 + 81x^2 - 146x + 80$

144. The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15{,}000}\left(600x^2 - x^3\right), \qquad 0 \le x \le 600$$

Where x and R are in thousands of dollars.
Is there a point of diminishing returns for this function?

145. Find the point of diminishing returns (x, y) for the function

$$R(x) - x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

146. A county realty group estimates that the number of housings starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?

147. Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x) = x^3 - 27x^2 + 240x + 750$. Where is $C(x)$ decreasing?

148. A manufacturer sells telephones with cost function $C(x) = 6.14x - 0.0002x^2, \quad 0 \le x \le 950$ and revenue function $R(x) = 9.2x - 0.002x^2, \quad 0 \le x \le 950$. Determine the interval(s) on which the profit function is increasing.

149. The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x) = 14x^2 - 4x + 1200$, where x is the processor speed in *MHz*. Determine the intervals where the cost function $C(x)$ is decreasing.

150. The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t) = \dfrac{t}{t^2 + 36}$. On what time interval is the concentration of the drug increasing?

151. Coughing forces, the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by:

$v(r) = k(R - r)r^2$, $0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant), and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

152. $P(x) = -x^3 + 15x^2 - 48x + 450$, $x \ge 3$ approximates the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousand of tires that must be sold to maximize profit.

153. $P(x) = -x^3 + 3x^2 + 360x + 5000$; $6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Section 3.3 – Applied Optimization

3.3–1 Solving Applied Optimization Problems

1. Read the problem
2. Draw a picture
3. Introduce variables
4. Write an equation for the unknown quantity
5. Test the critical points and endpoints in the domain of the unknown

Example 1

An open-top box is to be made by cutting small congruent squares from the corners of a 12-*in*. by 12-*in*. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution

$$V(x) = hlw$$
$$= x(12-2x)^2$$
$$= x\left(144-48x+4x^2\right)$$
$$= 4x^3-48x^2+144x \qquad 0<x<6$$

$$\frac{dV}{dx} = 12x^2-96x+144 = 0$$

$$\underline{x = 2,\ 6} \quad (CN)$$

$$V(2) = 4(2)^3-48(2)^2+144(2)$$
$$\underline{=128}$$

$$V(6) = 4(6)^3 - 48(6)^2 + 144(6)$$
$$= 0$$

The maximum volume is 128 in^3. The cutout squares are 2 *inches*.

Example 2

You have been asked to design a one liter can shaped like a right circular cylinder. What dimensions will use the least material?

Solution

Volume of can: $V = \pi r^2 h = 1\ liter$

$$= 1000\ cm^3$$

Surface area of can: $A = 2\pi r^2 + 2\pi rh$

$$\pi r^2 h = 1000$$
$$h = \frac{1000}{\pi r^2}$$
$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{2000}{r}$$
$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0$$

Solve for r

$$4\pi r = \frac{2000}{r^2}$$
$$r^3 = \frac{500}{\pi}$$
$$r = \sqrt[3]{\frac{500}{\pi}}\ cm$$
$$\approx 5.42\ cm$$

0	5.42
$A'(1) < 0$	$A'(6) > 0$
decreasing	***increasing***

$$h = \frac{1000}{\pi r^2}$$
$$= \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}}$$
$$= \frac{100}{\sqrt[3]{250\,\pi}} \quad cm$$
$$\approx 10.84 \quad cm \quad (h = 2r)$$

The graph is concave up on its domain.

The on 2-liter can that uses the least material has height equal to twice its radius.

$$r = \sqrt[3]{\frac{500}{\pi}} \quad cm \qquad h = \frac{100}{\sqrt[3]{250\,\pi}} \quad cm$$

$$r \approx 5.42\ cm \quad h \approx 10.84\ cm$$

Tall and Thin *Short and wide*

Example 3

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area of the rectangle can have, and what are its dimensions?

Solution

The equation of the circle is given by the equation: $x^2 + y^2 = 4$

Therefore, the semicircle is: $y = \sqrt{4 - x^2}$

The dimensions of semicircle:

$$length: 2x \quad height: \sqrt{4 - x^2}$$

Area:

$$A(x) = lh$$

$$= 2x\sqrt{4-x^2}$$

$$A'(x) = \frac{8-2x^2-2x^2}{\sqrt{4-x^2}} \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$= \frac{8-4x^2}{\sqrt{4-x^2}} = 0 \qquad \textit{Solve for x}$$

$$8-4x^2 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$A\left(\sqrt{2}\right) = 2\left(\sqrt{2}\right)\sqrt{4-\left(\sqrt{2}\right)^2}$$

$$= 4$$

$$A(2) = 2(2)\sqrt{4-(2)^2}$$

$$= 0$$

The area has a maximum value of 4 $unit^2$ when the height is $x = \sqrt{2}$ ***units*** and length $2x = 2\sqrt{2}$ ***units.***

Example 4

A right circular cylindrical is to be inscribed in a cone of base radius 4 *inches* and altitude of 10 *inches*. If the axes of the cylinder and the cine coincide.

What is the maximum volume of the right circular cylindrical and dimensions that maximize the volume?

Solution

$$\frac{h}{10} = \frac{4-r}{4}$$

$$h = \frac{5}{2}(4 - r)$$

$$V = \pi r^2 h$$

$$V(r) = \frac{5}{2}\pi r^2 (4 - r)$$

$$= \frac{5}{2}\pi\left(4r^2 - r^3\right)$$

$$\frac{dV}{dr} = \frac{5}{2}\pi\left(8r - 3r^2\right)$$

$$= \frac{5}{2}\pi\, r\,(8 - 3r) = 0$$

$$\cancel{r = 0} \qquad r = \frac{8}{3} \quad in$$

$$h = \frac{5}{2}\left(4 - \frac{8}{3}\right)$$

$$= \frac{10}{3} \quad in$$

$$V = \pi\left(\frac{64}{9}\right)\left(\frac{10}{3}\right)$$

$$= \frac{640\,\pi}{27} \quad in^3$$

The cylinder has the dimensions of radius $\frac{8}{3}$ in and height $\frac{10}{3}$ ft with a maximum volume of $\frac{640\,\pi}{27}$ in^3

Example 5

You are in a boat 2 mi from the nearest point on a straight shoreline; that point is 6 mi from a place that you wish to be at. If you can row at a rate of 3 mph and run at rate of 5 mph.
How should you proceed in order to arrive at your destination in the shortest amount of time?

Solution

$$Time = \frac{distance}{rate}$$

$$= \frac{distance}{rate}(boat) + \frac{distance}{rate}(run)$$

$$= T_b + T_r$$

row distance: $y = \sqrt{2^2 + (6 - x)^2}$

$$= \sqrt{x^2 - 12x + 40}$$

Given row rate: 3 *mph*

Time row: $T_b = \frac{1}{3}\sqrt{x^2 - 12x + 40}$

Running distance: x

Given runs rate: 5 *mph*

Time Running: $T_r = \frac{x}{5}$

$$T = T_b + T_r$$

$$T(x) = \frac{x}{5} + \frac{1}{3}\sqrt{x^2 - 12x + 40}$$

$$\frac{dT}{dx} = \frac{1}{5} + \frac{1}{6}\frac{2x - 12}{\sqrt{x^2 - 12x + 40}} = 0 \qquad \left(\sqrt{U}\right)' = \frac{U'}{2\sqrt{U}}$$

$$\frac{1}{3}\frac{x - 6}{\sqrt{x^2 - 12x + 40}} = -\frac{1}{5}$$

$$5(x - 6) = -3\sqrt{x^2 - 12x + 40}$$

$$(5x - 30)^2 = \left(3\sqrt{x^2 - 12x + 40}\right)^2$$

$$25x^2 - 300x + 900 = 9\left(x^2 - 12x + 40\right)$$

$$25x^2 - 300x + 900 = 9x^2 - 108x + 360$$

$$16x^2 - 192x + 540 = 0$$

$$4x^2 - 48x + 135 = 0$$

$$x = \frac{48 \pm \sqrt{2{,}304 - 2{,}160}}{8}$$

$$= \frac{48 \pm \sqrt{144}}{8}$$

$$= \begin{cases} \frac{48 - 12}{8} = \frac{9}{2} \\ \frac{48 + 12}{8} = \cancel{\frac{15}{2}} > 6 \end{cases}$$

$$row\ (sailing) = \sqrt{4 + \left(6 - \frac{9}{2}\right)^2}$$

$$= \sqrt{4+\frac{9}{4}}$$

$$= \frac{5}{2}$$

$$T\left(\frac{9}{2}\right) = \frac{9}{10} + \frac{5}{6}$$

$$= \frac{26}{15} \quad hours$$

To minimize the time: sailing $\frac{5}{2}$ *miles* and running $\frac{9}{2}$ *miles* with a total time of $\frac{26}{15}$ *hours*

$$15 \overline{)\,26} \quad 1.7\overline{3}$$

15

110

105

50

45

5

$.733 * 60 = 43.98 \qquad .98 \times 60 = 59$

It took you 1 *hr* 43 min 59 sec

Example 6

You have 1,000 *ft* of fencing. If you have a river on one side of your property.

What is the dimension of the rectangular that maximizes the area?

Find the maximum area of your property.

Solution

Since, the river is on one side, then the perimeter of the fencing should be summation of 3-sides.

$$P = l + 2w = 1{,}000$$

$$l = 1{,}000 - 2w \quad (1)$$

The area is giving by:

$$A = lw$$

$$A(w) = (1{,}000 - 2w)w$$

$$= -2w^2 + 1{,}000w$$

$$A'(w) = -4w + 1{,}000 = 0$$

$$w = 250 \quad ft$$

$$(1) \rightarrow \quad l = 500 \quad ft$$

$$Area = (500)(250)$$
$$= 125{,}000 \quad ft^2$$

The dimensions are $250 \ ft \times 500 \ ft$ with area of $125{,}000 \ ft^2$

3.3–2 Example from *Economics*

$R(x) =$ Revenue from selling x items

$C(x) =$ Cost of producing the x items

$P(x) = R(x) - C(x)$ Profit from producing and selling x items

At a production level yielding maximum profit, marginal revenue equals to marginal cost.

3.3–3 Marginal Analysis

Profit = P Revenue = R Cost = C $P = R - C$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Example 7

Suppose that $r(x)=9x$ and $c(x)=x^3-6x^2+15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution

$r'(x)=9 \qquad c'(x)=3x^2-12x+15$

To find the intersection between the 2 derivatives, set $r'(x)=c'(x)$

$$3x^2-12x+15=9$$
$$3x^2-12x+6=0$$

$$\begin{cases} x=2-\sqrt{2} & \approx .586 \\ x=2+\sqrt{2} & \approx 3.414 \end{cases}$$

The possible productions are $x\approx 0.586$ *or* $x\approx 3.414$ million.

OR

$$p(x)=r(x)-c(x)=9x-x^3+6x^2-15x$$

$$p(x)=-x^3+6x^2-6x$$

$$p'(x)=-3x^2+12x-6$$

$$p''(x)=-6x+12=0$$

point of inflection $\underline{x=2}$

The maximum profit is $x=2+\sqrt{2}$ about $x\approx 3.414$ *million*

The maximum lost is $x=2-\sqrt{2}$ about $x\approx .586$ *million*

Exercises *Section* 3.3 – Applied Optimization

1. Find two nonnegative numbers x and y for which $2x + y = 30$, such that xy^2 is maximized.

2. The product of two numbers is 72. Minimize the sum of the second number and twice the first number

3. Find two numbers x and y such that their sum is 480 and $x^2 y$ is maximized.

4. Find two positive real numbers such that the sum of the first number and the second number is 44 and their product is a maximum.

5. Find two positive real numbers such that their product is 3,200 and $x + 2y$ is as small as possible.

6. A rectangular page in a textbook (with width x and length y) has an area of $98\ in^2$, top and bottom margins set at 1 *in*., left and right margins set at $\frac{1}{2}$ *in.* The printable area of the page is the rectangle that lies within the margins. What are the dimensions of the page that maximize the printable area?

7. What point on the graph of $f(x) = \frac{5}{2} - x^2$ is closest to the origin? (*Hint*: You can minimize the square of the distance.)

8. Find the point (x_0, y_0) on the graph of $y = x^3 + 6x^2 + 3x - 5$, at which the slope of the tangent line has it minimum value.

9. A line segment of length 10 joins the points $(0, p)$ and $(q, 0)$ to form a triangle in the first quadrant. Find the values of p and q that maximize the area of the triangle.

10. A metal cistern in the shape of a right circular cylinder with volume $V = 50\ m^3$ needs to be painted each year to reduce corrosion. The paint is applied only to surfaces exposed to the elements (the outside cylinder wall and the circular top). Find the dimensions r and h of the cylinder that minimize the area of the painted surfaces.

11. Verify the function $V = 27x - \frac{1}{4}x^3$ has an absolute maximum when $x = 6$. What is the maximum volume?

12. A net enclosure for golf practice is open at one end. The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

13. If the price charged for a candy bar is $p(x)$ cents, then x thousand candy bars will be sold in a certain city, where $p(x) = 82 - \frac{x}{20}$.
How many candy bars must be sold to maximize revenue?

14. $S(x) = -x^3 + 6x^2 + 288x + 4000$; $4 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

15. A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

16. A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$3 per foot for two opposite sides, and \$4 per foot for the other two sides. Find the dimensions of the field of area 730 square feet that would be the cheapest to enclose.

17. A rectangular bird sanctuary with one side along a straight river is to be constructed so that it contains $250\ ft^2$ of area. Find the dimensions of the rectangle to minimize the amount of fence necessary to enclose the remaining three sides.

18. A manufacturer wants to design an open box that has a square base and a surface area of 108 in^2. What dimensions will produce a box with a maximum volume?

19. A company wants to manufacture cylinder aluminum can with a volume $1000cm^3$. What should the radius and height of the can be to minimize the amount of aluminum used?

20. What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

21. You are planning to make an open rectangular box from an 8-in. by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

22. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

23. A piece of cardboard measures 10-*in*. by 15-*in*. Two equal squares are removed from the corners of 10-*in*. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

a) Write a formula $V(x)$ for the volume of the box

b) Find the domain of V for the problem situation and graph V over this domain

c) Use the graphical or analytically method to find the maximum volume and the value of x that gives it.

24. An open box of maximum volume is to be made from a square piece of material, 24 *inches* on a side, by cutting equal squares from the corners and turning up the sides.

a) Write the volume V as a function of x.

b) Find the critical number of the function and find the maximum value.

c) Graph the function and verify the maximum volume from the graph.

25. A cardboard 10 *in*. by 10 *in*., square corners are cut out so that the sides can be folded up to make a box.

a) What are the dimensions of the box that yield to maximize the volume?

b) What is the maximum volume?

26. From a 50 *in*. by 50 *in*. sheet of aluminum, square corners are cut out so that the sides can be folded up to make a box.

a) What dimensions will yield a box of maximum volume?
b) What is the maximum volume?

27. A manufacturer wants to design an open box having square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

28. From a 20 *in*. by 20 *in*. thin piece of cardboard, square corners are cut out so that the sides can be folded up to make a box.

a) What dimensions will yield a box of maximum volume?
b) What is the maximum volume?

29. An open box is to be made from a 24 *in*. by 36 *in*. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?
a) What dimensions will yield a box of maximum volume?
b) What is the maximum volume?

30. A rectangular box with a square base, an open top, and a volume of $216\ in^3$ is to be constructed.

a) What dimensions of the box should be to minimize the surface area?
b) What is the minimum surface area?

31. A rectangular box with a square base, an open top, and a volume of $216\ in^3$ is to be constructed. Suppose the cost of the material for the base is $0.20\ \$/in^2$ and the cost of the material for the sides is $0.30\ \$/in^2$.

a) What dimensions of the box should be to minimize the cost?
b) What is the cost of the box?

32. A company plans to produce a one- compartment vertical file by bending the long side of a 10-*in*. by 18-*in*. sheet of metal along two lines to form a ⊔ − shape. How tall should the file be in order to maximize the volume that it can hold?

33. A rectangular solid (with a square base) has a surface area of 337.5 cm^2. Find the dimensions that will result in a solid with maximum volume.

34. A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 *inches*.

a) Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?

b) Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?

35. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 *inches*.

Find the dimensions of the package of maximum volume that can be sent.

36. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 84 *inches*.

a) Find the dimensions of the package of maximum volume that can be sent.
b) What is the maximum volume of the package?

37. A page is to contain 30 *square inches* of print. The margins at the top and bottom of the page are 2 *inches* wide. The margins on the sides are 1 *inch* wide. What dimensions will minimize the amount of paper used?

38. A rectangular page is to contain 24 *squares inches* of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ *inches*, and the margins on the left and right are to be 1 *inch*. What should the dimensions of the page be so that the least amount of paper used?

39. A rectangular page will contain 54 in^2 of print. The margins at the top and bottom of the page are 1.5 *inches* wide. The margins on each side are 1 *inch* wide. What should the dimensions of the page be to minimize the amount of paper used?

40. A rectangle has its base, on the *x*-axis and its upper vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

41. Find the points of $y = 4 - x^2$ that are closet to (0, 3)

42. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,\ 2)$?

43. Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle with sides of 3 and 4 *units*.

44. A rectangle is bounded by the *x*- and *y*-axes and the graph of $y = \frac{1}{2}(6 - x)$. What length and width should the rectangle have so that its area is a maximum?

45. A right triangle is formed in the first quadrant by the *x*- and *y*-axes and a line through the point $(1,\ 2)$.

a) Write the length *L* of the hypotenuse as a function of *x*.

b) Graph the function and approximate *x* graphically such that the length of the hypotenuse is a minimum.

c) Find the vertices of the triangle such that its area is a minimum.

46. Two factories are located at the coordinates $(-x,\ 0)$ and $(x,\ 0)$, and their power is at $(0,\ h)$. Find *y* such that the total length of power line from the power supply to the factories is a minimum.

47. A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

48. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

49. The slant height of the cone is 3 *m*. How large should the indicated angle be to maximize the cone's volume?

50. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6.

a) Solve the area as a function of h.
b) Solve the area as a function of α.
c) Identify the type of triangle of maximum area.

51. A rectangle is bounded by the *x-axis* and the semicircle $y = \sqrt{25 - x^2}$

What length and width should the rectangle have so that its area is a maximum?

52. What are the dimensions of the lightest open-top right circular cylindrical can, that will hold a volume of 1000 cm^3?

53. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius *r*.

54. The figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base.

What values of *r* and *h* will give the smaller cone the largest possible volume?

55. Sometimes the solution of a *max-min* problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius *r* and height *h* is inscribed in a right circular cone of radius *R* and height *H*. Find the value of *r* (in terms of *R* and *H*) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$

56. A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

What values of r and h will give the cylinder the largest possible volume?

57. A *rhombus* is inscribed in a rectangle that is w *meters* wide with a perimeter of 40 *m*. Each vertex of the rhombus is a midpoint of a side of the rectangle.

a) Express the area of the *rhombus* as a function of the rectangle's width.
b) Find the dimensions that maximize the area.
c) What the maximum area of the rhombus?

58. A rectangular box with a square base of height y *feet* and the square with length x *feet* is inscribed in a right circular cone with a height of h *feet* and a base with radius r *feet*.

a) Write the volume of the rectangular box as a function of x where h and r are constants.
b) What values of x and y will give the box the largest possible volume when cone with a height of 10 *feet* and a base with radius 6 *feet*?

59. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the lightest. Neglect the thickness of the frame.

60. A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 *feet* of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

61. A Norman window has the shape of a rectangle surmounted by a semicircle. The exterior perimeter of the window is 48 *feet*.

Find the height h and the radius r that will allow the maximum amount of light to enter the window?

62. A Norman window has the shape of a rectangle surmounted by a semicircle. It requires 24 *feet* of trim on the outer edges.

What dimensions will allow the maximum amount of light to enter a house?

63. The cost per hour for fuel to run a train is $\frac{v^2}{4}$ *dollars*, where v is the speed of the train in *miles* per *hour*. (Note that the cost goes up as the square of the speed.) Other costs, including labor are \$300 per *hour*. How fast should the train travel on a 360-*mile* trip to minimize the total cost for the trip?

64. You are 2 *mi* offshore in a boat and wishes to reach a coastal village 6 *mi* down a straight shoreline from the point nearest the boat. You can row 2 *mph* and can walk 5 *mph*. Where should you land the boat to reach the village in the least amount of time?

65. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

66. The height above the ground of an object moving vertically is given by $s = -16t^2 + 96t + 112$ with s in feet and t in seconds. Find

a) The object's velocity when $t = 0$
b) Its maximum height and when it occurs
c) Its velocity when $s = 0$

67. Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 *cm* and dimensions x *cm* and y *cm* is to be rolled into a cylinder as shown above. What values of x and y give the largest volume?
b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in the figure. What values of x and y give the largest volume?

68. A cylindrical container can with a volume of $500\ cm^3$.

a) What dimensions will minimize the material needed to produce each can?
b) What is minimize the surface area?

69. Find the radius *r* and the height *h* of a cylindrical can of total surface area 160 cm^2 whose volume is as large as possible.

70. The 8-*feet* wall stands 27 *feet* from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

71. A small frictionless cart, attached to the wall by a spring, is pulled 10 *cm* from its rest position and released at time $t = 0$ to roll back and forth for 4 *sec*. Its position at time t is $s = 10\cos \pi t$

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?
b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?

72. The owner of a retail lumber store wants to construct a fence an outdoor storage is adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if

a) 240 *feet* fencing material are used.
b) 400 *feet* fencing material are used.

73. A carpenter is building a rectangular shed with a fixed perimeter of 54 *ft*.

a) what are the dimensions of the largest shed that can be built?
b) What is its area?

74. A rancher wants to enclose two rectangle areas near a river, one for cattle and one sheep. There are 240 *yd* of fencing available. What is the largest total area that can be enclosed?

75. A rectangular garden is to be constructed using 60 *feet* of stone wall. If the house forms one side of the rectangle.

a) What is the maximum area that the mason can enclose?
b) What should the dimensions of the patio be in order to yield this area?

76. A rectangle is to be inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$

What should be the dimension of the rectangle be to maximize its area? What is the maximum area?

77. A rectangular garden measures 80 *feet* by 60 *feet*. A large path of uniform width is to be added along both shorter sides and one longer side of the garden. The landscape designer doing the work wants to double the garden's area with the addition of this path. How wide should the path be?

78. A landscaper has enough stone to enclose a rectangular pond next to exiting garden wall of the house with 24 *feet* of stone wall. If the garden wall forms one side of the rectangle.

a) What is the maximum area that the landscaper can enclose?
b) What dimensions of the pond will yield this area?

79. Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then release.

a) If it takes only 1.4 times as much energy to fly over water as land, how far up the shore (x, in miles) should the bird head to minimize the total energy expended in returning to the nesting area?
b) If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?

80. A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs $10 to store one bottle for one year and $40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

81. A 300-room hotel in Las Vegas is filled to capacity every night at $80 a room. For each $1 increase in rent, 3 fewer rooms are rented. If each rented room costs $10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

82. A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company $0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs $200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

83. A university student center sells 1,600 cups of coffee per day at a price of $2.40.

a) A market survey shows that for every $0.05 reduction in price, 50 more cups of coffee will be sold. How much should be the student center charge for a cup of coffee in order to maximize revenue?

b) A different market survey shows that for every $0.10 reduction in the original $2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee to maximize revenue?

84. Suppose you are standing on the shore of a circular pond with radius 1 *mi*, and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 *mi/hr.* from your current position to another point *P* on the shore and then walk at 3 *mi/hr.* along the shore to the terminal point. How should you choose *P* to minimize the total time for the trip?

85. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

86. Two posts, one 12 *feet* high and the other 28 *feet* high, stand 30 *feet* apart. They are to be stayed by two wires, attached to a single stake, running from ground level on the top of each post. Where should the stake be placed to use the least amount of wire?

87. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 m^2 to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?

88. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area when the total perimeter is 16 *feet*.

89. A wooden beam has a rectangular cross section of height h and width w, the strength S of the beam is directly proportional to the width and the square of the height.

What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 *inches*? $S = kwh^2$ $\left(k: \text{ proportional constant}\right)$

90. A light source is located over the center of a circular table of diameter 4 *feet*. Find the height h of the light source such that the illumination I at the perimeter of the table is maximum when

$$I = \frac{k \sin \alpha}{s^2}$$

where s is the slant height
α is the angle at which the light strikes the table
k is a constant.

91. When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called refraction and is defined by ***Snell's Law of Refraction***,

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

Where θ_1 and θ_2 are the magnitudes of the angles.

v_1 and v_2 are the velocities of light in the two media.

Show that the light waves traveling from P to Q follow the path of the minimum time.

92. You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 3 *mph* and walk 4 *mph*. Toward what point on the coast should you row in order to reach Q in the least time?

93. You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 2 mph and walk 4 mph. Toward what point on the coast should you row in order to reach Q in the least time?

94. A sector with central angle θ is cut from a circle of radius 12 *inches*, and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

95. A rancher has 400 *feet* of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

96. The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin\theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 *feet* by 24 *feet*, with a 10-*foot* ceiling. Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.

97. Consider a room in the shape of a cube, 4 *meters* on each side. A bug at point P wants to walk to point Q at the opposite corner. Determine the shortest path.

98. The line joining P and Q crosses the two parallel lines. The point R is d units from P. How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?

99. Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 *cm*. the three resulting flaps are bent up, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.

100. You must get from a point P on the straight shore of a lake to a stranded swimmer who is 50 *m* from a point Q on the shore that is 50 *m* from you. If you can swim at a speed of 2 *m/s* and run at a speed of 4 *m/s*, at what point along the shore, x meters from Q, should you stop running and start swimming if you want to reach the swimmer in the minimum time?

a) Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$

b) Find the critical point of T on (0, 50)

c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d) Graph the function T to check your work.

101. Consider the function $f(x) = ax^2 + bx + c$ with $a \neq 0$. Explain geometrically why f has exactly one absolute extreme value on $(-\infty, \infty)$. Find the critical points to determine the value of x at which f has an extreme value.

102. An 8-*foot*-tall fence runs parallel to the side of a house 3 *feet* away. What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent.

103. A man wishes to get from an initial point on the shore of a circular pond with radius 1 *mi* to a point on the shore directly opposite (on the other end of the diameter). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.

a) If he swims at 2 *mi/hr.* and walks at 4 *mi/hr.*, what are the minimum and maximum times for the trip?

b) If he swims at 2 *mi/hr.* and walks at 1.5 *mi/hr.*, what are the minimum and maximum times for the trip?

c) If he swims at 2 *mi/hr.*, what is the minimum walking speed for which it is quickest to walk the entire distance?

104. A boat on the ocean is 4 *mi* from the nearest point on a straight shoreline; that point is 6 *mi* from a restaurant on the shore. A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.

a) If she walks at 3 *mi/hr.* and rows at 2 *mi/hr.*, at which point on the shore should she land to minimize the total travel time?

b) If she walks at 3 *mi/hr*. what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?

105. A cone is constructed by cutting a sector of angle θ from a circular sheet of metal with radius 20 *cm*. the cut sheet is then folded up and wheeled. What angle θ maximizes the volume of the cone?

Find the radius and height of the cone with maximum volume that can be formed in this way.

106. Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one: while shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a hole in the top of the barrel to the edge of the base of the barrel.

Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies the radius r and height h of the barrel. For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?

107. A load must be suspended 6 *m* below a high ceiling using cables attached to two supports that are 2 *m* apart.

How far below the ceiling (x) should the cables be joined to minimize the total length of cable used?

108. An island is 3.5 *mi* from the nearest point on a straight shoreline; that point is 8 *mi* from a power station. A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume that is costs \$2,400/*mi* to lay underwater cable and \$1,200/*mi* to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the project?

109. A crank of radius r rotates with an angular frequency ω. It is connected to a piston by a connecting rod of length L. the acceleration of the piston varies with the position of the crank according to the function

$$a(\theta) = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{L}\right)$$

For fixed ω and r, find the values of θ, with $0 \le \theta \le 2\pi$, for which the acceleration of the piston is a maximum and minimum.

110. Rod OA rotates about the fixed-point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston.

Where θ is the angle of rotation of rod OA.

a) Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$

b) For which the distance x is a maximum and minimum

111. A rain gutter is made from sheets of metal 9 *in* wide. The gutters have a 3-*in* base and two 3-*in* sides, folded up at an angle θ.

What angle θ maximizes the cross-sectional area of the gutter?

112. The cross sections of a trapezoid with 9 *in* wide base and two 4-*in* sides, folded up at an angle θ

a) Write the cross-sectional area A as a function of θ.
b) Find the angle θ maximizes the cross-sectional area?
c) Find the maximum cross-sectional area

113. The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 10 *feet* long. Determine the angle of elevation θ of the sides such that the area of the cross sections is a maximum by completing the following.

a) Analytically complete six rows of a table such as the one below

***Base* 1**	***Base* 2**	***Altitude***	***Area***
10	$10+20\cos 10°$	$10\sin 10°$	≈ 34.47
10	$10+20\cos 20°$	$10\sin 20°$	≈ 66.34

b) Write the cross-sectional area A as a function of θ.
c) Find the critical number of the function.
d) Find the angle that will yield the maximum cross-sectional area.
e) Estimate the maximum cross-sectional area.
f) Graph the function and verify the maximum cross-sectional area.
g) What is the maximum volume if the width of the canal is 2,500 *feet*?
h) Redo part (*a*) with three sides are 6 *feet* long. Find the angle that will yield the maximum cross-sectional area and the maximum area.

114. Two poles of heights m and n are separated by a horizontal distance d. A rope is stretched from the top of one pole to the ground and then to the top of the other pole. Show that the configuration that requires the least amount of rope occurs when $\theta_1 = \theta_2$

115. Fermat's principle states that when light travels between two points in the same medium (at a constant speed), it travels on the path that minimizes the travel time. Show that when light from a source A reflects off of a surface and is received at point B, the angle of incidence equals the angle of reflection, or $\theta_1 = \theta_2$

116. An island is 2 *miles* north of its closed point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 *miles* west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 *mph* and swims at a rate of 3 *mph*.

How far should be the visitor run before swimming to minimize the time it takes to reach the island?

117. An island is 1 *mile* from shore, and the distance from the cabin to the point on the shore closest to the island is 15 *miles*. Suppose you swim at a rate of 2.5 *mph* and run at a rate of 6 *mph*.

a) Find a function for the time it takes you to get from the cabinet to the island.

b) How far should be you run before swimming to minimize the time it takes to reach the island?

118. The temperature $T(t)$, in degrees Fahrenheit, during the day can be modeled by the equation $T(t) = -0.7t^2 + 9.4t + 59.3$, where t is the number of hours after 6:00 *AM*.

a) At what time the temperature a maximum?
b) What is the maximum temperature?

119. When a softball player swings a bat, the amount of energy $E(t)$, in *joules*, that is transferred to the bat can be approximated by the function

$$E(t) = -279.67t^2 + 82.86t$$

Where $0 \le t \le 0.3$ and t is measured in *seconds*. According to this model, what is the maximum energy of the bat?

120. Some softball fields are built in a parabolic mound shape so that water will drain off the field. A model for the shape of a certain field is given by

$$h(x) = -0.0002348x^2 + 0.0375x$$

Where $h(x)$ is the height, in *feet*, of the field at a distance of x *feet* from one sideline. Find the maximum height of the field.

121. The fuel efficiency for a certain midsize car is given by

$$E(v) = -0.018v^2 + 1.476v + 3.4$$

Where $E(v)$ is the fuel efficiency in *miles* per *gallon* for a car traveling v in *miles* per *hour*.

a) What speed will yield the maximum fuel efficiency?
b) What is the maximum fuel efficiency for this car?

122. A car rental company charges its costumers p dollars per day, where $60 \le p \le 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

123. The velocity of the cough depends on the size of the object. Suppose a person has a windpipe with a 20-*mm* radius. If an object has a radius r, in *mm*, then the velocity V, in *mm/sec*, needed to remove the object by a cough is given by

$$V(r) = k\left(20r^2 - r^3\right), \quad 0 \le r \le 20$$

Where k is some positive constant. For what size object is the maximum velocity required to remove the object?

124. An electronics company determines that its total weekly profit, from production and sale of x radio is given by

$$P(x) = \frac{1500}{x^2 - 6x + 10}$$

Find the number of radios, x, for which the total weekly profit is a maximum.

125. A stereo manufacturer determines that to sell x units of new stereo, the price per unit, in dollars, is given by

$$P(x) = 1{,}000 - x$$

The manufacturer also determines that the total cost of producing x units is given by

$$C(x) = 3{,}000 + 20x$$

a) Find the total revenue $R(x)$.

b) Find the total profit $P(x)$.

c) How many units must the company produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?

126. For a dosage of $x\ cm^3$ of a certain drug, the resulting blood pressure B is approximated by

$$B(x) = 305x^2 - 1{,}830x^3, \quad 0 \le x \le 0.16$$

Find the maximum blood pressure and the dosage at which occurs.

127. Two poles are connected by a wire that is also connected to the ground. The first pole is 20 *feet* tall, and the second pole is 10 *feet* tall. There is a distance of 30 *feet* between poles.

Where should the wire be anchored to the ground to minimize the amount of wire needed?

128. You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 *feet* to 6 *feet*.

What is the longest item that can be carried horizontally around the corner?

129. A company makes rectangular solid candy bars that measures 5 *inches* by 2 *inches* by 0.5 *inch*. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 *inch* and makes length of the candy bar 3 *inches* longer than the width?

130. A company makes rectangular solid candy bars that measures 5 *inches* by 2 *inches* by 0.5 *inch*. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%.

What should the dimensions be for the new candy bar if the company keeps the height at 0.5 *inch* and makes length of the candy bar 2.5 times as long as its width?

131. The height of an arch is given by

$$h(x) = -\frac{3}{64}x^2 + 27, \quad -24 \le x \le 24$$

Where $|x|$ is the horizontal distance in *feet* from the center of the arch to the ground.

a) What is the maximum height of the arch?
b) What is the height of the arch 10 *feet* to the right of center?
c) How far from the center is the arch 8 *feet* tall?

132. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln\left(\frac{1}{x}\right)$. If the radius of the core is 1 *cm*, what insulation thickness h will allow the greatest transmission speed?

133. You are under contract to build a solar station at ground level on the east-west line between the two buildings. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{50-x}{30}\right)$$

Then find the value of x that maximizes θ.

134. Suppose that a light at A is in a medium in which light travels at a speed v_1 and the point B is in a medium in which light travels at a speed v_2. Using Fermat's Principle, which states that light travels along the path that requires the minimum travel, show that the path taken between points A and B satisfies

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \qquad (\textit{Snell's Law})$$

135. How far from huge TV screen should you stand to view it best if the TV screen is 4 *ft* high and the bottom of it is 2 *ft* above your eyes level?

136. An ice cream cone with a height of one unit holds a sphere of ice cream. What is the radius of the ice cream sphere that maximizes the amount of the ice cream inside the cone?

You can see why this question results in an optimization problem: If the radius is small, much of the sphere is inside the cone, but the volume of the sphere is small. Alternatively, if the radius is large, the volume of the sphere is large, but only a small fraction of the sphere is inside the cone. Somewhere between these extremes, there should an optimal radius.

We assume that the cone has a base angle of 2θ and that the ice cream sphere is tangent to the sides of the cone. The solution requires the formula for the volume of a spherical cap. A cap of height h sliced from a sphere of radius r has volume of

$$V = \frac{\pi}{3} h^2 (3r - h)$$

To optimize the problem from the given information is to find a function. In this case, the objective function is the volume of the ice cream sphere that is inside the cone. Two cases must be considered. Let a be the distance between the center of the sphere and the top edge of the cone, where $a > 0$ means the center is below the top of the cone and $a < 0$ means the center is below the top of the cone.

a) In the case that $a > 0$, show that $r = (1 + a)\sin\theta$. (The line from the center of the sphere to the point of tangency is perpendicular to the cone.)

b) In the case that $a > 0$, show that the volume of ice cream inside the cone is $V(r) = \frac{\pi}{3}(r - a)^2 (2r + a)$, where $a = \frac{r}{\sin\theta} - 1$

c) In the case that $a < 0$, show that $r = (1 - |a|)\sin\theta = (1 + a)\sin\theta$, which is the same relationship as in the case that $a > 0$.

d) For $a < 0$, show that the volume of the ice cream inside the cone is $V(r) = \frac{\pi}{3}(r-a)^2(2r+a)$, where $a = \frac{r}{\sin\theta} - 1$. Thus, the volume function is the same for both $a > 0$ and $a < 0$.

e) Argue that the maximum value of r that needs to be considered occurs when the bottom of the sphere is at the top of the cone ($a > 0$). In this case $r_{\max} = a = \frac{\sin\theta}{1-\sin\theta}$.

f) Argue that the minimum value of r that needs to be considered occurs when the top of the sphere is at the top of the cone ($a < 0$). In this case $r_{\min} = -a = \frac{\sin\theta}{1+\sin\theta}$.

Section 3.4 – L'Hôpital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerator and denominators both approach zero or $\pm\infty$. The rule is known today as ***L'Hôpital's Rule***, after a French mathematician Guillaume de L'Hôpital.

$(1661 - 1704)$

His name is firmly associated with l' Hôpital's rule for calculating limits involving indeterminate forms 0/0 and ∞/∞. Although the rule did not originate with L' Hôpital, it appeared in print for the first time in his 1696 treatise on the infinitesimal calculus, entitled:

"*Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*"

This book was a first systematic exposition of differential calculus. Several editions and translations to other languages were published and it became a model for subsequent treatments of calculus.

Notes: $\frac{0}{\neq 0} = 0$ & $\frac{\neq 0}{0} = +/-\infty$

3.4–1 Indeterminate form 0/0

3.4–2 *Theorem* – L' Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Assuming the limit on the right side of this equation exists.

Example 1

Find the $\lim_{x \to 0} \frac{3x - \sin x}{x}$

Solution

$$\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{3-\cos x}{1}$$

$$=\frac{3-\cos x}{1}\Bigg|_{x=0}$$

$$=2$$

Example 2

Find the $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

Solution

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}$$

$$=\frac{1}{2\sqrt{1+x}}\Bigg|_{x=0}$$

$$=\frac{1}{2}$$

Example 3

Find the $\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$

Solution

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-3/2}}{2}$$

$$=\frac{-\frac{1}{4}(1+x)^{-3/2}}{2}\Bigg|_{x=0}$$

$$= -\frac{1}{8}$$

Example 4

Find the $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

Solution

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6}$$

$$= \frac{1}{6}$$

Example 5

Use l'Hôpital Rule to find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1+2x}$$

$$= \frac{\sin x}{1+2x}\Big|_{x=0}$$

$$= \frac{0}{1}$$

$$= 0$$

Example 6

Use l'Hôpital Rule to find $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2}$

Solution

$$\lim_{x\to 0}\frac{\sin x}{x^2}=\frac{0}{0}=\lim_{x\to 0}\frac{\cos x}{2x}$$

$$=\frac{1}{0}$$

$$=\infty$$

Example 7

Use l'Hôpital Rule to find $\lim_{x\to 0^-}\frac{\sin x}{x^2}$

Solution

$$\lim_{x\to 0^-}\frac{\sin x}{x^2}=\frac{0}{0}$$

$$=\lim_{x\to 0^-}\frac{\cos x}{2x}$$

$$=\frac{1}{0^-}$$

$$=-\infty$$

3.4–3 *Indeterminate* form ∞ / ∞, ∞ – 0, ∞ – ∞

L'Hôpital Rule applies to the indeterminate form ∞/∞, 0/0. If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example 8

Find the limits of these ∞ / ∞ forms: $\lim_{x\to \pi/2}\frac{\sec x}{1+\tan x}$

Solution

$$\lim_{x\to \pi/2}\frac{\sec x}{1+\tan x}=\frac{\infty}{\infty}$$

$$=\lim_{x\to \pi/2}\frac{\sec x\tan x}{\sec^2 x}$$

$$=\lim_{x\to \pi/2}\frac{\tan x}{\sec x}$$

$$= \lim_{x \to \pi/2} \frac{\sin x}{\cos x} \cos x$$

$$= \lim_{x \to \pi/2} \sin x$$

$$= 1$$

Example 9

Find the limits of these ∞ / ∞ forms: $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$

Solution

$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{1/x}{1/\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x}}$$

$$= 0$$

Example 10

Find the limits of these ∞ / ∞ forms: $\lim_{x \to \infty} \frac{e^x}{x^2}$

Solution

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{e^x}{2}$$

$$= \infty$$

Example 11

Find the limits of these $\infty \cdot 0$ forms: $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)$

Solution

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\infty\cdot 0$$

$$=\lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) \qquad \textit{Let } h=\frac{1}{x}$$

$$=\lim_{h\to 0^+}\left(\frac{\sin h}{h}\right)$$

$$=1$$

Example 12

Find the limits of these $\infty \cdot 0$ forms: $\lim\limits_{x\to 0^+}\sqrt{x}\,\ln x$

Solution

$$\lim_{x\to 0^+}\sqrt{x}\,\ln x=0\bullet(-\infty)$$

$$=\lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}} \qquad =\frac{\infty}{\infty}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}$$

$$=\lim_{x\to 0^+}\left(-2\sqrt{x}\right)$$

$$=0$$

Example 13

Find the limits of these $\infty - \infty$ form: $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty$$

$$= \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \frac{0}{0}$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x + x\cos x}\right) = \frac{0}{0}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{\cos x + \cos x - x\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{2\cos x - x\sin x}\right)$$

$$= \frac{0}{2}$$

$$= 0$$

3.4–4 Indeterminate Powers

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$

Example 14

Apply l'Hôpital Rule to show that $\lim_{x\to 0^+} (1+x)^{1/x} = e$

Solution

$$\lim_{x\to 0^+} (1+x)^{1/x} = 1^\infty$$

$$\ln f(x) = \ln(1+x)^{1/x}$$ *Using Power Rule*

$$= \frac{1}{x}\ln(1+x)$$

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+}(1+x)^{1/x}$$

$$= e^1$$

$$\underline{= e}$$

Example 15

Find $\lim_{x\to\infty} x^{1/x}$

Solution

$$\lim_{x\to\infty} x^{1/x} = \infty^0$$

$$\ln f(x) = \ln x^{1/x}$$ *Using Power Rule*

$$= \frac{\ln x}{x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \frac{0}{1}$$

$$\underline{= 0}$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^0$$

$$\underline{= 1}$$

Exercises *Section* 3.4 – L'Hôpital's Rule

$(1-138)$ Apply l'Hôpital Rule to evaluate

1. $\lim\limits_{x\to -2} \dfrac{x+2}{x^2-4}$

2. $\lim\limits_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

3. $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5}$

4. $\lim\limits_{t\to 0} \dfrac{\sin 5t}{2t}$

5. $\lim\limits_{\theta\to -\pi/3} \dfrac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)}$

6. $\lim\limits_{x\to 0} \dfrac{x^2}{\ln(\sec x)}$

7. $\lim\limits_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$

8. $\lim\limits_{x\to 0} \dfrac{3^x-1}{2^x-1}$

9. $\lim\limits_{x\to 0^+} (\ln x - \ln\sin x)$

10. $\lim\limits_{x\to 0} \dfrac{\left(e^x-1\right)^2}{x\sin x}$

11. $\lim\limits_{x\to \pi/2^-} \dfrac{1+\tan x}{\sec x}$

12. $\lim\limits_{x\to \infty} \dfrac{4x^3-6x^2+1}{2x^3-10x+3}$

13. $\lim\limits_{x\to 0} \dfrac{3\sin 4x}{5x}$

14. $\lim\limits_{x\to 2\pi} \dfrac{x\sin x+x^2-4\pi^2}{x-2\pi}$

15. $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

16. $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2}$

17. $\lim\limits_{x\to -1} \dfrac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}$

18. $\lim\limits_{x\to 1} \dfrac{x^n-1}{x-1}$ $n\in\mathbb{Z}^+$

19. $\lim\limits_{x\to 1^-} (1-x)\tan\left(\dfrac{\pi x}{2}\right)$

20. $\lim\limits_{x\to \infty} \dfrac{3}{x}\csc\dfrac{5}{x}$

21. $\lim\limits_{x\to \pi/4} \dfrac{\tan x-\cot x}{x-\frac{\pi}{4}}$

22. $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{8x^2}$

23. $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

24. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

25. $\lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$ *x is a real number*

26. $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{3x+2}-2}{x-2}$

27. $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

28. $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

29. $\lim\limits_{x\to \infty} \dfrac{8-4x^2}{3x^3+x-1}$

30. $\lim\limits_{x\to \pi/2} \dfrac{2\tan x}{\sec^2 x}$

31. $\lim\limits_{x\to 0} \dfrac{e^x-x-1}{5x^2}$

32. $\lim\limits_{x\to 0} \dfrac{e^x-\sin x-1}{x^4+8x^3+12x^2}$

33. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$

34. $\lim\limits_{x\to\infty} \dfrac{e^{3x}}{3e^{3x}+5}$

35. $\lim\limits_{x\to\infty} \dfrac{\ln(3x+5)}{\ln(7x+3)+1}$

36. $\lim\limits_{x\to\infty} \dfrac{\ln\left(3x+5e^{x}\right)}{\ln\left(7x+3e^{2x}\right)}$

37. $\lim\limits_{x\to\infty} \dfrac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}$

38. $\lim\limits_{x\to 1^+} x^{1/(x-1)}$

39. $\lim\limits_{x\to e^+} (\ln x)^{1/(x-e)}$

40. $\lim\limits_{x\to\infty} (1+2x)^{1/(2\ln x)}$

41. $\lim\limits_{x\to\infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

42. $\lim\limits_{t\to 2} \dfrac{t^3-t^2-2t}{t^2-4}$

43. $\lim\limits_{x\to 0} \dfrac{1-\cos 6x}{2x}$

44. $\lim\limits_{x\to\infty} \dfrac{5x^2+2x-5}{\sqrt{x^4-1}}$

45. $\lim\limits_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2}$

46. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

47. $\lim\limits_{\theta\to 0} (2\theta\cot 3\theta)$

48. $\lim\limits_{x\to 0} \dfrac{e^{-2x}-1+2x}{x^2}$

49. $\lim\limits_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}$

50. $\lim\limits_{y\to 0^+} \dfrac{\ln^{10} y}{\sqrt{y}}$

51. $\lim\limits_{\theta\to 0} \dfrac{3\sin 8\theta}{8\sin 3\theta}$

52. $\lim\limits_{x\to\infty} \dfrac{\ln x^{100}}{\sqrt{x}}$

53. $\lim\limits_{x\to 0} \csc x \sin^{-1} x$

54. $\lim\limits_{x\to\infty} \dfrac{\ln^3 x}{\sqrt{x}}$

55. $\lim\limits_{x\to\infty} \ln\left(\dfrac{x+1}{x-1}\right)$

56. $\lim\limits_{x\to 0^+} (1+x)^{\cot x}$

57. $\lim\limits_{x\to\frac{\pi}{2}^+} (\sin x)^{\tan x}$

58. $\lim\limits_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x}$

59. $\lim\limits_{x\to 0^+} |\ln x|^{x}$

60. $\lim\limits_{x\to\infty} x^{1/x}$

61. $\lim\limits_{x\to\infty} \left(1-\dfrac{3}{x}\right)^{x}$

62. $\lim\limits_{x\to\infty} \left(\dfrac{2}{\pi}\tan^{-1} x\right)^{x}$

63. $\lim\limits_{x\to 1} (x-1)^{\sin \pi x}$

64. $\lim\limits_{x\to\infty} \dfrac{2x^5-x+1}{5x^6+x}$

65. $\lim\limits_{x\to\infty} \dfrac{4x^4-\sqrt{x}}{2x^4+x^{-1}}$

66. $\lim\limits_{x\to 0} \dfrac{1-\cos x^n}{x^{2n}}$

67. $\lim\limits_{x\to 0} \dfrac{1-\cos^n x}{x^2}$

68. $\lim\limits_{x\to 0} \dfrac{1-\cos x^n}{x^2}$

69. $\lim\limits_{x\to 0} \dfrac{3x}{\tan 4x}$

70. $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx}$

71. $\lim\limits_{x\to 2} \dfrac{\ln(2x-3)}{x^2-4}$

72. $\lim\limits_{x\to 0} \dfrac{1-\cos ax}{1-\cos bx}$

73. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{\tan^{-1} x}$

74. $\lim\limits_{x\to 1} \dfrac{x^{1/3}-1}{x^{2/3}-1}$

75. $\lim\limits_{x\to 0} x\cot x$

76. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\ln\left(1+x^2\right)}$

77. $\lim\limits_{x\to \pi} \dfrac{\sin^2 x}{x-\pi}$

78. $\lim\limits_{x\to 0} \dfrac{10^x-e^x}{x}$

79. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos 3x}{\pi-2x}$

80. $\lim\limits_{x\to 1} \dfrac{\ln(ex)-1}{\sin \pi x}$

81. $\lim\limits_{x\to \infty} x\sin\dfrac{1}{x}$

82. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

83. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$

84. $\lim\limits_{x\to 0} \dfrac{2-x^2-2\cos x}{x^4}$

85. $\lim\limits_{x\to 0^+} \dfrac{\sin^2 x}{\tan x - x}$

86. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln(\sin x)}{\cos x}$

87. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x}{x}$

88. $\lim\limits_{x\to 1^-} \dfrac{\arccos x}{x-1}$

89. $\lim\limits_{x\to \infty} x\left(2\tan^{-1} x-\pi\right)$

90. $\lim\limits_{x\to \frac{\pi}{2}^+} x(\sec x-\tan x)$

91. $\lim\limits_{x\to 0} \left(\dfrac{1}{x}-\dfrac{1}{xe^{ax}}\right)$

92. $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

93. $\lim\limits_{x\to \pi} \dfrac{\cos x+1}{(x-\pi)^2}$

94. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{7x^3}$

95. $\lim\limits_{x\to \infty} \dfrac{\tan^{-1} x-\frac{\pi}{2}}{\frac{1}{x}}$

96. $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

97. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

98. $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{\sin^2 \pi x}$

99. $\lim\limits_{x\to 2} \dfrac{(3x+2)^{1/3}-2}{x-2}$

100. $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

101. $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

102. $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{1}{3}(2x-\pi)\tan x$

103. $\lim\limits_{x\to \infty} x\ln\left(1+\dfrac{1}{x}\right)$

104. $\lim\limits_{x\to\frac{\pi}{2}^-} \left(\frac{\pi}{2}-x\right)\sec x$

105. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{\sin\frac{1}{x}}$

106. $\lim\limits_{x\to 0^+} \sin x\sqrt{\dfrac{1-x}{x}}$

107. $\lim\limits_{x\to 0} \left(\cot x-\frac{1}{x}\right)$

108. $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2+1}\right)$

109. $\lim\limits_{\theta\to\frac{\pi}{2}^-} \left(\tan\theta-\sec\theta\right)$

110. $\lim\limits_{x\to 0^+} \ln x^{2x}$

111. $\lim\limits_{x\to 0} \ln\left(1+4x\right)^{3/x}$

112. $\lim\limits_{\theta\to\frac{\pi}{2}^-} \ln\left(\tan\theta\right)^{\cos\theta}$

113. $\lim\limits_{x\to 0^+} \left(1+x\right)^{\cot x}$

114. $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{\ln x}$

115. $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^{x}$

116. $\lim\limits_{x\to 0} \left(e^{5x}+x\right)^{1/x}$

117. $\lim\limits_{x\to 0} \left(e^{ax}+x\right)^{1/x}$

118. $\lim\limits_{x\to 0} \left(2^{ax}+x\right)^{1/x}$

119. $\lim\limits_{x\to 0^+} \left(\tan x\right)^{x}$

120. $\lim\limits_{x\to 0} x\csc x$

121. $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2+4x}\right)$

122. $\lim\limits_{\theta\to 0^+} \left(\sin\theta\right)^{\tan\theta}$

123. $\lim\limits_{x\to\infty} \left(1+\frac{10}{x^2}\right)^{x^2}$

124. $\lim\limits_{x\to\infty} \dfrac{100x^3-3}{x^4-2}$

125. $\lim\limits_{x\to\infty} \dfrac{2x^3-x^2+1}{5x^3+2x}$

126. $\lim\limits_{x\to 6} \dfrac{\sqrt[5]{5x+2}-2}{\frac{1}{x}-\frac{1}{6}}$

127. $\lim\limits_{t\to\pi/2^+} \dfrac{\tan 3t}{\sec 5\mathrm{t}}$

128. $\lim\limits_{x\to\infty} \left(\sqrt{x-2}-\sqrt{x-4}\right)$

129. $\lim\limits_{x\to\pi/2} \left(\pi-2x\right)\tan x$

130. $\lim\limits_{x\to\infty} x^3\left(\frac{1}{x}-\sin\frac{1}{x}\right)$

131. $\lim\limits_{x\to\infty} \left(x^2e^{1/x}-x^2-x\right)$

132. $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1}-\dfrac{1}{\sqrt{x-1}}\right)$

133. $\lim\limits_{x\to 0^+} x^{\ln x}$

134. $\lim\limits_{x\to\infty} \dfrac{\log_2 x}{\log_3 x}$

135. $\lim\limits_{x\to\infty} \left(\log_2 x-\log_3 x\right)$

136. $\lim\limits_{n\to\infty} \dfrac{1+2+\cdots+n}{n^2}$

137. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$

138. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$

(139 – 312) Evaluate the given limit using any method

139. $\lim\limits_{x\to 6} \dfrac{\sqrt{x-2}-2}{x^2-36}$

140. $\lim\limits_{x\to 2} \dfrac{x-2}{6x^2-10x-4}$

141. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{x+1}-1-\frac{1}{3}x}{x^2}$

142. $\lim\limits_{x\to 1} \dfrac{\sqrt[4]{x}-1}{\sqrt[3]{x}-1}$

143. $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{x}$

144. $\lim\limits_{x\to 0} \dfrac{x^3}{\sin x - x}$

145. $\lim\limits_{x\to 0} \dfrac{3x}{\tan x}$

146. $\lim\limits_{x\to 0} \dfrac{\sin^{-1}(3x)}{x}$

147. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$

148. $\lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{\sin\frac{\pi}{x}}$

149. $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2\sin x}{\sin 3x - 3\sin x}$

150. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin x}$

151. $\lim\limits_{x\to 0} \dfrac{\sin x^3}{\sin^3 x}$

152. $\lim\limits_{x\to \infty} \dfrac{\ln\left(3+e^{2x}\right)}{\ln\left(1+e^x\right)}$

153. $\lim\limits_{x\to 1} \dfrac{4x^2-5x+1}{\ln x}$

154. $\lim\limits_{x\to \pi} \dfrac{\ln(\cos 2x)}{(x-\pi)^2}$

155. $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

156. $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\ln(\tan x)}{\sin x - \cos x}$

157. $\lim\limits_{x\to 0} \dfrac{\left(e^x-1\right)^3}{x+2+(x-2)e^x}$

158. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}}{\sin 5x}$

159. $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{1-\cos \pi x}$

160. $\lim\limits_{x\to 0} \dfrac{e^x-1}{\sin 5x}$

161. $\lim\limits_{x\to 0} \dfrac{\sin^2 x+8x}{e^{2x}-1}$

162. $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2}$

163. $\lim\limits_{\theta\to \infty} \dfrac{\theta\sin\theta+\theta^2-4\pi^2}{\theta-2\pi}$

164. $\lim\limits_{\theta\to 0} \theta\csc\theta$

165. $\lim\limits_{\theta\to 0} \dfrac{\theta\sin(\sin\theta)}{1-\cos(\sin\theta)}$

166. $\lim\limits_{x\to a} \dfrac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$

167. $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1}{x\sin x}$

168. $\lim\limits_{x\to 0^+} \dfrac{\tan^{-1}x}{1-\cos 2x}$

169. $\lim\limits_{x\to 16^+} \dfrac{\sqrt[4]{x-16}}{\sqrt[4]{x}-2}$

170. $\lim\limits_{x\to 0} \dfrac{2x}{\tan^{-1}x}$

171. $\lim\limits_{x\to 0} \dfrac{\ln x}{x^2-x}$

172. $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x-\sin x}$

173. $\lim\limits_{x\to 2^+} \dfrac{\ln(x-1)}{(x-2)^2}$

174. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{\sin\frac{1}{x}}$

175. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x\sin x}$

176. $\lim\limits_{x\to 0} \dfrac{\sin x^2-\sin^2 x}{x^4}$

177. $\lim\limits_{x\to \pi} \dfrac{1+\cos x}{(x-\pi)^2}$

178. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x\tan x}$

179. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$

180. $\lim\limits_{x\to 0} \dfrac{1-\cos\left(2\sqrt{ax}\right)}{2x^2}$

181. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{1-\cos x}$

182. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{x-\sin x}{1-\cos x}$

183. $\lim\limits_{x\to \pi} \dfrac{\sin^2 x}{1+\cos 3x}$

184. $\lim\limits_{x\to 0} \dfrac{\sin^2 x+2\cos x-2}{\cos^2 x-x\sin x-1}$

185. $\lim\limits_{x\to \infty} \dfrac{\sin^2 x+2\cos x-2}{\cos^2 x-x\sin x-1}$

186. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin^2 x+2\cos x-2}{\cos^2 x-x\sin x-1}$

187. $\lim\limits_{x\to 2} \dfrac{3\sqrt[3]{x-1}-x-1}{3(x-2)^2}$

188. $\lim\limits_{x\to 0} \dfrac{x-\arctan x}{x^3}$

189. $\lim\limits_{x\to \infty} \dfrac{\tan^2\frac{1}{x}}{\ln^2\left(1+\frac{4}{x}\right)}$

190. $\lim\limits_{x\to \infty} \dfrac{\ln(\ln(\ln x))}{\ln x}$

191. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^{3x}}$

192. $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{e^{3x}-1}$

193. $\lim\limits_{x\to 0} \dfrac{2\cosh x-2}{1-\cos 4x}$

194. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x^2}$

195. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$

196. $\lim\limits_{x\to 1} \dfrac{x^3+3e^{1-x}-4}{x-\ln x-1}$

197. $\lim\limits_{x\to 0} \dfrac{x^2\tan x}{\tan x-x}$

198. $\lim\limits_{x\to 0} \dfrac{\sin x-\tan x}{x^3}$

199. $\lim\limits_{x\to 1} \dfrac{\sin(\pi x)}{\ln x}$

200. $\lim\limits_{x\to 0} \dfrac{\sin x-x}{x^2}$

201. $\lim\limits_{x\to \infty} (\sinh x-\cosh x)$

202. $\lim\limits_{x\to \infty} (\sinh x+\cosh x)$

203. $\lim\limits_{x\to \infty} \dfrac{\sinh x-\cosh x}{e^x}$

204. $\lim\limits_{x\to \infty} (\tanh x)^x$

205. $\lim\limits_{x\to \infty} \tanh x$

206. $\lim\limits_{x\to -\infty} \tanh x$

207. $\lim\limits_{x\to \infty} \coth x$

208. $\lim\limits_{x\to 0^-} \coth x$

209. $\lim\limits_{x\to 0^+} \coth x$

210. $\lim\limits_{x\to \infty} \operatorname{sech} x$

211. $\lim\limits_{x\to\infty} \operatorname{csch} x$

212. $\lim\limits_{x\to-\infty} \operatorname{csch} x$

213. $\lim\limits_{x\to\infty} \sinh x$

214. $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x}$

215. $\lim\limits_{x\to\infty} \dfrac{\sinh x}{x}$

216. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$

217. $\lim\limits_{x\to\infty} \cosh x$

218. $\lim\limits_{x\to\infty} \dfrac{\cosh x}{e^x}$

219. $\lim\limits_{x\to 0} \dfrac{x}{\tan x}$

220. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\cot x}$

221. $\lim\limits_{x\to 0^+} x\ln x$

222. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$

223. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$

224. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2}$

225. $\lim\limits_{x\to\infty} \dfrac{x^2}{\ln x}$

226. $\lim\limits_{x\to 0^+} x\ln\left(x^4\right)$

227. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x}\right)$

228. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

229. $\lim\limits_{x\to\infty} x^{1/x}$

230. $\lim\limits_{x\to\infty} x^{1/\ln x}$

231. $\lim\limits_{x\to\infty} x^{\sin x}$

232. $\lim\limits_{x\to 0} x^{1/\cos x}$

233. $\lim\limits_{x\to\infty} \left(1-\dfrac{1}{x}\right)^x$

234. $\lim\limits_{x\to 0} x^{2/x}$

235. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\sin x}$

236. $\lim\limits_{x\to\infty} x e^{1/x}$

237. $\lim\limits_{x\to 0} \dfrac{\tan x}{\sqrt{x}}$

238. $\lim\limits_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$

239. $\lim\limits_{x\to 1} \dfrac{(x-1)^2}{\ln x}$

240. $\lim\limits_{x\to 1} \dfrac{e^{x-1}-1}{x-1}$

241. $\lim\limits_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2}$

242. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\cos x + e^x - 1}$

243. $\lim\limits_{x\to 1} \dfrac{x\sin(x-1)}{2x^2 - x - 2}$

244. $\lim\limits_{x\to 0^+} \dfrac{\arctan(2x)}{\ln x}$

245. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

246. $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2x}{x - \sin x}$

247. $\lim\limits_{x\to 0} \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

248. $\lim\limits_{x\to\infty} x\sin\left(\dfrac{\pi}{x}\right)$

249. $\lim\limits_{x\to\infty} \sqrt{x}\, e^{-x/2}$

250. $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$

251. $\lim\limits_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$

252. $\lim\limits_{x\to 0^+}(4x+1)^{\cot x}$

253. $\lim\limits_{x\to 1}(2-x)^{\tan(\pi x/2)}$

254. $\lim\limits_{t\to a^+}\dfrac{\cos t\ \ln(t-a)}{\ln\left(e^t-e^a\right)}$

255. $\lim\limits_{x\to\infty}\left(\dfrac{2x-3}{2x+5}\right)^{2x+1}$

256. $\lim\limits_{x\to -3}\dfrac{x^2+2x-3}{2x^2+3x-9}$

257. $\lim\limits_{x\to 1}\dfrac{x^2-3x+2}{x^2-2x+1}$

258. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{\tan x-x}$

259. $\lim\limits_{x\to 0}\dfrac{\cos x+x}{x}$

260. $\lim\limits_{x\to 1}\dfrac{\ln x}{x}$

261. $\lim\limits_{\theta\to 0^+}\dfrac{\ln\theta}{\cot\theta}$

262. $\lim\limits_{\theta\to\frac{\pi}{2}^-}\dfrac{2+\sec\theta}{2\tan\theta}$

263. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^2}$

264. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x+1}{\cos^2 x}$

265. $\lim\limits_{x\to 0}\dfrac{\sin x+1}{\cos^2 x}$

266. $\lim\limits_{x\to 0}\dfrac{x+1-e^x}{x^2}$

267. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}2x}{\sin^{-1}x}$

268. $\lim\limits_{x\to 0}\dfrac{2x}{\arctan x}$

269. $\lim\limits_{x\to 0}\dfrac{x-\tan^{-1}x}{x\sin x}$

270. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}}{\tan^{-1}x}$

271. $\lim\limits_{x\to\infty}\dfrac{2e^{3x}+\ln x}{e^{3x}+x^2}$

272. . $\lim\limits_{x\to\infty}\dfrac{\ln(\ln x)}{\ln x}$

273. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n}$

274. $\lim\limits_{x\to 0}\dfrac{\cos^2 x-x\sin x-1}{\sin^2 x+2\cos x-2}$

275. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\pi-2x}{\cos x}$

276. $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{2x}}$

277. $\lim\limits_{x\to 0}\left(\dfrac{1}{e^x-1}-\dfrac{1}{x}\right)$

278. $\lim\limits_{x\to 0^+}\left(e^x-1\right)^x$

279. $\lim\limits_{x\to 0^+}(1-x)^{\ln x}$

280. $\lim\limits_{x\to\frac{\pi}{2}^-}(1+\cos x)^{\tan x}$

281. $\lim\limits_{x\to 0}(1+ax)^{b/x}$

282. $\lim\limits_{x\to\infty}\left(\sqrt{x^4+5x^2+3}-x^2\right)$

283. $\lim\limits_{x\to 0}(x+\cos 2x)^{\csc x}$

284. $\lim\limits_{x\to\infty}e^{-x}\ln x$

285. $\lim\limits_{x\to 0}(\cos x)^{x+1}$

286. $\lim\limits_{x\to 1} \dfrac{x^a - 1}{x^b - 1}$

287. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

288. $\lim\limits_{x\to 0} \dfrac{\sin mx}{\sin nx}$

289. $\lim\limits_{x\to 1} \dfrac{x^2 + 5}{3x + 4}$

290. $\lim\limits_{x\to \infty} \dfrac{x^2 + 5}{3x + 4}$

291. $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{4\tan x}{1 + \sec x}$

292. $\lim\limits_{x\to \infty} \dfrac{e^{2x}}{x^2}$

293. $\lim\limits_{x\to 4} \dfrac{x - 4}{\sqrt[3]{x + 4} - 2}$

294. $\lim\limits_{x\to 2} \dfrac{x^2 - 2x}{x^2 - 6x + 8}$

295. $\lim\limits_{x\to \infty} \dfrac{x^2 - 2x}{x^2 - 6x + 8}$

296. $\lim\limits_{x\to 0} \dfrac{\cos x + 2x - 1}{3x}$

297. $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos 2x}$

298. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^2}$

299. $\lim\limits_{x\to \infty} \dfrac{10}{5 + 4^{-x}}$

300. $\lim\limits_{x\to \infty} \left(2\sqrt{x} - \sqrt{x + 10}\right)$

301. $\lim\limits_{x\to \infty} \left(\ln(2x + 3) - \ln(5x + 2)\right)$

302. $\lim\limits_{x\to \infty} \left(\ln\left(\sqrt{9x^2 + 5}\right) - \ln(x)\right)$

303. $\lim\limits_{x\to \infty} \left(\ln(2x) - \ln\left(\sqrt{4x^2 + 1}\right)\right)$

304. $\lim\limits_{x\to 1} \left(3\tan\left(\frac{\pi}{2}x\right) \ln(x)\right)$

305. $\lim\limits_{x\to \frac{11\pi}{2}} \left(\left(x - \frac{11\pi}{2}\right) \tan\left(\frac{x}{11}\right)\right)$

306. $\lim\limits_{x\to \infty} x^{7/x}$

307. $\lim\limits_{x\to 0^+} x^{7/x}$

308. $\lim\limits_{x\to 0} (8x)^{\sin 3x}$

309. $\lim\limits_{x\to \infty} \left(\dfrac{x}{x + 9}\right)^x$

310. $\lim\limits_{x\to \infty} \left(\dfrac{x + 1}{x}\right)^{x^2}$

311. $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x^2}\right)^{2x}$

312. $\lim\limits_{x\to \infty} \dfrac{\log_{12} x}{\ln x}$

313. According to Michaelis-Menten equation, when an enzyme is combined with a substrate of concentration s, the reaction rate is:

$$v(s) = \frac{V s}{K + s}$$

Where, V : represents the maximum velocity, at maximum (saturating) substrate concentration K. (Michaelis constant).

a) Find the limiting reaction rate as the concentration s approaches infinite.

b) Find the reaction rate $v(K)$.

c) For certain reaction, $K = 1{,}200\ mM$ and $V = 0.3$. For which concentration s is $v(s)$ equal to 75% of its limiting value?

314. The functions $f(x) = \left(x^x\right)^x$ *and* $g(x) = x^{\left(x^x\right)}$ are different functions. For example, $f(3) = 19{,}683$ and $g(3) \approx 7.6 \times 10^{12}$. Determine whether $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^+} g(x)$ are intermediate forms and evaluate the limits.

315. Consider the function $g(x) = \left(1 + \frac{1}{x}\right)^{x+a}$. show that if $0 \le a < \frac{1}{2}$, then $g(x) \to e$ from below as $x \to \infty$; if $\frac{1}{2} \le a < 1$, then $g(x) \to e$ from above as $x \to \infty$

316. Let $f(x) = (a + x)^x$, where $a > 0$

a) What is the domain of f (in terms of a)?

b) Describe the end behavior of f (near the left boundary of its domain and as $x \to \infty$).

c) Compute f'.

d) Show that f has a single local minimum at the point z that satisfies $(z + a)\ln(z + a) + z = 0$

e) Describe how $f(z)$ varies as a increases.

Section 3.5 – Newton's Method and Linear Approximation

3.5–1 Procedure for *Newton's Method*

The goal of Newton's method, also called the ***Newton-Raphson*** method, for estimating a solution of an equation $f(x)=0$ is to produce a sequence of approximations that approach the solution.

We begin with the first number x_0 of the sequence. Then the function is approximated by its tangent line, and one computes the *x*-intercept of this tangent line.

At each step, the method approximates a zero of *f* with a zero of one of its linearization.

> Initial estimates, x_0, the method than uses the tangent curve $y=f(x)$ @ $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the *x*-axis. The number x_1 usually a better approximation to the solution that is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the *x*-axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

The point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)\cdot(x - x_n)$$

We can find where it crosses the *x*-axis by setting *y* = 0.

$$0 = f(x_n) + f'(x_n)\cdot(x - x_n)$$

$$x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x = x_n - \frac{f(x_n)}{f'(x_n)}}$$

3.5–2 *Newton*'s Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \textit{if} \quad f'(x_n) \neq 0$$

Example 1

Find the positive root of the equation $f(x) = x^2 - 2 = 0$

Solution

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= x_n - \frac{x_n}{2} + \frac{1}{x_n}$$

$$= \frac{x_n}{2} + \frac{1}{x_n}$$

Example 2

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution

$$x^3 - x = 1$$

$$x^3 - x - 1 = 0$$

$$f(x) = x^3 - x - 1$$

$$\begin{cases} f(1) = -1 \\ f(2) = 5 \end{cases}$$

$$f'(x) = 3x^2 - 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	–1	2	1.5
1	1.5	0.875	5.75	1.347826087
2	1.347826087	0.100682173	4.449905482	1.325200399
3	1.325200399	0.002058362	4.268468292	1.324718174
4	1.324718174	0.000000924	4.264634722	1.324717957
5	1.324717957	–1.8672 E–13	4.264632999	1.324717957

The result: $x = 1.324717957$

3.5–3 Square Root Using Newton's Method

To find the square root of a number $x = \sqrt{a}$ using Newton's method.

Let the function be $f(x) = x^2 - a$.

We begin with the first number $x_0 = a$ of the sequence.

Exercise

Use the Newton's method to approximate $\sqrt{7}$

Solution

Let $f(x) = x^2 - 7$

$f'(x) = 2x$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	7.0000000000	42.0000000000	12.0000000000	4.0000000000
1	4.0000000000	9.0000000000	8.0000000000	2.8750000000
2	2.8750000000	1.2656250000	5.7500000000	2.6548913043
3	2.6548913043	0.0484478379	5.3097826087	2.6457670442
4	2.6457670442	0.0000832521	5.2915340884	2.6457513111
5	2.6457513111	0.0000000003	5.2915026222	2.6457513111

$x \approx 2.6457513111$

$\sqrt{7} \approx 2.6457513111$ $(Calculator)$

3.5–4 Linear Approximation

A line tangent to a graph of a function f at a point $(x, f(x))$ is used to approximate the value of f at points near x.

3.5–5 *Definition*

Suppose f is differentiable on an interval I, containing the point a. The linear approximation to f at a is the linear function

$$L(x) = f(a) + f'(a)(x-a), \quad \textit{for } x \textit{ in } I$$

Tangent line at $(a, f(a))$:
$L(x) = f(a) + f'(a)(x-a)$

Example 3

Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$

Solution

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(1) + f'(1)(x-1) \\ &= 1 + \frac{1}{2}(x-1) \\ &= \frac{1}{2}(x+1) \end{aligned}$$

$$\begin{aligned} \sqrt{1.1} &\approx L(1.1) \\ &= \frac{1}{2}\left(\frac{11}{10} + 1\right) \end{aligned}$$

$$= \frac{21}{20}$$

$$= 1.05$$

The exact value $\sqrt{1.1} \approx 1.0488$

$$Error = \frac{1.05 - 1.0488}{1.0488}$$

$$= 0.001144 \quad (0.11\%)$$

The error in the approximation decreases as $x \to 1$

3.5–6 Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately.

$$\Delta y \approx f'(a)\Delta x \qquad \textit{where } a + \Delta x \textit{ is in } I.$$

3.5–7 Rolle's *Theorem*

Suppose that $y = f(x)$ is continuous at every point of the closed interval [a, b] and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least on number c in (a, b) at which $f'(c) = 0$.

Proof

Being continuous, f assumes absolute maximum and minimum values on [a, b]. These can occur only

1. At interior points where f' is zero,
2. At interior points where f' does not exist,
3. At the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.
If either maximum or the minimum occurs at a point c between a and b. then $f' = 0$.
If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b).

Example 4

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution

$f(x) = x^3 + 3x + 1$

$f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem the equation has one real solution in the open interval (–1, 0).

$f'(x) = 3x^2 + 3 > 0$ (***Always positive***).

Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero.

Therefore, f has no more than one zero.

3.5–8 The Mean Value *Theorem*

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Example 5

The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $\frac{4-0}{2-0} = 2$.

In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$.

However, it is not always easy to find c algebraically, even though we know it always exists.

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

3.5–9 Corollary

If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

$f(x) = x^2 + C$

Exercises *Section* 3.5 – Newton's Method & Linear Approximation

1. Use Newton's method to estimate a root of $x^3+3x+1=0$. Start with $x_0=0$ and then find x_2

2. Use Newton's method to estimate a root of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2.

3. Use Newton's method to estimate a root of $2x-x^2+1=0$. Start with $x_0=0$ for the left-hand zero and with $x_0=2$ for the zero on the right. Then, in each case, find x_2.

4. Use Newton's method to estimate a root of $x^4-2=0$. Start with $x_0=1$ and then find x_2.

5. Use Newton's method to estimate a root of $f(x)=x^3-3x+1$ in the interval $[1,\ 2]$. let $x_0=2$ and find x_5.

(6 – 13) Use the Newton's method to approximate the roots to ten digits of

6. $f(x)=3x^3-4x^2+1$

7. $f(x)=e^{-2x}+2e^x-6$

8. $f(x)=2x^5-6x^3-4x+2$

9. $f(x)=e^x$

10. $f(x)=x^3+2x+1$

11. $f(x)=x^2+1$

12. $f(x)=\sin x$

13. $f(x)=x^3+3xe^x$

14. Use the Newton's method to approximate $\sqrt{2}$.

15. Use the Newton's method to approximate $\sqrt{3}$.

16. Use the Newton's method to approximate $\sqrt{5}$.

17. A poorly chosen value of x_0 can lead to the unexpected results. The graph of $f(x)=x^3-\sin x$ indicates that there are three roots of $f(x)=0$: they are $x=0$ and two roots near $x=1$ *and* $x=-1$.

a) Verify by using Newton's method to approximates the known root $x = 0$ by using an initial value of $x = 0.49$. Calculate the approximation $x_1, x_2, x_3, \ldots$ until two consecutive values agree to 6 decimal places. What happens and why?

b) What happens if you use an initial value of $x = 0.4$?

c) What happens if you use an initial value of $x = 0.6$?

18. Let $f(x) = \dfrac{x}{x^2+1}$

a) Show that Newton's method takes the form $x_{n+1} = \dfrac{2x_n^3}{x_n^2 - 1}$

b) Let $x_0 = \dfrac{1}{\sqrt{3}}$ and then find the exact values of $x_1, x_2, x_3, x_4, x_5, \ldots$

c) Graph $f(x) = \dfrac{x}{x^2+1}$; $-1 \le x \le 1$; $-0.5 \le f(x) \le 0.5$ and illustrate how $x_1, x_2, x_3,$ *and* x_4 were found. Does Newton's method lead to an approximate solution to $\dfrac{x}{x^2+1} = 0$ if $x_0 = \dfrac{1}{\sqrt{3}}$? Why or why not?

Summary

Critical Number(s)

$f'(x)=0$ Solve for the variable.

Increasing and Decreasing Functions

Suppose that f is continuous on [*a*, *b*] and differentiable on (*a*, *b*).

- If $f'(x)>0$ for all *x* in (*a*, *b*), then *f* is increasing on (*a*, *b*)
- If $f'(x)<0$ for all *x* in (*a*, *b*), then *f* is decreasing on (*a*, *b*)
- If $f'(x)=0$ for all *x* in (*a*, *b*), then *f* is constant on (*a*, *b*)

Local Extrema

- If f' changes from negative to positive at *c*, then f has a local minimum (***LMIN***).
- If f' changes from positive to negative at *c*, then f has a local maximum (***LMAX***).
- If f' doesn't change sign at *c*, then f has no local extremum at *c*.

Concavity

Let *f* be function whose second derivative exists on an open interval *I*.

- If $f''(x)>0$ for all *x* in I, then *f* is ***concave up*** on I.
- If $f''(x)<0$ for all *x* in I, then *f* is ***concave down*** on I.

***L'Hôpital's* Rule** $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Exercises *Lecture 3 – Review*

(1 – 10) Determine all critical points of each function and find the extreme values (absolute and local) of the function and where they occur.

1. $f(x) = \sin 2x + 3$; $[-\pi, \pi]$

2. $f(x) = 2x^3 - 3x^2 - 36x + 12$

3. $f(x) = 4x^{1/2} - x^{5/2}$; $[0, 4]$

4. $f(x) = 2x \ln x + 10$; $(0, 4)$

5. $f(x) = x^{1/3}\left(9 - x^2\right)$; $[-4, 4]$

6. $f(x) = \dfrac{x^2 + 8}{x + 1}$; $[-5, 5]$

7. $f(x) = x^3 + 2x + \tan x$

8. $f(x) = \csc x + \cot x$

9. $f(x) = (x + 7)\sqrt[3]{11 - 3x}$

10. $f(x) = -x^3 + 6x^2 - 9x + 3$

(11 – 25) Find the critical numbers and the open intervals on which the function is increasing or decreasing and the extreme points.

11. $f(x) = x^3 - 3x + 2$

12. $y = x^3 - 3x^2 + 3$

13. $y = -x^3 + 6x^2 - 9x + 3$

14. $f(x) = \sqrt{9 - x^2}$

15. $f(x) = x\sqrt{3 - x}$

16. $f(x) = \dfrac{x^2}{x^2 + 4}$

17. $f(x) = \left(4 - x^2\right)^{2/3}$

18. $f(x) = x(x - 1)e^{-x}$

19. $f(x) = \dfrac{\cos \pi x}{1 + x^2}$ *on* $[-2, 2]$

20. $f(x) = \sin^2 x - 3x$

21. $f(x) = \dfrac{x^2 - x + 1}{x}$

22. $f(x) = \dfrac{3x}{x^2 + 3}$

23. $f(x) = \dfrac{x + 1}{x - 3}$

24. $f(x) = \dfrac{x^2 + 1}{x}$

25. $f(x) = \dfrac{x^3 + 2}{2x}$

(26 – 30) Find the point(s) of inflection.

26. $f(x) = \frac{1}{8}x^4 - x^3 - \frac{1}{2}x^2 + 5x$

27. $f(x) = x^3 + 3x^2 - 9x - 8$

28. $f(x) = x^3(8 - x)$

29. $f(x) = \sin^2 x - 3x - 5$

30. $f(x) = x^2\left(4 - x^2\right)$

$(\mathbf{31-40})$ Determine the intervals on which the graph of the function is concave upward or concave downward.

31. $f(x) = -x^3 + 6x^2 - 9x + 3$

32. $f(x) = x^2\left(2x^2 - 9\right)$

33. $f(x) = x^4 - 2x^3 + 1$

34. $f(x) = x - 3x^{2/3}$

35. $y = x\sqrt{3-x}$

36. $y = x\sqrt{4-x^2}$

37. $f(x) = \sin^2 x - 3x - 5$

38. $f(x) = \dfrac{x^2+x}{4-x^2}$

39. $f(x) = \dfrac{2x+5}{3x-1}$

40. $f(x) = e^{2x}$

$(\mathbf{41-52})$ Use L'Hôpital Rule to find

41. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

42. $\lim\limits_{x\to 1} \dfrac{\sin(n\pi x)}{\ln x}$

43. $\lim\limits_{x\to\infty} \left(\dfrac{x^3}{x^2-1} - \dfrac{x^3}{x^2+1}\right)$

44. $\lim\limits_{x\to 4} \dfrac{\sin^2(\pi x)}{e^{x-4}+3-x}$

45. $\lim\limits_{x\to 0} \dfrac{2^{-\sin x}-1}{e^x-1}$

46. $\lim\limits_{x\to\infty} \left(\dfrac{e^x+1}{e^x-1}\right)^{\ln x}$

47. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{\frac{1}{x}}$

48. $\lim\limits_{x\to 2\pi} \dfrac{x\sin x + x^2 - 4\pi^2}{x-2\pi}$

49. $\lim\limits_{y\to 2} \dfrac{y^2+y-6}{\sqrt{8-y^2}-y}$

50. $\lim\limits_{y\to\infty} \dfrac{y^2+y-6}{\sqrt{8-y^2}-y^2}$

51. $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}}{x^2}$

52. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

53. Given the rational function $f(x) = \dfrac{ax+b}{x^2-1}$

Find the values of a and b such that the function has a local extreme at $(3,\ 1)$.

Determine what type of extreme.

54. The sum of two nonnegative numbers is 36. Find the numbers if the sum of their square roots is to be as large as possible.

55. The sum of two nonnegative numbers is 36. Find the numbers if the differences of their square roots are to be as large as possible.

56. The sum of two nonnegative numbers is 20. Find the numbers

a) If the product of one number and the square root of the other is to be large as possible

b) If one number plus the square root of the other is to be as large as possible

57. A rectangular page will contain 64 in^2 of print. The margins at the top and bottom of the page are $\frac{3}{4}$ *inch* wide. The margins on each side are 1 *inch* wide.

What should the dimensions of the page be to minimize the amount of paper used?

58. The slant height of the cone is 6 *feet*. How large should the indicated angle be to maximize the cone's volume?

59. An open box is made from a 10-*in* by 16-*in* of tin by cutting a square from each corner and folding up the edges. What is the length of the sides of the squares?

60. A rectangular box with a square base, an open top, and a volume of $864\ in^3$ is to be constructed.

a) What dimensions of the box should be to minimize the surface area?

b) What is the minimum surface area?

61. A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 15 *feet* and a base with radius 5 *feet*.

What values of r and h will give the cylinder the largest possible volume?

62. Find the height and radius of the largest right circular cylinder that can be out in a sphere of radius $\sqrt{3}$

63. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

64. A customer has asked you to design an open-top rectangular stainless-steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

65. A right triangle has legs of length h and r and a hypotenuse of length 4. It is revolved about the leg of length h to sweep out a right circular cone. What values of h and r maximize the volume of the cone? (Volume of the cone $= \frac{1}{3}\pi r^2 h$)

66. Suppose the resident population P (in *millions*) of the United States can be modeled by

$$P(t) = 0.00000583t^3 + 0.005003t^2 + 0.13776t + 4.658; \quad -4 \le t \le 197,$$

where $t = 0$ corresponds to 1,800. Analytically find the minimum and maximum populations in the U.S. for $-4 \le t \le 197$

67. The number of milligrams x of a medication in the bloodstream t hours after a dose is taken can be modeled by $x(t) = \dfrac{2{,}000\,t}{t^2+3}$; $t > 0$. Find the maximum value of x. Round your answer to two decimal places

68. A rectangular box with a square base is to be formed from a square piece of paper with 42" sides. If a square piece with side a is cut from each corner of the paper and the sides are folded up to from an open box, the volume of the box is $V = (42-2x)^2 x$. What value of $\boldsymbol{x}$ will maximize the volume of the box?

69. Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 400 square *feet*.

70. How far from huge TV screen should you stand to view it best if the TV screen is 10 *feet* high and the bottom of it is 2 *feet* above your eyes level?

71. You are moving into a new apartment and notice there is a corner where the hallway narrows from ***a*** *feet* to ***b*** *feet*.

What is the longest item that can be carried horizontally around the corner?

72. A wire 100 inches long is to be cut into two pieces. One piece will be bent into a shape of a circle and the other into the shape of an equilateral triangle.

Where should the wire be cut in order that the sum of the areas of the circle and triangle is

a) Maximum,

b) Minimum?

73. You threw a ball diagonally into a river along a straight edge. The distance from the ball landing point *A*, which is perpendicular distance 32 *feet* from point *B* on the shore, where *B* is a distance 50 *feet* from you standing point *C* with your dog. Your dog can run $15\ ft/\sec$ on the ground and swim at $3\ ft/\sec$

How far should be your dog run before swimming to minimize the time it takes your dog to reach the ball?

Practice *Exam*

Find the ***critical numbers*** of the following function on the given intervals. Identify the ***absolute maximum*** and ***absolute minimum*** values (if possible).

1. $f(\theta)=\sin\theta,\quad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$

2. $f(x)=x^2-1\qquad -1\le x\le 2$

Find the ***critical numbers*** and the open intervals on which the function is ***increasing*** or ***decreasing***.

3. $f(x)=x\sqrt{4-x^2}$

4. $g(x)=x^4-4x^3+4x^2$

Find the ***critical numbers*** and identify the function's ***local extreme*** values.

5. $f(x)=\frac{12x}{x^2+1}$

6. $f(x)=-4x^3-8x^2+32$

Find the ***points of inflection*** and where the graph is ***concave up*** and ***down***.

7. $f(x)=\frac{1}{2}x^4-3x^2+4x+1$

8. $f(x)=-4x^3-8x^2+32$

Evaluate the following limits using l'Hôpital's *R*ule

9. $\lim\limits_{x\to 1}\frac{x^3-1}{4x^3-x-3}$

10. $\lim\limits_{x\to 1}\frac{x^a-1}{x^b-1}$

11. $\lim\limits_{x\to 0}\frac{\sin x-2}{7x^3}$

12. $\lim\limits_{x\to 0}\frac{e^{-2x}-1+2x}{x^2}$

13. $\lim\limits_{x\to\infty}\frac{2x^5-x+1}{5x^6+x}$

14. $\lim\limits_{x\to\infty}\left(1-\frac{3}{x}\right)^x$

15. Whenever a person coughs, the radius of the main air passages to the lungs decreases. We wish to determine at what radius the velocity of the expelled air will be maximum. Let v denote the function which gives the velocity of air through a particular air passage at a radius r of that passage. Under usual conditions, it has been found that the following relationship prevails:

$$v(r)=cr^2\left(r_0-r\right)$$

where c is a positive constant and r_0 is the radius of the air passage at atmosphere pressure.

16. A rectangular page is to contain 24 *squares inches* of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ *inches*, and the margins on the left and right are to be 1 *inch*. What should the ***dimensions*** of the page be so that the least amount of paper used.

17. A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way. (Volume of the cone $= \frac{1}{3}\pi r^2 h$)

18. A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 *inches*. Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume.

Lecture Four

Integrations

Section 4.1 – Antiderivatives

4.1–1 Definition

In the mathematical, engineering and sciences models are developed to assist in the understanding of physical phenomena. These models often yield an equation that contains a derivative of an unknown function.

Until now, we worked and studied the derivative:

$$f(x) = x^3 + C \qquad \Rightarrow \qquad f'(x) = 3x^2$$

The antiderivatives are the opposite of the derivatives, inverse derivative. We have the derivative and will try to find the function if exist. Not all derivatives have antiderivatives.

A Function F is an ***antiderivative*** of a function f on an interval I if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I.$$

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

Where C is an arbitrary constant.

4.1–2 Notation for Antiderivatives and indefinite *integrals*

The notation $$\int f(x)\, dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.

That is $F'(x) = f(x)$ for all x in the domain of f.

$$\int f(x)\,dx \qquad \textit{Indefinite integral}$$

4.1–3 Basic Integration Rules

$$\int k\,dx = kx + C$$

$$\int k f(x)\,dx = k\int f(x)\,dx$$

$$\int \left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int \left(f(x)-g(x)\right)dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Example 1

Find indefinite integral. $\int \frac{1}{x^2}\,dx$

Solution

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$$

$$= \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -\frac{1}{x} + C$$

Example 2

Find indefinite integral. $\int \sqrt[3]{x}\, dx$

Solution

$$\int \sqrt[3]{x}\, dx = \int x^{1/3}\, dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$= \frac{3}{4} x^{4/3} + C$$

or $$= \frac{3}{4} x \sqrt[3]{x} + C$$

Example 3

Evaluate $\int \left(4x^3 - 5x + 2\right) dx$

Solution

$$\int \left(4x^3 - 5x + 2\right) dx = \int 4x^3\, dx - \int 5x\, dx + \int 2\, dx$$

$$= 4\frac{x^4}{4} - 5\frac{x^2}{2} + 2x + C$$

$$= x^4 - \frac{5}{2}x^2 + 2x + C$$

Example 4

Evaluate $\int \left(x^2 - 2x + 5\right) dx$

Solution

$$\int \left(x^2 - 2x + 5\right) dx = \frac{x^3}{3} - x^2 + 5x + C$$

***Example* 5**

Evaluate $\displaystyle\int \left(\sqrt{x} - \sqrt[3]{x^2} + 2\right) dx$

Solution

$$\int \left(\sqrt{x} - \sqrt[3]{x^2} + 2\right) dx = \int \left(x^{1/2} - x^{2/3} + 2\right) dx$$

$$= \frac{2}{3} x^{3/2} - \frac{3}{5} x^{5/3} + 2x + C$$

$$= \frac{2}{3} x\sqrt{x} - \frac{3}{5} x\sqrt[3]{x^2} + 2x + C$$

4.1–4 Trigonometry Integrals

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C \qquad \int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C \qquad \int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C$$

***Example* 6**

Evaluate $\displaystyle\int \sin x\, dx$

Solution

$$\int \sin x\, dx = -\cos x + C$$

***Example* 7**

Evaluate $\displaystyle\int \cos 3x\, dx$

Solution

$$\int \cos 3x \, dx = \frac{1}{3}\sin 3x + C$$

Example 8

Evaluate $\int \sec^2 x \, dx$

Solution

$$\int \sec^2 x \, dx = \tan x + C$$

Example 9

Evaluate $\int \csc x \cot x \, dx$

Solution

$$\int \csc x \cot x \, dx = -\csc x + C$$

4.1–5 *Definition*

The ***natural logarithm*** is the function given by

$$\ln x = \int_1^x \frac{1}{t} \, dt, \quad x > 0$$

Zero width: $\ln 1 = \int_1^1 \frac{1}{t} dt = 0$

4.1–6 *Definition*

The ***number e*** is that number in the domain of the natural logarithm satisfying

$$\ln e = 1 \quad \textit{and} \quad \int_1^e \frac{1}{t} dt = 1$$

4.1–7 Other Indefinite Integrals

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \quad \rightarrow \quad \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \rightarrow \quad \int \frac{dx}{x} = \ln|x| + C$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \rightarrow \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{x^2 + a^2} \quad \rightarrow \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \rightarrow \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example 10

Evaluate $\displaystyle\int e^{-10x}\,dx$

Solution

$$\int e^{-10x}\,dx = -\frac{1}{10}e^{-10x} + C$$

Example 11

Evaluate $\displaystyle\int \frac{5}{x}\,dx$

Solution

$$\int \frac{5}{x}\,dx = 5\ln|x| + C$$

Example 12

Evaluate $\displaystyle\int \frac{4}{\sqrt{9-x^2}}\,dx$

Solution

$$\int \frac{4}{\sqrt{9-x^2}}\,dx = 4\sin^{-1}\left(\frac{x}{3}\right)+C \qquad a^2=9 \quad \rightarrow \quad a=3$$

Example 13

Evaluate $\displaystyle\int \frac{dx}{16x^2+1}$

Solution

$$\int \frac{dx}{16x^2+1} = \frac{1}{16}\int \frac{dx}{x^2+\frac{1}{16}}$$

$$= \frac{1}{16}\int \frac{dx}{x^2+\left(\frac{1}{4}\right)^2}$$

$$= \left(\frac{1}{16}\right)4\ \tan^{-1}(4x)+C$$

$$= \frac{1}{4}\ \tan^{-1}(4x)+C$$

Example 14

Evaluate $\displaystyle\int \left(\frac{2}{x}+x^3-\sin x\right)dx$

Solution

$$\int \left(\frac{2}{x}+x^3-\sin x\right)dx = 2\ln|x|+\frac{1}{4}x^4+\cos x+C$$

Example 15

Evaluate $\displaystyle\int \left(e^x+\frac{1}{2}x^2+\tan x\sec x\right)dx$

Solution

$$\int \left(e^x + \frac{1}{2}x^2 + \tan x \sec x\right) dx = e^x + \frac{1}{6}x^3 + \sec x + C$$

Example 16

Evaluate $\displaystyle\int \left(\frac{1}{x^2} + \frac{1}{2\sqrt{x}}\right) dx$

Solution

$$\int \left(\frac{1}{x^2} + \frac{1}{2\sqrt{x}}\right) dx = -\frac{1}{x} + \sqrt{x} + C$$

Example 17

Evaluate $\displaystyle\int \left(\theta - \cos 5\theta + \csc^2 \theta\right) d\theta$

Solution

$$\int \left(\theta - \cos 5\theta + \csc^2 \theta\right) d\theta = \frac{1}{2}\theta^2 + \frac{1}{5}\sin 5\theta - \cot \theta + C$$

Example 18

Evaluate $\displaystyle\int \cosh x \, dx$

Solution

$$\int \cosh x \, dx = \sinh x + C$$

Exercises *Section* 4.1 – Antiderivatives

(1 – 128) Find each indefinite integral.

1. $\int v^2\,dv$

2. $\int x^{1/2}\,dx$

3. $\int 4y^{-3}\,dy$

4. $\int \left(x^3-4x+2\right)dx$

5. $\int \left(3z^2-4z+5\right)dz$

6. $\int \left(x^2-1\right)^2 dx$

7. $\int \frac{x^2+1}{\sqrt{x}}\,dx$

8. $\int \left(\sqrt[4]{x^3}+1\right)dx$

9. $\int \sqrt{x}\,(x+1)\,dx$

10. $\int (1+3t)\,t^2\,dt$

11. $\int \frac{x^2-5}{x^2}\,dx$

12. $\int (-40x+250)\,dx$

13. $\int \frac{x+2}{\sqrt{x}}\,dx$

14. $\int \left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx$

15. $\int \left(\sqrt{x}+\sqrt[3]{x}\right)dx$

16. $\int 2x\left(1-x^{-3}\right)dx$

17. $\int \left(\frac{4+\sqrt{t}}{t^3}\right)dt$

18. $\int (-2\cos t)\,dt$

19. $\int 7\sin\frac{\theta}{3}\,d\theta$

20. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

21. $\int \left(4\sec x\tan x-2\sec^2 x\right)dx$

22. $\int (2\cos 2x-3\sin 3x)\,dx$

23. $\int \left(1+\tan^2\theta\right)d\theta$

24. $\int \frac{\csc\theta}{\csc\theta-\sin\theta}\,d\theta$

25. $\int \left(2e^x-3e^{-2x}\right)dx$

26. $\int \frac{dx}{\sqrt{9-x^2}}$

27. $\int \frac{dx}{9+3x^2}$

28. $\int \frac{4x^2-3x+2}{x^2}\,dx$

29. $\int \left(x^8-3x^3+1\right)dx$

30. $\int (2x+1)^2\,dx$

31. $\int \frac{x+1}{x}\,dx$

32. $\int \left(\frac{1}{x^2}-\frac{2}{x^{5/2}}\right)dx$

33. $\int \frac{x^4-2\sqrt{x}+2}{x^2}\,dx$

34. $\int (1+\cos 3\theta)\,d\theta$

35. $\int 2\sec^2\theta\,d\theta$

36. $\int \sec 2x\tan 2x\,dx$

37. $\int 2e^{2x}dx$

38. $\int \frac{12}{x}\,dx$

39. $\int \frac{dx}{\sqrt{1-x^2}}$

40. $\displaystyle\int \frac{dx}{x^2+1}$

41. $\displaystyle\int \frac{1+\tan\theta}{\sec\theta}\,d\theta$

42. $\displaystyle\int \left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx$

43. $\displaystyle\int \left(x^{-3}+7e^{5x}+\frac{4}{x}\right)dx$

44. $\displaystyle\int \left(\frac{2}{x}+\frac{x}{2}\right)dx$

45. $\displaystyle\int \frac{1}{ax}\,dx$

46. $\displaystyle\int x\sqrt{x}\,dx$

47. $\displaystyle\int \left(\frac{2}{\sqrt{x}}+2\sqrt{x}\right)dx$

48. $\displaystyle\int \left(x-2x^2+\frac{1}{2x}\right)dx$

49. $\displaystyle\int \left(\frac{7}{2x^3}-\sqrt[3]{x}\right)dx$

50. $\displaystyle\int 3e^{-2x}\,dx$

51. $\displaystyle\int e^{-x}\,dx$

52. $\displaystyle\int e\,dx$

53. $\displaystyle\int \frac{7}{2e^{2x}}\,dx$

54. $\displaystyle\int -3\left(e^{2x}+1\right)dx$

55. $\displaystyle\int \left(-3e^{-x}+2x-\frac{1}{2}e^{5x}\right)dx$

56. $\displaystyle\int \left(\sqrt[4]{x^3}+1\right)dx$

57. $\displaystyle\int \left(5x^4+3x^2+2x+5\right)dx$

58. $\displaystyle\int \left(5x^{4/3}+3x^{2/3}+2x^{1/3}\right)dx$

59. $\displaystyle\int \left(5x^{-4/3}+3x^{-2/3}+2x^{-1/3}\right)dx$

60. $\displaystyle\int \frac{x^4-3x^2+5}{x^4}\,dx$

61. $\displaystyle\int \left(\frac{3}{x^7}-\frac{5}{x^6}\right)dx$

62. $\displaystyle\int \frac{x+8}{\sqrt{x}}\,dx$

63. $\displaystyle\int \frac{x^2+8}{\sqrt[3]{x}}\,dx$

64. $\displaystyle\int \cos\left(\frac{5\pi}{3}x\right)dx$

65. $\displaystyle\int \sin\left(\frac{2x}{3}\right)dx$

66. $\displaystyle\int \left(5\cos x+4\sin x+3\sec^2 x\right)dx$

67. $\displaystyle\int \sec\theta\left(\sec\theta+\tan\theta\right)d\theta$

68. $\displaystyle\int \left(\tan^2\theta+1\right)d\theta$

69. $\int \left(\cos^4\theta - \sin^4\theta\right)\, d\theta$

70. $\int \left(\cos^2\theta - \sin^2\theta\right)\, d\theta$

71. $\int \left(\cos^2\theta + \sin^2\theta\right)\, d\theta$

72. $\int \left(\cos 2x\, \cos 4x - \sin 2x\, \sin 4x\right)\, dx$

73. $\int \left(\sin 2x\, \cos 4x - \cos 2x\, \sin 4x\right)\, dx$

74. $\int \left(\sin 3x\, \cos 2x + \cos 3x\, \sin 2x\right)\, dx$

75. $\int \cos 2x\, \sin 2x\, dx$

76. $\int \left(2\cos^2 x - 1\right)\, dx$

77. $\int \left(1 - 2\sin^2 x\right)\, dx$

78. $\int e^{-5x}\, dx$

79. $\int 4e^{4x}\, dx$

80. $\int \left(2\sin\theta - 5e^{\theta}\right)\, d\theta$

81. $\int \left(\frac{3}{x} + \sec^2 x\right)\, dx$

82. $\int \left(\sin x + 2^x\right)\, dx$

83. $\int \left(2x - 3^x\right)\, dx$

84. $\int \left(4x - \frac{3}{x} - \csc^2 x\right)\, dx$

85. $\int \left(e^{4x} - \frac{3}{x} + 2\csc x \cot x\right)\, dx$

86. $\int (a+b)\, e^{(a+b)x}\, dx$

87. $\int \left(a^2 - b^2\right) e^{(a-b)x}\, dx$

88. $\int x\, dt$

89. $\int \pi\, dt$

90. $\int r\cos t\, dt$

91. $\int (20t + 5)\, dt$

92. $\int \frac{t-1}{t^{3/2}}\, dt$

93. $\int \left(\frac{1}{t^2} + \frac{1}{t} - e^t\right) dt$

94. $\int \frac{dx}{x^2 + 3}$

95. $\int (13 - x)\, dx$

96. $\int \frac{dx}{\sqrt{5 - x^2}}$

97. $\int \sinh x\, dx$

98. $\int \left(9x^2 - 4x + 3\right) dx$

99. $\int \left(\frac{1}{x^3} - \frac{2}{x^2}\right) dx$

100. $\int 5\pi \, dx$

101. $\int (x+7) \, dx$

102. $\int \left(2x - 3x^2\right) dx$

103. $\int \left(8x^3 - 9x^2 + 4\right) dx$

104. $\int \left(x^5 - 4\right) dx$

105. $\int \left(6x^{3/2} - 7x + 2\right) dx$

106. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$

107. $\int \left(\sqrt[3]{x^2}\right) dx$

108. $\int \left(4\sqrt[4]{x^3} - 9x^3\right) dx$

109. $\int \frac{x+6}{\sqrt{x}} \, dx$

110. $\int \frac{dx}{9+x^2}$

111. $\int \frac{1}{2} \frac{dx}{x}$

112. $\int \frac{x^2+2x-3}{x^3} \, dx$

113. $\int \left(2x^2 - 1\right)^2 dx$

114. $\int (1+3t) \, t^2 \, dt$

115. $\int t^2 \sqrt{t} \, dt$

116. $\int (5\cos x + 4\sin x) \, dx$

117. $\int \left(x^2 - \cos x\right) dx$

118. $\int (1 - \csc x \cot x) \, dx$

119. $\int \left(\theta^2 + \sec^2 \theta\right) d\theta$

120. $\int \left(\sec^2 \theta - \sin \theta\right) d\theta$

121. $\int \frac{\cos x}{1 - \cos^2 x} \, dx$

122. $\int 5e^{5x} \, dx$

123. $\int e^{-2x} \, dx$

124. $\int \left(x^3 + 1\right)\left(2x^2 + 8x - 5\right) dx$

125. $\int \left(x - \frac{1}{x}\right)^2 dx$

126. $\int (3x+1)(2x-1) \, dx$

127. $\int (2x+1)(2x-1) \, dx$

128. $\int \frac{x^3 + 3x^2 - 9x - 2}{x-2} \, dx$

(129 – 136) Find the function with the following property

129. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

130. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \quad x > 0$

131. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

132. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

133. $f'(x) = 3x^2 - 1 \quad \& \quad f(0) = 10$

134. $f'(t) = \sin t + 2t \quad \& \quad f(0) = 5$

135. $f'(x) = x^2 + x^{-2} \quad \& \quad f(1) = 1$

136. $f'(x) = \sin^2 x \quad \& \quad f(1) = 1$

137. Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

138. Derive the position function if a ball is thrown upward with initial velocity of 32 *feet* per second from an initial height of 48 *feet*. When does the ball hit the ground? With what velocity does the ball hit the ground?

139. Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

Section 4.2 – Area Under Curve

The ***definite integral*** is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths, and more…

4.2–1 *Area*

To find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$.

$$Area \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}$$

$$\approx 0.875$$

$$Area \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4}$$

$$\approx 0.78125$$

$$Area \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4}$$

$$\approx 0.53125$$

$$Area \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4}$$

$$\approx 0.671875$$

In each case of the computations, the interval $[a, b]$ over which the function f is defined was subdivided into n equal subintervals (also called ***length***) $\Delta x = \frac{b-a}{n}$, and f was evaluated at a point in each subinterval. The finite sums can be given by the form:

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x$$

4.2–2 Distance Traveled

The distance formula is given by: $distance = velocity \times time$

Example 1

The velocity function of a projectile fired straight up into the air is $f(t) = 160 - 9.8t \ m/\sec$. Use the summation technique to estimate how far the projectile rises during the first 3 *sec*. How close do the sums come to the exact value of 435.9 *m*?

Solution

i. $\Delta t = 1 \text{ sec} \quad \rightarrow \quad t = 0,\ 1,\ 2$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= f(0)\Delta t + f(1)\Delta t + f(2)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(1))(1) + (160 - 9.8(2))(1)$$
$$= 450.6 \quad m$$

ii. $\Delta t = 1 \text{ sec} \quad \rightarrow \quad t = 1,\ 2,\ 3$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= (160 - 9.8(1))(1) + (160 - 9.8(2))(1) + (160 - 9.8(3))(1)$$
$$= 421.2 \quad m$$

iii. $\Delta t = 0.5 \text{ sec} \quad \rightarrow \quad t = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1)$$
$$+ (160 - 9.8(2))(1) + (160 - 9.8(2.5))(1)$$
$$\approx 443.25 \quad m$$

iv. $\Delta t = 0.5 \text{ sec} \quad \rightarrow \quad t = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) + (160 - 9.8(2))(1)$$
$$+ (160 - 9.8(2.5))(1) + (160 - 9.8(3))(1)$$
$$\approx 428.55 \quad m$$

The true value is 435.9 if you use more subintervals $\Delta t = 0.25\ sec$, the interval 436.13 & 435.67.

The projectile rose about 436 m during the first 3 sec of flight.

4.2–3 Average Value of a Nonnegative Continuous Function

The average value of a collection of n numbers $x_1, x_2, \cdots, x_n$ is obtained by adding them together and dividing by n.

Example 2

Estimate the average value of the function $f(x) = \sin x$ on the interval [0, π].

Solution

To get the upper sum approximation with 8 rectangles of equal width $\Delta x = \frac{\pi}{8}$.

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) \cdot \frac{\pi}{8}$$

$$\approx 2.365\ unit^2$$

4.2–4 Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter $\sum$ (capital ***sigma***, corresponding to our letter *S*)

Example 3

Sigma Notation	*Written*	*Value of the Sum*
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{4} (-1)^k \cdot k$	$(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4$	$-1+2-3+4 = \underline{2}$
$\sum_{k=1}^{3} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$	$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \underline{\frac{23}{12}}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \underline{\frac{139}{12}}$

Example 4

We can write:

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2 = \sum_{k=1}^{10} k^2$$

Example 5

Express the sum: 1+ 3 +5 +7 + 9 in sigma notation.

Solution

Starting with $k=0$: $1+3+5+7+9 = \sum_{k=0}^{4} (2k+1)$

Starting with $k=1$: $1+3+5+7+9 = \sum_{k=1}^{5} (2k-1)$

4.2–5 *Theorem* on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer ***n***,

$$(1) \quad \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$(2) \quad \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$(3) \quad \sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$$

$$(4) \quad \sum_{k=1}^{n} c = n \cdot c$$

Proof

$$\sum_{k=1}^{n}\left(a_k+b_k\right)=\left(a_1+b_1\right)+\left(a_2+b_2\right)+\ldots+\left(a_n+b_n\right)$$

$$=\left(a_1+a_2+\ldots+a_n\right)+\left(b_1+b_2+\ldots+b_n\right)$$

$$=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

***Example* 6**

(1) $$\sum_{k=1}^{n}(k+4)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}4$$

$$=\sum_{k=1}^{n}k+4\cdot n$$

(2) $$\sum_{k=1}^{n}\left(3k-k^2\right)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$$

(3) $$\sum_{k=1}^{n}\left(-a_k\right)=\sum_{k=1}^{n}(-1)\cdot\left(a_k\right)$$

$$=(-1)\cdot\sum_{k=1}^{n}\left(a_k\right)$$

$$=-\sum_{k=1}^{n}a_k$$

(4) $$\sum_{k=1}^{n}\frac{1}{n}=n\cdot\frac{1}{n}$$

$$=1$$

Example 7

Show that the sum of the first n integers is $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Solution

The sum of the first 4 integers is: $\displaystyle\sum_{k=1}^{4} k = \frac{4(5)}{2}$

$$= 10$$

To prove the formula in general:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\ \hline n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 \end{array} \rightarrow n(n+1)$$

Since it is twice the desired quantity, the sum of the first n integers is $\dfrac{n(n+1)}{2}$

$$\boxed{\begin{aligned} \sum_{k=1}^{n} k^2 &= 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2 \end{aligned}}$$

4.2–6 Limits of Finite Sums

Example 8

Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval [0, 1] on the x-axis using equal-width rectangles whose width approach zero and whose number approaches infinity.

Solution

The lower sum approximation using n rectangles of equal width: $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$

By subdividing the interval [0, 1] into n equal width subintervals:

$$[0,\ 1]=\left[0,\ \frac{1}{n}\right],\ \left[\frac{1}{n},\ \frac{2}{n}\right],\ \left[\frac{2}{n},\ \frac{3}{n}\right],\ \ldots,\ \left[\frac{n-1}{n},\ \frac{n}{n}\right]=\left[\frac{k-1}{n},\ \frac{k}{n}\right]$$

$$f\left(\frac{k}{n}\right)=1-\left(\frac{k}{n}\right)^2$$

$$\left[f\left(\frac{1}{n}\right)\right]\cdot\left(\frac{1}{n}\right)+\left[f\left(\frac{2}{n}\right)\right]\cdot\left(\frac{1}{n}\right)+\cdots+\left[f\left(\frac{n}{n}\right)\right]\cdot\left(\frac{1}{n}\right)$$

We can write this in sigma notation:

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\left(\frac{1}{n}\right)=\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}\right)^2\right]\cdot\left(\frac{1}{n}\right)$$

$$=\sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right)$$

$$=\sum_{k=1}^{n}\frac{1}{n}-\sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$=n\cdot\frac{1}{n}-\frac{1}{n^3}\sum_{k=1}^{n}k^2$$

$$=1-\frac{1}{n^3}\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}$$

$$=1-\frac{2n^3+3n^2+n}{6n^3}$$

$$=\frac{6n^3-2n^3-3n^2-n}{6n^3}$$

$$=\frac{4n^3-3n^2-n}{6n^3}$$

$$\lim_{x\to\infty}\left(\frac{4n^3-3n^2-n}{6n^3}\right)=\frac{4}{6}\ \ =\frac{2}{3}$$

The lower sum approximation converges to $\frac{2}{3}$

The upper sum approximation also converges to $\frac{2}{3}$

4.2–7 *Review*

4.2–8 *Definition* of Arithmetic Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence if there is a real number ***d*** such that for every positive integer ***k***,

$$a_{k+1} = a_k + d$$

The number $d = a_{k+1} - a_k$ is called the ***common difference*** of the sequence.

The nth Term of an Arithmetic Sequence: $\boxed{a_n = a_1 + (n-1)d}$

Example 9

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

Number of terms: $n = 5$

Difference in terms: $d = 11 - 4 = 7$

$$a_n = a_1 + (n-1)d$$

$$\begin{aligned} a_n &= 4 + (n-1)7 \\ &= 4 + 7n - 7 \\ &= \underline{7n - 3} \end{aligned}$$

$$4+11+18+25+32 = \sum_{n=1}^{5} (7n-3)$$

4.2–9 *Theorem*

Formulas for S_n

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference ***d***, then the *n*th partial sum S_n (that is, the sum of the first ***n*** terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad or \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

4.2–10 *Definition* of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the nth Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: $6,\ -12,\ 24,\ -48,\ \ldots,\ (-2)^{n-1}(6),\ \ldots$ is $= \frac{-12}{6} = -2$

Example 10

Express the sum in terms of summation notation (Answers are not unique.)

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$$

Solution

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} = \frac{1}{4} - \frac{1}{4}\frac{1}{3^1} + \frac{1}{4}\frac{1}{3^2} - \frac{1}{4}\frac{1}{3^3}$$

$$= \sum_{n=1}^{4} (-1)^{n+1} \frac{1}{4} \left(\frac{1}{3}\right)^{n-1}$$

4.2–11 *Theorem*: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$\boxed{S_n = a_1 \frac{1 - r^n}{1 - r}}$$

4.2–12 *Riemann* Sums

The theory of limits of finite approximations was made precise by the German mathematician ***Bernhard Riemann***.

(1826 – 1866)

Moved from Gottingen to Berlin in 1846 to study under Jacobi, Dirichlet and Eisenstein. In 1849, he returned to Gottingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851. Riemann On Gauss's recommendation Riemann was appointed to a post in Gottingen. Riemann's paper Uber die Hypotheses welche der Geometrie zu Grunde Liegen, written in 1854, became a classic of mathematics, and its results were incorporated into Albert Einstein's relativistic theory of gravitation.

We introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral.

Let a closed interval [*a*, *b*] be partitioned by points $a < x_1 < x_2 < \cdots < x_{n-1} < b$

To make the notation consistent, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The set: $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$ is called a partition of [*a*, *b*].

The partition *P* divides [*a*, *b*] into *n* closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

These products are:

$$S_P = \sum_{k=1}^{n} f\left(c_k\right)\Delta x_k$$

The sum S_P is called a ***Riemann sum*** *for f on the interval* [*a*, *b*], and c_k in the subintervals.

If we choose *n* subintervals all having equal width $\Delta x = \frac{b-a}{n}$ to partition [*a*, *b*], then choose the point c_k to be the right-hand endpoints of each subintervals when forming the Riemann sum. This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\cdot\left(\frac{b-a}{n}\right)$$

Example 11

The set $P = \{0,\ 0.2,\ 0.6,\ 1,\ 1.5,\ 2\}$ is a partition of [0, 2]

Solution

There are five subintervals of *P*: [0, 0.2], [0.2, 0.6], [0.6, 1], [1, 1.5], and [1.5, 2]

The lengths of the subintervals are:

$$\Delta x_1 = 0.2 \quad \Delta x_2 = 0.4 \quad \Delta x_3 = 0.4 \quad \Delta x_4 = 0.5 \quad \Delta x_5 = 0.5$$

The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$

Exercises *Section* 4.2 – Area Under Curve

(1 – 2) Use finite approximations to estimate the area under the graph of the function using

a) A lower sum with two rectangles of equal width.
b) A lower sum with four rectangles of equal width.
c) A upper sum with two rectangles of equal width.
d) A upper sum with four rectangles of equal width.

1. $f(x) = \frac{1}{x}$ *between* $x = 1$ *and* $x = 5$

2. $f(x) = 4 - x^2$ *between* $x = -2$ *and* $x = 2$

3. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t) = \frac{1}{2} + \sin^2 \pi t \quad on \quad [0,\ 2]$$

4. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$

(5 – 8) Write the sums without sigma notation. Then evaluate them:

5. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **6.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$ **7.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$ **8.** $\displaystyle\sum_{k=1}^{4} (-1)^k \cos k\pi$

9. Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

10. Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

11. Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

12. Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

13. Suppose that $\sum_{k=1}^{n} a_k = -5$ *and* $\sum_{k=1}^{n} b_k = 6$. Find the value of $\sum_{k=1}^{n}\left(b_k - 2a_k\right)$

$(\mathbf{14}-\mathbf{21})$ Evaluate the sums

14. $\sum_{k=1}^{10} k^3$

15. $\sum_{k=1}^{7} (-2k)$

16. $\sum_{k=1}^{5} \frac{\pi k}{15}$

17. $\sum_{k=1}^{5} k(3k+5)$

18. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

19. $\sum_{k=1}^{500} 7$

20. $\sum_{k=18}^{71} k(k-1)$

21. $\sum_{k=1}^{n} \left(\frac{1}{n}+2n\right)$

22. Graph the function $f(x) = x^2 - 1$ over the given interval $[0,\ 2]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$, given c_k is the

a) Left-hand endpoint

b) Right-hand endpoint

c) Midpoint of kth subinterval.

(Make a separate sketch for each set of rectangles.)

Section 4.3 – Definite Integral

4.3–1 *Definition*

Let $f(x)$ be a function defined on a closed interval [*a*, *b*]. We say that a number *J* is the ***definite integral* of *f* over [*a*, *b*]** and that *J* is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

> Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of [a, b] with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon$$

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums.

Integral of *f* from *a* to *b*.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)dx$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)dx$$

4.3–2 *Theorem* – Integrability of Continuous Functions

If a function f is continuous over the interval $[a,\ b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f in integrable over $[a,\ b]$

4.3–3 Properties of Definite Integrals

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \int_a^a f(x)dx = 0$$

4.3–4 *Theorem*

When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules:

Order of Integration: $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Zero Width Interval: $$\int_a^a f(x)dx = 0$$

Constant Multiple: $$\int_a^b k f(x)dx = k\int_a^b f(x)dx$$

Sum and Difference: $$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Additivity: $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Max-Min Inequality: If f has ***maximum*** value *max* f and ***minimum*** value *min* f on $[a, b]$, then

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$

Domination: $f(x) \ge g(x)$ *on* $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$$

Zero Width Interval:

$$\int_a^a f(x)\,dx = 0$$

Constant Multiple: ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

Sum: (*Areas add*)

$$\int_a^b \big(f(x)+g(x)\big)\,dx$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Additive for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Max-Min Inequality:	***Domination***
$\min f \cdot (b-a) \le \int_a^b f(x)\,dx$ $\le \max f \cdot (b-a)$	$f(x) \ge g(x)$ *on* $[a,b]$ $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Example 1

Suppose that $\int_{-1}^{1} f(x)\,dx = 5,\quad \int_{1}^{4} f(x)\,dx = -2,\quad \int_{-1}^{1} h(x)\,dx = 7.$

Find:

a) $\int_{4}^{1} f(x)\,dx$

b) $\int_{-1}^{1} \left(2f(x) + 3h(x)\right) dx$

Solution

a) $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx$

$= -(-2)$

$= 2$

b) $\int_{-1}^{1} \left[2f(x) + 3h(x)\right] dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

$= 2(5) + 3(7)$

$= 31$

***Example* 2**

Show that the value of $\int_0^1 \sqrt{1+\cos x}\ dx$ is less than or equal to $\sqrt{2}$

Solution

$\min f \cdot (b-a)$: is the lower bound

$\max f \cdot (b-a)$: is the upper bound

The maximum value of $\sqrt{1+\cos x}$ on $[0,\ 1]$ is $\sqrt{1+1}=\sqrt{2}$

So, $\int_0^1 \sqrt{1+\cos x}\ dx \ \le\ \sqrt{2}\cdot(1-0)\ =\sqrt{2}$

4.3–5 Mean Value Theorem for Definite Integrals

If f is continuous on $[a,\ b]$, then some point c in $[a,\ b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

4.3–6 *Theorem* – The Fundamental Theorem of Calculus, P–1

If f is continuous on $[a,\ b]$, then $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,\ b]$, and differentiable on $(a,\ b)$ and its derivative is $f(x)$:

$$F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

4.3–7 *Theorem* – The Fundamental Theorem of Calculus, P–2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Example 3

Evaluate the definite integral $\displaystyle\int_0^{\pi} \cos x\, dx$

Solution

$$\int_0^{\pi} \cos x\, dx = \sin x \Big|_0^{\pi}$$
$$= \sin \pi - \sin 0$$
$$= 0$$

Example 4

Evaluate the definite integral $\displaystyle\int_{-\frac{\pi}{4}}^{0} \sec x \tan x\, dx$

Solution

$$\int_{-\frac{\pi}{4}}^{0} \sec x \tan x\, dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$$
$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$
$$= 1 - \sqrt{2}$$

Example 5

Evaluate the definite integral $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

Solution

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = x^{3/2} + \frac{4}{x}\Big|_1^4$$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$

$$= (9) - (5)$$

$$= 4$$

4.3–8 Definite Integrals of Symmetric Functions

Theorem

Let f be continuous on the symmetric interval $[-a, a]$

- ✓ If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- ✓ If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Example 6

Evaluate $\int_{-2}^{2} \left(x^4 - 4x^2 + 6\right) dx$

Solution

Since $f(-x) = f(x) \Rightarrow f(x)$ *is even*

$$\int_{-2}^{2} \left(x^4 - 4x^2 + 6\right) dx = 2\int_0^2 \left(x^4 - 4x^2 + 6\right) dx$$

$$= 2\left(\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right|_0^2$$

$$= 2\left[\left(\frac{2^5}{5} - \frac{4}{3}2^3 + 6(2)\right) - 0\right]$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right)$$

$$= \frac{232}{15}$$

Example 7

Evaluate $\displaystyle\int_{-4}^{4}\left(x^5 - 4x^3 + 6x\right)dx$

Solution

Since $f(-x) = -f(x) \quad \Rightarrow \quad f(x)$ *is odd*

$$\int_{-4}^{4}\left(x^5 - 4x^3 + 6x\right)dx = 0$$

4.3–9 *Area* Under the Graph of a Nonnegative Function

Definition

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$Area = \int_a^b f(x)\,dx$$

Example 8

Compute $\displaystyle\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Solution

To Compute the definite integral, we consider the partition P subdivides the interval $[0, b]$ into n subintervals of equal width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.

$$P = \left\{0,\ \frac{b}{n},\ \frac{2b}{n},\ \frac{3b}{n},\ \ldots,\ \frac{nb}{n}\right\} \quad \textit{and} \quad c_k = \frac{kb}{n}$$

$$\begin{aligned}
\sum_{k=1}^{n} f\left(c_k\right)\Delta x &= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n} \\
&= \sum_{k=1}^{n} \frac{kb^2}{n^2} \\
&= \frac{b^2}{n^2}\sum_{k=1}^{n} k \\
&= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2} \\
&= \frac{b^2}{2}\left(1+\frac{1}{n}\right)
\end{aligned}$$

$$\lim_{n\to\infty}\left[\frac{b^2}{2}\left(1+\frac{1}{n}\right)\right] = \frac{b^2}{2}$$

$$\int_0^b x\,dx = \frac{b^2}{2}$$

$$A = \int_0^b x\,dx = \frac{b^2}{2}$$

$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx \quad \int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad a < b$$

$$= -\int_0^a x\,dx + \int_0^b x\,dx$$

$$= -\frac{a^2}{2} + \frac{b^2}{2}$$

$$= \frac{1}{2}\left(b^2 - a^2\right)$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} \qquad a < b$$

Example 9

Shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ are reflected across the x-axis. For each function, compute

a) The definite integral over the interval [–2, 2]

b) The area between the graph and the x-axis over [–2, 2]

Solution

a) $$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \left(x^2 - 4\right)dx$$

$$= 2\left(\frac{x^3}{3} - 4x\right|_0^2$$

$$= 2\left(\frac{8}{3} - 8\right)$$

$$= -\frac{32}{3}$$

$$\int_{-2}^{2} g(x)\,dx = \int_{-2}^{2} \left(4 - x^2\right)dx$$

$$= 2\left(4x - \frac{x^3}{3}\right|_0^2$$

$$= 2\left(8 - \frac{8}{3}\right)$$

$$= \frac{32}{3}$$

b) In both cases, the area between the curve and the x-axis over [–2, 2] is $\frac{32}{3}$ ***unit***^{2}.

Example 10

Shows the graph of $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

a) The definite integral of $f(x)$ over [0, 2π]

b) The area between the graph and the *x*-axis over [0, 2π]

Solution

a) $$\int_0^{2\pi} \sin x\, dx = -\cos x \Big|_0^{2\pi}$$

$$= -(\cos 2\pi - \cos 0)$$

$$= -(1-1)$$

$$= 0$$

b) The area between the graph and the axis is obtained by adding the absolute values

$$Area = \left| \int_0^{\pi} \sin x\, dx \right| + \left| \int_{\pi}^{2\pi} \sin x\, dx \right|$$

$$= \left|-\cos x\right| \Big|_0^{\pi} + \left|-\cos x\right| \Big|_{\pi}^{2\pi}$$

$$= \left|-(\cos \pi - \cos 0)\right| + \left|-(\cos 2\pi - \cos \pi)\right|$$

$$= \left|-(-1-1)\right| + \left|-(1-(-1))\right|$$

$$= |2| + |-2|$$

$$= 4 \ \ unit^2$$

Summary

To find the area between the graph of $y = f(x)$ and the *x*-axis over the interval [*a*, *b*]:

1. Subdivide [*a*, *b*] at the zeros of *f*.
2. Integrate *f* over each subinterval.
3. Add the absolute values of the integrals.

Example 11

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2$

Solution

The zeros of: $f(x) = x^3 - x^2 - 2x = 0$

$x\left(x^2 - x - 2\right) = 0$

$\underline{x = 0,\ -1,\ 2}$

$$\int_{-1}^{0} \left(x^3 - x^2 - 2x\right) dx = \frac{x^4}{4} - \frac{x^3}{3} - x^2 \Big|_{-1}^{0}$$

$$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)$$

$$= -\left(\frac{7}{12}\right)$$

$$\underline{= \frac{5}{12}}$$

$$\int_{0}^{2} \left(x^3 - x^2 - 2x\right) dx = \frac{x^4}{4} - \frac{x^3}{3} - x^2 \Big|_{0}^{2}$$

$$= \frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2 - 0$$

$$= 4 - \frac{8}{3} - 4$$

$$\underline{= -\frac{8}{3}}$$

$$Area = \int_{-1}^{0} \left(x^3 - x^2 - 2x\right) dx - \int_{0}^{2} \left(x^3 - x^2 - 2x\right) dx$$

$$= \frac{5}{12} - \left(-\frac{8}{3}\right)$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$\underline{= \frac{37}{12}\ unit^2}$$

Example 12

Find the area of the region between the x-axis and the graph of $f(x) = 2x^2 - 5x + 3, \quad -1 \le x \le 2$

Solution

$$2x^2 - 5x + 3 = 0$$

$$x_{1,2} = 1, \ \frac{3}{2}$$

Since **0** is not part of the solution, then we can use zero to test the sign in front the integral.

$$Area = +\int_{-1}^{1}\left(2x^2 - 5x + 3\right)dx - \int_{1}^{\frac{3}{2}}\left(2x^2 - 5x + 3\right)dx + \int_{\frac{3}{2}}^{2}\left(2x^2 - 5x + 3\right)dx$$

$$= \frac{2}{3}x^3 - \frac{5}{2}x^2 + 3x \Big|_{-1}^{1} - \left(\frac{2}{3}x^3 - \frac{5}{2}x^2 + 3x\right)\Big|_{1}^{\frac{3}{2}} + \left(\frac{2}{3}x^3 - \frac{5}{2}x^2 + 3x\right)\Big|_{\frac{3}{2}}^{2}$$

$$= \left(\frac{2}{3} - \frac{5}{2} + 3\right) - \left(-\frac{2}{3} - \frac{5}{2} - 3\right) - \left(\frac{9}{4} - \frac{45}{8} + \frac{9}{2} - \left(\frac{2}{3} - \frac{5}{2} + 3\right)\right) + \frac{16}{3} - 10 + 6 - \left(\frac{9}{4} - \frac{45}{8} + \frac{9}{2}\right)$$

$$= \frac{4 - 15 + 18}{6} + \frac{4 + 15 + 18}{6} - \frac{18 - 45 + 36}{8} + \frac{4 - 15 + 18}{6} + \frac{16 - 12}{3} - \frac{18 - 45 + 36}{8}$$

$$= \frac{7}{6} + \frac{37}{6} - \frac{9}{8} + \frac{7}{6} + \frac{4}{3} - \frac{9}{8}$$

$$= \frac{59}{6} - \frac{9}{4}$$

$$= \frac{91}{12} \ unit^2$$

4.3–10 *Theorem* – The Net Change Theorem

The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of tis rate of change:

$$F(b) - F(a) = \int_{a}^{b} F'(x)\,dx$$

Example 13

Consider the analysis of a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time t during its motion was given as $v(t) = 160 - 32t$ $ft/\sec$

a) Find the displacement of the rock during the time period $0 \le t \le 8$

b) Find the total distance traveled during this time period.

Solution

a) *displacement*: $s(t) = \int_0^8 v(t)\,dt$

$$= \int_0^8 (160 - 32t)\,dt$$

$$= 160t - 16t^2 \Big|_0^8$$

$$= \left(160(8) - 16(8)^2\right) - \left(160(0) - 16(0)^2\right)$$

$$= 256 \quad ft$$

The height if the rock is 256 *feet* above the ground 8 *sec* after the explosion.

b) $v(t) = 160 - 32t = 0 \quad \rightarrow \quad t = 5\ sec$

The velocity is positive over the time [0, 5] and negative over [5, 8]

$$\int_0^8 |v(t)|\,dt = \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt$$

$$= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt$$

$$= \left(160t - 16t^2\right|_0^5 - \left(160t - 16t^2\right|_5^8$$

$$= \left[\left(160(5) - 16(5)^2\right) - \left(160(0) - 16(0)^2\right)\right]$$

$$- \left[\left(160(8) - 16(8)^2\right) - \left(160(5) - 16(5)^2\right)\right]$$

$$= 400 - (-144)$$

$$= 544 \quad m$$

∴ The total distance traveled is $544 \quad m$

4.3–11 Average Value of a Continuous Function Revisited

The average value of a nonnegative continuous function f over an interval $[a,\ b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$.

$$Average = \frac{1}{b-a}\int_a^b f(x)\,dx$$

4.3–12 *Definition*

If f is integrable on $[a,\ b]$, then its *average value* on $[a,\ b]$, also called its ***mean***, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 14

Find the average value of $f(x) = \sqrt{4-x^2}$ *on* $[-2,\ 2]$

Solution

$f(x) = \sqrt{4-x^2}$ is a function of an upper semicircle with a radius 2 and centered at the origin.

The area between the semicircle and the x-axis from –2 to 2 can be computed using the geometry formula:

$$\begin{aligned} Area &= \frac{1}{2}\pi r^2 \\ &= \frac{1}{2}\pi(2)^2 \\ &= 2\pi \quad unit^2 \end{aligned}$$

$$Area = \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 2\pi \quad unit^2$$

$$av(f) = \frac{1}{2-(-2)}\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}(2\pi)$$

$$= \frac{\pi}{2}$$

Example 15

We can model the voltage in the electrical wiring of a typical home with the sine function

$$V = V_{\max} \sin 120\pi t$$

Which express the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 Hz (*hertz*)).

The positive constant $V_{\max}$ is the ***peak voltage***.

Solution

The average value of V over the half-cycle from 0 to $\frac{1}{120}$ ***sec*** is

$$V_{av} = \frac{1}{\frac{1}{120} - 0} \int_0^{1/120} V_{\max} \sin 120\pi t \ \ dt$$

$$= 120\, V_{\max} \int_0^{1/120} \sin 120\pi t \ \ dt$$

$$= 120 V_{\max} \left(-\frac{1}{120\pi} \cos 120\pi t \right|_0^{1/120}$$

$$= -\frac{1}{\pi} V_{\max} \left(\cos\left(120\pi \frac{1}{120}\right) - \cos(120\pi \cdot 0) \right)$$

$$= -\frac{1}{\pi} V_{\max} (\cos\pi - \cos 0)$$

$$= -\frac{1}{\pi} V_{\max} (-1 - 1)$$

$$= \frac{2}{\pi} V_{\max}$$

The average value of the voltage over a full cycle is zero.

To measure the voltage effectively, we can use an instrument the square root of the average value of the square of the voltage, namely:

$$V_{rms} = \sqrt{\left(V^2\right)_{av}}$$

"*rms*" : root mean square.

$$V_{av}^2 = \frac{1}{\frac{1}{60} - 0} \int_0^{1/60} \left(V_{\max}\right)^2 \sin^2 120\pi t \ \ dt$$

$$= \frac{\left(V_{\max}\right)^2}{2}$$

The *rms* voltage is:

$$V_{rms} = \sqrt{\left(V^2\right)_{av}}$$

$$= \frac{V_{\max}}{\sqrt{2}}$$

The "115 volts *ac*" means that the *rms* voltage is 115. The peak voltage is:

$$V_{\max} = \sqrt{2}\ V_{rms}$$

$$= \sqrt{2}\ (115)$$

$$\approx 163 \ \ volts$$

Exercises *Section* 4.3 – Definite Integral

$(\mathbf{1-70})$ Evaluate the integrals

1. $\displaystyle\int_0^3 (2x+1)\,dx$

2. $\displaystyle\int_0^2 x(x-3)\,dx$

3. $\displaystyle\int_0^4 \left(3x-\frac{x^3}{4}\right)dx$

4. $\displaystyle\int_{-2}^2 \left(x^3-2x+3\right)dx$

5. $\displaystyle\int_0^1 \left(x^2+\sqrt{x}\right)dx$

6. $\displaystyle\int_0^{\pi/3} 4\sec u\,\tan u\,du$

7. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta\,\cot\theta\,d\theta$

8. $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

9. $\displaystyle\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

10. $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx$

11. $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

12. $\displaystyle\int_0^{\pi/3} (\cos x+\sec x)^2\,dx$

13. $\displaystyle\int_0^{\pi} \frac{1}{2}\left(\cos x+|\cos x|\right)dx$

14. $\displaystyle\int_0^1 2x\left(4-x^2\right)dx$

15. $\displaystyle\int_0^4 (8-2x)\,dx$

16. $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$

17. $\displaystyle\int_{-4}^2 (2x+4)\,dx$

18. $\displaystyle\int_0^2 (1-x)\,dx$

19. $\displaystyle\int_0^2 \left(x^2-2\right)dx$

20. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

21. $\displaystyle\int_1^7 \frac{dx}{x}$

22. $\displaystyle\int_4^9 3\sqrt{x}\,dx$

23. $\displaystyle\int_{-2}^3 \left(x^2-x-6\right)dx$

24. $\displaystyle\int_0^1 \left(1-\sqrt{x}\right)dx$

25. $\displaystyle\int_{0}^{\pi/4} 2\cos x\, dx$

26. $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx$

27. $\displaystyle\int_{0}^{\ln 8} e^{x}\, dx$

28. $\displaystyle\int_{1}^{4} \left(\frac{x-1}{x}\right) dx$

29. $\displaystyle\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right) dx$

30. $\displaystyle\int_{0}^{2} \frac{dx}{x^2+4}$

31. $\displaystyle\int_{0}^{1} \left(4x^{21} - 2x^{16} + 1\right) dx$

32. $\displaystyle\int_{1}^{8} \left(\frac{2}{3x} - \frac{8}{x^2}\right) dx$

33. $\displaystyle\int_{-2}^{2} x^{9}\, dx$

34. $\displaystyle\int_{-200}^{200} 2x^{5}\, dx$

35. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$

36. $\displaystyle\int_{-2}^{2} \left(x^{9} - 3x^{5} + 2x^{2} - 10\right) dx$

37. $\displaystyle\int_{-\pi/2}^{\pi/2} \left(\cos 2x + \cos x \sin x - 3\sin x^{5}\right) dx$

38. $\displaystyle\int_{-2}^{2} \left(3x^{4} - 2x + 1\right) dx$

39. $\displaystyle\int_{0}^{10} \left(1 + \frac{z}{2}\right) dz$

40. $\displaystyle\int_{0}^{9-r^2} \left(1 + \frac{z}{9}\right) dz$

41. $\displaystyle\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz$

42. $\displaystyle\int_{0}^{1} \pi\left(\left(x^2+2\right)^2 - \left(\frac{1}{2}x+1\right)^2\right) dx$

43. $\displaystyle\int_{-1}^{2} \left(x^{3}+1\right)^2 dx$

44. $\displaystyle\int_{-2}^{2} (x-1)^3\, dx$

45. $\displaystyle\int_{0}^{2} 2\left(4 - x^2\right) dx$

46. $\displaystyle\int_{0}^{8} \pi\left(4x^{2/3} - \frac{1}{4}x^2\right) dx$

47. $\displaystyle\int_{-1}^{2} 2\pi(3-x)\left(2 + x - x^2\right) dx$

48. $\displaystyle\int_{0}^{2} 2\pi x\left(2x^{3} - x^{4}\right) dx$

49. $\displaystyle\int_{0}^{a} x\left(\sqrt{a^2 - x^2}\right)^2 dx$

50. $\displaystyle\int_{0}^{4} e^{-x}\, dx$

51. $\int_{-2}^{2} \left(t^2 - 4\right) dt$

52. $\int_{1}^{9} \left(\frac{t-1}{\sqrt{t}}\right) dt$

53. $\int_{0}^{x} \cos t \, dt$

54. $\int_{x}^{x^2} \sin t \, dt$

55. $\int_{1}^{4} \frac{1}{2\sqrt{t}} \, dt$

56. $\int_{x}^{2x} t^3 \, dt$

57. $\int_{1}^{x} e^{-t} \, dt$

58. $\int_{0}^{\sqrt{x}} t \, dt$

59. $\int_{-3}^{3} \left(t^2 - 9\right)\left(t^2 - 4\right) dt$

60. $\int_{1}^{4} \frac{2-\sqrt{t}}{t^2} \, dt$

61. $\int_{1}^{4} \left(3\sqrt{\theta} + 1\right)\left(\sqrt{\theta} - 2\right) d\theta$

62. $\int_{1}^{4} \left(\sqrt{\theta} - 2\right)^2 d\theta$

63. $\int_{0}^{\pi} \left(6 + 8\cos\theta + 2\cos 2\theta\right) d\theta$

64. $\int_{1}^{2} \sqrt{\frac{5}{4} e^{\theta}} \, d\theta$

65. $\int_{0}^{\frac{\pi}{4}} \left(2 + \sec^2\theta\right) d\theta$

66. $\int_{0}^{\frac{\pi}{4}} \left(\sin\theta + \cos\theta\right) d\theta$

67. $\int_{0}^{6} \left|x^2 - 4\right| dx$

68. $\int_{0}^{6} \left|4 - x^2\right| dx$

69. $\int_{-3}^{3} \left|x^2 - 3x + 2\right| dx$

70. $\int_{-3}^{3} \left|x^2 + 3x + 2\right| dx$

71. If f is an odd function, why is $\int_{-a}^{a} f(x) \, dx = 0$?

72. If f is an even function, why is $\int_{-a}^{a} f(x) \, dx = 2\int_{0}^{a} f(x) \, dx$

73. Is x^{12} an even or odd function? Is $\sin\left(x^2\right)$ an even or odd function?

(74 – 78) Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integrals

74. $\int_{-4}^{4} f(x)\,dx$

75. $\int_{-4}^{4} 3g(x)\,dx$

76. $\int_{0}^{1} 8x\,f\left(4x^2\right)\,dx$

77. $\int_{-2}^{2} 3x\,f(x)\,dx$

78. $\int_{-4}^{4} \left(4f(x) - 3g(x)\right)\,dx$

(79 – 80) Suppose that f is an even function with $\int_0^8 f(x)dx = 9$. Evaluate the integral

79. $\int_{-1}^{1} x\,f\left(x^2\right)\,dx$

80. $\int_{-2}^{2} x^2 f\left(x^3\right)\,dx$

(81 – 82) Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)dx = \pi$. Evaluate the integral

81. $\int_{0}^{\frac{\pi}{2p}} (\cos px)\,f(\sin px)\,dx$

82. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)\,f(\sin x)\,dx$

(83 – 105) Find the total area between the region between the given graph and the x-axis

83. $y = -x^2 - 2x, \quad -3 \le x \le 2$

84. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

85. $y = x^{1/3} - x, \quad -1 \le x \le 8$

86. $f(x) = x^2 + 1, \quad 2 \le x \le 3$

87. $y = 4x - 8, \quad -4 \le x \le 8$

88. $y = -3x, \quad -2 \le x \le 2$

89. $y = 3x + 6, \quad 0 \le x \le 6$

90. $y = 1 - |x|, \quad -2 \le x \le 2$

91. $f(x) = 1 - \frac{x^2}{4}, \quad -2 \le x \le 3$

92. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

93. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

94. $y = 6\cos x$, $-\frac{\pi}{2} \le x \le \pi$

95. $f(x) = \frac{1}{x}$, $-2 \le x \le -1$

96. $f(x) = x^2 + 4x + 3$, $-2 \le x \le 2$

97. $f(x) = 3x^2 + 5x + 2$, $0 \le x \le 2$

98. $f(x) = 3x^2 + 5x + 2$, $-1 \le x \le 2$

99. $f(x) = 2x^2 - 4x + 2$, $0 \le x \le 3$

100. $f(x) = x^2 - 4x + 3$, $0 \le x \le 3$

101. $f(x) = x^2 + 4x + 3$, $-3 \le x \le 0$

102. $f(x) = x^2 - 3x + 2$, $0 \le x \le 2$

103. $f(x) = x^2 + 3x + 2$, $-2 \le x \le 0$

104. $f(x) = 2x^2 - 4x + 2$, $0 \le x \le 2$

105. $f(x) = 2x^2 + 4x + 2$, $-1 \le x \le 1$

106. Find the area of the region above the *x-axis* bounded by $y = 4 - x^2$

107. Find the area of the region below the *x-axis* bounded by $y = x^4 - 16$

108. Find the area of the region above the *x-axis* bounded by $y = 16 - x^2$

(105 – 110) Compute the area of the region bounded by the graph of f and the *x-axis* on the given interval.

109. $f(x) = \frac{1}{x^2 + 1}$ *on* $\left[-1, \sqrt{3}\right]$

110. $f(x) = 2\sin\frac{x}{4}$ *on* $[0, 2\pi]$

111. $f(x) = 16 - x^2$ *on* $[-4, 4]$

112. $f(x) = x^3 - x$ *on* $[-1, 0]$

113. $f(x) = x^2 - x$ *on* $[0, 3]$

114. $f(x) = x^4 - x^2$ *on* $[-1, 1]$

(115 – 118) Find the average value of the following functions on the given interval.

115. $f(x) = x^3$ *on* $[-1, 1]$

116. $f(x) = \frac{1}{x^2 + 1}$ *on* $[-1, 1]$

117. $f(x) = \frac{1}{x}$ *on* $[1, e]$

118. $f(x) = e^{2x}$ *on* $[0, \ln 2]$

119. Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

120. Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where ***r*** is measured in thousands of dollars and ***x*** in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

121. The height H (*ft*) of a palm tree after growing for ***t*** years is given by

$$H(t) = \sqrt{t+1} + 5t^{1/3} \quad for \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.
b) Find the tree's average height for $0 \le t \le 8$

122. Without evaluating the integrals, explain why the following statement is true for positive integers n.

$$\int_0^1 x^n \, dx + \int_0^1 \sqrt[n]{x} \, dx = 1$$

(***hint***: *graph/ relation*)

123. The Gateway Arch in St. Louis is 630 *ft* high and has a 630-*ft* base. Its shape can be modeled by the parabola

$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right)$$

Find the average height of the arch above the ground.

124. The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are 2 a in the x-direction and 2 b in the y-direction is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at (0, 0). Integrate over the interval $[-a,\ a]$ to show that the average value of d^2 is $\frac{a^2+2b^2}{3}$

b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2.

c) Assuming $0 < b < a$, the coordinates of the Sun are $\left(\sqrt{a^2-b^2},\ 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a,\ a]$ to show that the average value of D^2 is $\frac{4a^2-b^2}{3}$.

125. A particle moves along a line with a velocity given by $v(t)=5\sin\pi t$ starting with an initial position $s(0)=0$. Find the displacement of the particle between $t=0$ and $t=2$, which is given by $s(t)=\int_0^2 v(t)\ dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)|dt$.

126. A baseball is launched into the outfield on a parabolic trajectory given by $y=0.01x(200-x)$. Find the average height of the baseball over the horizontal extent of its flight.

127. Find the average value of f shown in the figure on the interval $[1,\ 6]$ and then find the point(s) c in $(1,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals

128. Find the average value of f shown in the figure on the interval $[2,\ 6]$ and then find the point(s) c in $(2,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals

Section 4.4 – Substitution Rule

4.4–1 *Substitution*: Running the Chain Rule Backwards

The Chain rule formula is:

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}$$

We can see that $\frac{u^{n+1}}{n+1}$ is an antiderivative of the function $u^n\frac{du}{dx}$. Therefore, if we integrate both sides

$$\int u^n\frac{du}{dx}\,dx = \int \frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)dx$$

$$\boxed{\int u^n du=\frac{u^{n+1}}{n+1}+C}$$

Example 1

Find the integral $\int\left(x^3+x\right)^5\left(3x^2+1\right)dx$

Solution

Let: $u=x^3+x$

$$du=\frac{du}{dx}dx$$
$$=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)dx=\int u^5\,du$$
$$=\frac{u^6}{6}+C$$
$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

Direct/straight substitution

$$d\left(x^3+x\right)=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)\,dx=\int\left(x^3+x\right)^5 d\left(x^3+x\right)$$

$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

Example 2

Find the integral $\int\sqrt{2x+1}\,dx$

Solution

$$d\left(2x+1\right)=2\,dx$$

$$\int\sqrt{2x+1}\,dx=\frac{1}{2}\int\left(2x+1\right)^{1/2}d\left(2x+1\right)$$

$$=\frac{1}{3}\left(2x+1\right)^{3/2}+C$$

4.4–2 *Theorem* – The Substitution Rule

If $u=g\left(x\right)$ is differentiable function whose range is an interval *I*, and *f* is continuous on *I*, then

$$\int f\left(\left(g\left(x\right)\right)\right)g'\left(x\right)dx=\int f\left(u\right)du$$

Proof

By the Chain Rule, $F\left(g\left(x\right)\right)$ is an antiderivative of $f\left(g\left(x\right)\right)\cdot g'\left(x\right)$ whenever *F* is an antiderivative of *f*:

$$\frac{d}{dx}F\left(g\left(x\right)\right)=F'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

$$=f\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

If we make the substitution $u=g\left(x\right)$, then

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx=\int\frac{d}{dx}F\left(g\left(x\right)\right)dx$$

$$= F\left(g\left(x\right)\right) + C$$
$$= F\left(u\right) + C$$
$$= \int F'\left(u\right) du$$
$$= \int f\left(u\right) du$$

4.4–3 Integral of *ln*

Definition

The ***number e*** is that number in the domain of the ***natural logarithm*** satisfying

$$\ln e = 1 \quad and \quad \int_1^e \frac{1}{t} dt = 1$$

The ***natural logarithm*** of a number $x > 0$, denoted by $\ln x$, is defined as

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

If u is a differentiable function that is never zero $\displaystyle\int \frac{1}{u}\, du = \ln|u| + C$

Example 3

Evaluate the integral $\displaystyle\int \frac{dx}{ax+b}$

Solution

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{d\left(ax+b\right)}{ax+b} \qquad d\left(ax+b\right) = a\, dx$$

$$= \frac{1}{a}\ln\left|ax+b\right| + C$$

Example 4

Evaluate the integral $\int \tan x\, dx$

Solution

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx \qquad d(\cos x) = -\sin x\, dx \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$= -\int \frac{d(\cos x)}{\cos x}$$

$$= -\ln|\cos x| + C$$

$$= \ln \frac{1}{|\cos x|} + C$$

$$= \ln|\sec x| + C$$

Example 5

Evaluate the integral $\int_0^2 \frac{2x}{x^2-5}\, dx$

Solution

$$d\left(x^2-5\right) = 2x dx$$

$$\int_0^2 \frac{2x}{x^2-5}\, dx = \int_0^2 \frac{d\left(x^2-5\right)}{x^2-5}$$

$$= \ln\left|x^2-5\right| \Big|_0^2$$

$$= \ln|-1| - \ln|-5|$$

$$= -\ln 5$$

Example 6

Evaluate $\int_0^4 \frac{x}{x^2+9}\, dx$

Solution

$$d\left(x^2+9\right)=2x\,dx$$

$$\int_0^4 \frac{x}{x^2+9}\,dx=\frac{1}{2}\int_0^4 \frac{1}{x^2+9}\,d\left(x^2+9\right)$$

$$=\frac{1}{2}\ln\left(x^2+9\right)\Big|_0^4$$

$$=\frac{1}{2}\left(\ln 25-\ln 9\right)$$

$$=\frac{1}{2}\left(2\ln 5-2\ln 3\right)$$

$$=\ln\frac{5}{3}$$

Example 7

Find the integral $\displaystyle\int \sec^2(5t+1)\cdot 5\,dt$

Solution

$$d(5t+1)=5\,dt$$

$$\int \sec^2(5t+1)\cdot 5\,dt=\int \sec^2(5t+1)\,d(5t+1)$$

$$=\tan(5t+1)+C \qquad \frac{d}{du}(\tan u)=\sec^2 u$$

4.4–4 Antiderivative of the Exponential Function

The function $\ln x$, being *increasing* function of *x*. *Domain* (0, ∞) and *range* (–∞, ∞)

The inverse function $\ln^{-1} x$ with *Domain* (–∞, ∞) and *range* (0, ∞)

The function $\ln^{-1} x$ is usually denoted as $\exp x$ $\left(e^x\right)$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad (all\ x > 0) \qquad\qquad \ln\left(e^x\right) = x \quad (all\ x)$$

The derivative of $f(x) = e^x$ doesn't change $f'(x) = e^x$, therefore the antiderivative stays the same.

$$\int e^u du = e^u + C$$

Example 8

Evaluate the integral $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

Solution

$$d\left(1+e^x\right) = e^x dx$$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{1+e^x}\,d\left(1+e^x\right)$$

$$= \ln\left(1+e^x\right) + C$$

Example 9

Evaluate the integral $\displaystyle\int_0^{\ln 2} e^{3x}\,dx$

Solution

$$\int_0^{\ln 2} e^{3x}\,dx = \frac{1}{3}\int_0^{\ln 2} e^{3x}\,d(3x)$$

$$= \frac{1}{3}e^{3x}\Big|_0^{\ln 2}$$

$$= \frac{1}{3}\left(e^{3\ln 2} - e^{0}\right)$$

$$= \frac{1}{3}\left(e^{\ln 2^{3}} - e^{0}\right)$$

$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}$$

Example 10

Find the integral $\int \cos(7\theta+3)\, d\theta$

Solution

$$d(7\theta+3) = 7d\theta$$

$$\int \cos(7\theta+3)\, d\theta = \frac{1}{7}\int \cos(7\theta+3)\; d(7\theta+3)$$

$$= \frac{1}{7}\sin(7\theta+3) + C$$

Example 11

Find the integral $\int x^{2}\sin\left(x^{3}\right)dx$

Solution

$$d\left(x^{3}\right) = 3x^{2}\, dx$$

$$\int x^{2}\sin\left(x^{3}\right)dx = \frac{1}{3}\int \sin\left(x^{3}\right)d\left(x^{3}\right)$$

$$= -\frac{1}{3}\cos\left(x^{3}\right) + C$$

Example 12

Find the integral $\int x\,\sqrt{2x+1}\, dx$

Solution

Let: $u = 2x+1$

$du = 2dx$

$dx = \frac{1}{2}du$

$u = 2x+1$

$2x = u-1$

$x = \frac{u-1}{2}$

$$\int x\sqrt{2x+1}\,dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\,du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Example 13

Find the integral $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Solution

$d\left(z^2+1\right) = 2z\,dz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \left(z^2+1\right)^{-1/3} d\left(z^2+1\right)$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

Or

let: $u = \sqrt[3]{z^2+1}$

$u^3 = z^2+1$

$3u^2du = 2zdz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2 du}{u}$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

4.4–5 *Definition*

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx} \quad \rightarrow \quad \int a^u du = \frac{a^u}{\ln a} + C$$

Example 14

Evaluate the integral $\int 2^x\,dx$

Solution

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

Example 15

Evaluate the integral $\int 2^{\sin x}\cos x\,dx$

Solution

$$\int 2^{\sin x}\cos x\,dx = \int 2^{\sin x}\,d\left(\sin x\right)$$

$$= \frac{2^{\sin x}}{\ln 2} + C$$

Example 16

Evaluate the integral $\int x\,3^{x^2}\,dx$

Solution

$$\int x\,3^{x^2}\,dx = \frac{1}{2}\int 3^{x^2}\,d\left(x^2\right)$$

$$= \frac{1}{2}\frac{1}{\ln 3}3^{x^2} + C$$

Example 17

Evaluate $\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Solution

$$\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int_1^4 6^{-\sqrt{x}}\,d\left(-\sqrt{x}\right) \qquad d\left(-\sqrt{x}\right) = -\frac{1}{2\sqrt{x}}dx$$

$$= -\frac{2}{\ln 6}\,6^{-\sqrt{x}}\Big|_1^4$$

$$= -\frac{2}{\ln 6}\left(\frac{1}{36} - \frac{1}{6}\right)$$

$$= \frac{5}{18\ \ln 6}$$

Example 18

Evaluate the integral $\int \frac{\log_2 x}{x}\,dx$

Solution

$$\int \frac{\log_2 x}{x}\,dx = \int \frac{1}{x}\frac{\ln x}{\ln 2}\,dx$$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$$

$$= \frac{1}{\ln 2}\int \ln x\,d\left(\ln x\right)$$

$$= \frac{1}{\ln 2} \cdot \frac{1}{2}(\ln x)^2 + C$$

$$= \frac{(\ln x)^2}{2\ln 2} + C$$

4.4–6 Substitution Formula

Theorem

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof

Let F denote any antiderivative of f. Then

$$\int_a^b f(g(x)) \cdot g'(x) dx = F(g(x)) \Big|_{x=a}^{x=b}$$

$$= F(g(b)) - F(g(a))$$

$$= F(u) \Big|_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\, du$$

Example 19

Evaluate the definite integral $\displaystyle\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\, dx$

Solution

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\, dx = \int_{-1}^{1} \left(x^3+1\right)^{1/2} d\left(x^3+1\right) \qquad d\left(x^3+1\right) = 3x^2 dx$$

$$= \frac{2}{3}\left(x^3+1\right)^{3/2}\Big|_{-1}^{1}$$

$$= \frac{2}{3}\left(2^{3/2}-0^{3/2}\right)$$

$$= \frac{2}{3}2^{3/2} \qquad 2^{5/2}=\sqrt{2^5}=2^2\sqrt{2}$$

$$= \frac{4\sqrt{2}}{3}$$

Example 20

Evaluate the definite integral $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta$

Solution

$$\int_{\pi/4}^{\pi/2} \cot\theta\,\csc^2\theta\, d\theta = -\int_{\pi/4}^{\pi/2} \cot\theta\, d\left(\cot\theta\right) \qquad d\left(\cot\theta\right)=-\csc^2\theta\, d\theta$$

$$= -\frac{1}{2}\cot^2\theta\Big|_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{2}\left(\cot^2\frac{\pi}{2}-\cot^2\frac{\pi}{4}\right)$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Let $u=\cot\theta \quad\Rightarrow\quad du=-\csc^2\theta d\theta \rightarrow -du=\csc^2\theta d\theta$

$$\rightarrow \begin{cases} \theta=\frac{\pi}{4} & u=\cot\frac{\pi}{4}=1 \\ \theta=\frac{\pi}{2} & u=\cot\frac{\pi}{2}=0 \end{cases}$$

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta = \int_1^0 u\cdot(-du)$$

$$= -\int_1^0 u\, du$$

$$= -\frac{u^2}{2}\Big|_{1}^{0}$$

$$= -\left(0 - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Example 21

Evaluate the integral $\displaystyle\int_0^{\pi/6} \tan 2x\, dx$

Solution

$$\int_0^{\pi/6} \tan 2x\, dx = \frac{1}{2}\int_0^{\pi/6} \tan 2x\; d(2x)$$

$$= \frac{1}{2}\ln|\sec 2x|\Big|_{0}^{\pi/6}$$

$$= \frac{1}{2}\left[\ln\left|\sec 2\frac{\pi}{6}\right| - \ln|\sec 0|\right]$$

$$= \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{1}{2}\ln 2$$

4.4–7 *Integrals* of $\sin^2 x$ *and* $\cos^2 x$

Recall the double angles properties for the cosine and sine.

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

Example 22

Find the integral $\displaystyle\int \sin^2 x\, dx$

Solution

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 23

Find the integral $\int \cos^2 x\, dx$

Solution

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

4.4–8 *Inverse* Trigonometry Integration Formulas

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \left(Valid\ for\ \ u^2 < a^2\right)$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad (Valid\ for\ \ all\ u)$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \left(Valid\ for\ \ |u| > a > 0\right)$$

Example 24

Evaluate the integral $\int \frac{5}{\sqrt{9-4x^2}}\, dx$

Solution

$$\int \frac{5}{\sqrt{9-4x^2}}\,dx = \frac{5}{2}\int \frac{1}{\sqrt{9-(2x)^2}}\,d(2x) \qquad d(2x) = 2\,dx$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

***Example* 25**

Evaluate the integral $\displaystyle\int \frac{4}{25+x^2}\,dx$

Solution

$$\int \frac{4}{25+x^2}\,dx = \frac{4}{5}\arctan\left(\frac{x}{5}\right) + C$$

4.4–9 *Hyperbolic* Integration Formulas

$\int \sinh u\,du = \cosh u + C$	$\int \cosh u\,du = \sinh u + C$
$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$	$\int \operatorname{sech}^2 u\,du = \tanh u + C$
$\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$	$\int \operatorname{csch}^2 u\,du = -\coth u + C$

***Example* 26**

Evaluate the integral $\displaystyle\int \coth 5x\,dx$

Solution

$$\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx \qquad d(\sinh 5x) = 5\cosh 5x\,dx$$

$$= \frac{1}{5}\int \frac{d(\sinh 5x)}{\sinh 5x}$$

$$= \frac{1}{5}\ln|\sinh 5x| + C$$

Example 27

Evaluate the integral $\displaystyle\int_0^1 \sinh^2 x\,dx$

Solution

$$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2x - x\right|_0^1$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\sinh 2 - 1\right) - \left(\frac{1}{2}\sinh 0 - 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2 - 1\right)$$

$$= \frac{1}{4}\sinh 2 - \frac{1}{2} \qquad \approx 0.40672$$

Example 28

Evaluate the integral $\displaystyle\int_0^{\ln 2} 4e^x \sinh x\,dx$

Solution

$$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx$$

$$= 2\int_0^{\ln 2} \left(e^{2x} - 1\right)dx$$

$$= 2\left(\frac{1}{2}e^{2x} - x\right|_0^{\ln 2}$$

$$= 2\left[\left(\frac{1}{2}e^{2\ln 2} - \ln 2\right) - \left(\frac{1}{2}e^0 - 0\right)\right]$$

$$= 2\left(\frac{1}{2}e^{\ln 2^2} - \ln 2 - \frac{1}{2}\right)$$

$$= 3 - 2\ln 2 \qquad \approx 1.6137$$

Example 29

Evaluate the integral $\displaystyle\int x\coth\left(x^2\right)dx$

Solution

$$\int x\coth\left(x^2\right)dx = \frac{1}{2}\int \coth\left(x^2\right)d\left(x^2\right) \qquad d\left(x^2\right) = 2x\,dx$$

$$= \frac{1}{2}\ln\left(\sinh x^2\right) + C$$

4.4–10 Inverse *Hyperbolic* Integration Formulas

$\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \quad a > 0$	$\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \quad u > a > 0$
$\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, \quad u^2 < a^2$	$\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, \quad u^2 > a^2$
$\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \quad 0 < u < a$	$\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{u}{a}\right\rvert + C, \quad u \neq 0, a > 0$
$\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$	$\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left\lvert\frac{a+u}{a-u}\right\rvert + C$
$\displaystyle\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a+\sqrt{a^2 \pm u^2}}{\lvert u\rvert} + C$	

Example 30

Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$

Solution

$a=\sqrt{3},\quad u=2x \quad \to du=2dx$

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}=\int_0^1 \frac{d(2x)}{\sqrt{\left(\sqrt{3}\right)^2+(2x)^2}}$$

$$=\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg|_0^1$$

$$=\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)-\sinh^{-1}(0)$$

$$=\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) \qquad \approx 0.98665$$

Example 31

Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{x^2+16}}$

Solution

$$\int_0^3 \frac{dx}{\sqrt{x^2+16}}=\sinh^{-1}\frac{x}{4}\Bigg|_0^3$$

$$=\sinh^{-1}\frac{3}{4}-\sinh^{1}0$$

$$=\sinh^{-1}\frac{3}{4} \qquad \approx 0.639$$

Example 32

Evaluate $\displaystyle\int_9^{25} \frac{dx}{\sqrt{x}\,(4-x)}$

Solution

$u=\sqrt{x} \qquad \to \qquad u^2=x$

$du=\frac{1}{2}\frac{1}{\sqrt{x}}dx$

$$\int_9^{25} \frac{dx}{\sqrt{x}\,(4-x)}=2\int_9^{25} \frac{du}{4-u^2}$$

$$= 2\frac{1}{2}\coth^{-1}\frac{\sqrt{x}}{2}\Bigg|_{9}^{25}$$

$$= \coth^{-1}\frac{5}{2} - \coth^{-1}\frac{3}{2}$$

Exercises *Section* 4.4 – Substitution Rule

(1 – 5) Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx, \quad u = 2x+4$

2. $\int \frac{4x^3}{(x4+1)^2}\,dx, \quad u = x^4+1$

3. $\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$

4. $\int 12\left(y^4+4y^2+1\right)^2\left(y^3+2y\right)dx, \quad u = y^4+4y^2+1$

5. $\int \csc^2 2\theta\,\cot 2\theta\,d\theta \quad \rightarrow \begin{cases} a)\ \textit{Using} \quad u = \cot 2\theta \\ b)\ \textit{Using} \quad u = \csc 2\theta \end{cases}$

(6 – 136) Evaluate the integrals.

6. $\int \frac{1}{\sqrt{5s+4}}\,ds$

7. $\int \theta\,\sqrt[4]{1-\theta^2}\,d\theta$

8. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

9. $\int \tan^2 x\,\sec^2 x\,dx$

10. $\int \sin^5\frac{x}{3}\,\cos\frac{x}{3}\,dx$

11. $\int \tan^7\frac{x}{2}\,\sec^2\frac{x}{2}\,dx$

12. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

13. $\int x^{1/2}\,\sin\left(x^{3/2}+1\right)dx$

14. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

15. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

16. $\int \frac{\sec z\,\tan z}{\sqrt{\sec z}}\,dz$

17. $\int \frac{1}{\sqrt{t}}\,\cos\left(\sqrt{t}+3\right)dt$

18. $\int \frac{1}{\theta^2}\,\sin\frac{1}{\theta}\,\cos\frac{1}{\theta}\,d\theta$

19. $\int t^3\left(1+t^4\right)^3 dt$

20. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

21. $\int x^3\sqrt{x^2+1}\,dx$

22. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

23. $\int 2x\sqrt{x^2-2}\,dx$

24. $\displaystyle\int \frac{x}{\left(x^2-4\right)^3}\,dx$

25. $\displaystyle\int \frac{(2r-1)\,\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

26. $\displaystyle\int x^3\left(3x^4+1\right)^2\,dx$

27. $\displaystyle\int 2\left(3x^4+1\right)^2\,dx$

28. $\displaystyle\int 5x\sqrt{x^2-1}\,dx$

29. $\displaystyle\int \left(x^2-1\right)^3(2x)\,dx$

30. $\displaystyle\int \frac{6x}{\left(1+x^2\right)^3}\,dx$

31. $\displaystyle\int u^3\sqrt{u^4+2}\,du$

32. $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\,dt$

33. $\displaystyle\int \left(1+\frac{1}{t}\right)^3\frac{1}{t^2}\,dt$

34. $\displaystyle\int \left(7-3x-3x^2\right)(2x+1)\,dx$

35. $\displaystyle\int \sqrt{x}\left(4-x^{3/2}\right)^2\,dx$

36. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

37. $\displaystyle\int \sqrt{1-x}\,dx$

38. $\displaystyle\int x\sqrt{x^2+4}\,dx$

39. $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)\,d\theta$

40. $\displaystyle\int \cos^2(8\theta)\,d\theta$

41. $\displaystyle\int \sin^2(2\theta)\,d\theta$

42. $\displaystyle\int 8\cos^4 2\pi x\,dx$

43. $\displaystyle\int \sec x\,dx$

44. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

45. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}}$

46. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$

47. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

48. $\displaystyle\int \frac{dx}{4x^2+4x+2}$

49. $\displaystyle\int \frac{1}{6x-5}\,dx$

50. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}\,dx$

51. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

52. $\displaystyle\int \frac{x-3}{x+3}\,dx$

53. $\displaystyle\int \frac{3x}{x^2+4}\,dx$

54. $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$

55. $\displaystyle\int \frac{\sec x}{\sqrt{\ln(\sec x+\tan x)}}\,dx$

56. $\displaystyle\int 8e^{(x+1)}\,dx$

57. $\displaystyle\int 4xe^{x^2}\,dx$

58. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

59. $\displaystyle\int t^3\,e^{t^4}\,dt$

60. $\displaystyle\int e^{\sec \pi t}\sec \pi t\,\tan \pi t\,dt$

61. $\displaystyle\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx$

62. $\displaystyle\int (2x+1)\,e^{x^2+x}\,dx$

63. $\displaystyle\int \frac{dx}{1+e^x}$

64. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

65. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$

66. $\displaystyle\int \frac{1}{x^3}\,e^{1/4x^2}\,dx$

67. $\displaystyle\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}\,dx$

68. $\displaystyle\int \frac{-e^{3x}}{2-e^{3x}}\,dx$

69. $\displaystyle\int \frac{7\,e^{7x}}{3+e^{7x}}\,dx$

70. $\displaystyle\int \frac{2\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}\,dx$

71. $\displaystyle\int \frac{3^x}{3-3^x}\,dx$

72. $\displaystyle\int \frac{x\,2^{x^2}}{1+2^{x^2}}\,dx$

73. $\displaystyle\int \frac{dx}{x\left(\log_8 x\right)^2}$

74. $\displaystyle\int \frac{dx}{x\sqrt{25x^2-2}}$

75. $\displaystyle\int \frac{6\,dr}{\sqrt{4-(r+1)^2}}$

76. $\displaystyle\int \frac{dx}{2+(x-1)^2}$

77. $\displaystyle\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$

78. $\displaystyle\int \frac{dx}{\sqrt{-x^2+4x-3}}$

79. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

80. $\displaystyle\int \frac{x-2}{x^2-6x+10}\,dx$

81. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

82. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

83. $\displaystyle\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$

84. $\displaystyle\int \frac{\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}\,dx$

85. $\displaystyle\int \frac{dy}{\left(\sin^{-1}y\right)\sqrt{1+y^2}}$

86. $\displaystyle\int \frac{1}{\sqrt{x}(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}\,dx$

87. $\displaystyle\int 2x\left(x^2+1\right)^4\,dx$

88. $\displaystyle\int 8x\,\cos\left(4x^2+3\right)\,dx$

89. $\displaystyle\int \sin^3 x\,\cos x\,dx$

90. $\displaystyle\int (6x+1)\sqrt{3x^2+x}\,dx$

91. $\int 2x\left(x^2-1\right)^{99}\,dx$

92. $\int x\,e^{x^2}\,dx$

93. $\int \frac{2x^2}{\sqrt{1-4x^3}}\,dx$

94. $\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}\,dx$

95. $\int \left(x^2+x\right)^{10}(2x+1)\,dx$

96. $\int \frac{dx}{10x-3}$

97. $\int x^3\left(x^4+16\right)^6\,dx$

98. $\int \sin^{10}\theta\,\cos\theta\,d\theta$

99. $\int \frac{dx}{\sqrt{1-9x^2}}$

100. $\int x^9\,\sin x^{10}\,dx$

101. $\int \left(x^6-3x^2\right)^4\left(x^5-x\right)\,dx$

102. $\int \frac{x}{x-2}\,dx$

103. $\int \frac{dx}{1+4x^2}$

104. $\int \frac{3}{1+25y^2}\,dy$

105. $\int \frac{8x+6}{2x^2+3x}\,dx$

106. $\int \frac{x}{\sqrt{x-4}}\,dx$

107. $\int \frac{x^2}{(x+1)^4}\,dx$

108. $\int \frac{x}{\sqrt[3]{x+4}}\,dx$

109. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

110. $\int x\,\sqrt[3]{2x+1}\,dx$

111. $\int (x+1)\sqrt{3x+2}\,dx$

112. $\int \sin^2 x\,dx$

113. $\int \sin^2\left(\theta+\frac{\pi}{6}\right)\,d\theta$

114. $\int x\,\cos^2\left(x^2\right)\,dx$

115. $\int \sec 4x\,\tan 4x\,dx$

116. $\int \sec^2(10x)\,dx$

117. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

118. $\int \left(x^{3/2}+8\right)^5\sqrt{x}\,dx$

119. $\int \sin x\,\sec^8 x\,dx$

120. $\int \frac{e^{2x}}{e^{2x}+1}\,dx$

121. $\int \sec^3\theta\,\tan\theta\,d\theta$

122. $\int x\,\sin^4\left(x^2\right)\,\cos x^2\,dx$

123. $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

124. $\int \frac{x^2}{x^3+27}\,dx$

125. $\int y^2\left(3y^3+1\right)^4\,dy$

126. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

127. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

128. $\int \frac{dx}{\left(\tan^{-1}x\right)\left(1+x^2\right)}$

129. $\int \frac{\left(\tan^{-1}x\right)^5}{1+x^2}\,dx$

130. $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

131. $\int \sinh 2x\,dx$

132. $\int 4\cosh(3x-\ln 2)\,dx$

133. $\int \left(\sin^5 x + 3\sin^3 x - \sin x\right)\cos x \, dx$

134. $\int \frac{2}{x\sqrt{4x^2-1}} \, dx \quad \left(x > \frac{1}{2}\right)$

135. $\int x \, \sin\left(x^2\right) \, \cos^8\left(x^2\right) \, dx$

136. $\int \tan^{10}(4x) \, \sec^2(4x) \, dx$

137. $\int \tanh \frac{x}{7} dx$

138. $\int \coth \frac{\theta}{\sqrt{3}} d\theta$

139. $\int \operatorname{csch}^2(5-x) dx$

140. $\int \frac{\operatorname{sech}\sqrt{t} \, \tanh\sqrt{t}}{\sqrt{t}} dt$

141. $\int \frac{\operatorname{csch}(\ln t) \, \coth(\ln t)}{t} dt$

142. $\int \frac{\sinh x}{1+\cosh x} dx$

143. $\int \operatorname{sech}^2 x \, \tanh x \, dx$

144. $\int \coth^2 x \, \operatorname{csch}^2 x \, dx$

145. $\int \tanh^2 x \, dx$

146. $\int \frac{\sinh(\ln x)}{x} dx$

147. $\int \frac{dx}{8-x^2} \quad x > 2\sqrt{2}$

148. $\int \frac{dx}{\sqrt{x^2-16}}$

149. $\int \sinh \frac{x}{5} \, dx$

150. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx$

151. $\int \operatorname{sech}^2(2x-1) \, dx$

152. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx$

153. $\int \frac{\cosh x}{\sinh x} dx$

154. $\int x \operatorname{csch}^2 \frac{x^2}{2} \, dx$

155. $\int \operatorname{sech}^3 x \, \tanh x \, dx$

156. $\int \frac{\operatorname{csch}\left(\frac{1}{x}\right)\coth\left(\frac{1}{x}\right)}{x^2} \, dx$

157. $\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}} \, dx$

158. $\int \frac{x}{x^4+1} \, dx$

159. $\int \frac{2}{x\sqrt{1+4x^2}} \, dx$

160. $\int \frac{dx}{\sqrt{x^2-9}}$

161. $\int \cosh 2x \sinh^2 2x \, dx$

162. $\displaystyle\int \frac{1}{\sqrt{9x^2+25}}\,dx$

163. $\displaystyle\int \frac{1}{\sqrt{49-4x^2}}\,dx$

164. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-16}}\,dx$

165. $\displaystyle\int \frac{e^x}{16-e^{2x}}\,dx$

166. $\displaystyle\int \frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx$

167. $\displaystyle\int \cos^2\frac{t}{5}\sin^2\frac{t}{5}\,dt$

168. $\displaystyle\int \frac{x^2+2x-2}{\left(x^3+3x^2-6x\right)^2}\,dx$

169. $\displaystyle\int (\tan x)^{-3/2}\ \sec^2 x\,dx$

170. $\displaystyle\int \left(2\theta+1+2\cos(2\theta+1)\right)\,d\theta$

171. $\displaystyle\int \left(\frac{1}{\sqrt{2\theta-\pi}}+2\sec^2(2\theta-\pi)\right)\,d\theta$

172. $\displaystyle\int \frac{(t+1)^2-1}{t^4}\,dt$

173. $\displaystyle\int \sec\theta\tan\theta\sqrt{1+\sec\theta}\,d\theta$

174. $\displaystyle\int e^t\cos\left(3e^t-2\right)dt$

175. $\displaystyle\int e^y\csc\left(e^y+1\right)\cot\left(e^y+1\right)dy$

176. $\displaystyle\int \left(\csc^2 x\right)e^{\cot x}\,dx$

177. $\displaystyle\int 2^{\tan x}\sec^2 x\,dx$

178. $\displaystyle\int \frac{24}{y\sqrt{y^2-16}}\,dy$

179. $\displaystyle\int \frac{dx}{\sqrt{-x^2+4x-1}}$

180. $\displaystyle\int \frac{dt}{(3t+1)\sqrt{9t^2+6t}}$

181. $\displaystyle\int \sqrt{3x+5}\,dx$

182. $\displaystyle\int \frac{dx}{\tan 2x}$

183. $\displaystyle\int (\csc x-1)^2\,dx$

184. $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx$

185. $\displaystyle\int \frac{6x}{1+3x^2}\,dx$

186. $\displaystyle\int \sqrt[3]{3x-2}\,dx$

187. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

188. $\displaystyle\int \frac{dx}{2x+1}$

189. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

190. $\int \cos(1-5x)\,dx$

191. $\int e^{x/3}\,dx$

192. $\int \frac{dx}{\sec 2x}$

193. $\int \frac{\sec\sqrt{x}\ \tan\sqrt{x}}{\sqrt{x}}\,dx$

194. $\int \frac{2x}{1+x^4}\,dx$

195. $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

196. $\int \frac{\sec^2(\sin\theta)}{\sec\theta}\,d\theta$

197. $\int \frac{x^2+5}{x^2+4}\,dx$

198. $\int \cot 4\theta\ d\theta$

199. $\int \frac{dx}{\sin 2x}$

200. $\int \frac{dx}{x(\ln x)^2}$

201. $\int \frac{dx}{2-x}$

202. $\int \frac{\sec x\ \tan x}{1+\sec^2 x}\,dx$

203. $\int (1+\sec x)^2\,dx$

204. $\int \frac{\sec^2 x}{\tan x}\,dx$

205. $\int 10x^3 e^{x^4}\,dx$

206. $\int \frac{(\ln x)^3}{x}\,dx$

207. $\int \frac{\cos x}{\sin^2 x}\,dx$

208. $\int \frac{dx}{\sqrt{1-25x^2}}$

209. $\int \frac{1}{16+25x^2}\,dx$

210. $\int \csc^2(3-x)\,dx$

211. $\int \frac{\sin\theta}{1+\cos\theta}\,d\theta$

212. $\int \frac{\sec^2\theta}{\sqrt{1-\tan^2\theta}}\,d\theta$

213. $\int \frac{dx}{e^{4x}}$

214. $\int e^x \cos e^x\,dx$

215. $\int \sin\left(\frac{2x-5}{3}\right)dx$

216. $\int \frac{6\csc^2\theta}{1-3\cot\theta}\,d\theta$

217. $\int 6\theta^2 \cot\theta^3 \csc\theta^3\,d\theta$

218. $\int \sin\theta\ \sin(\cos\theta)\,d\theta$

219. $\int (\csc\theta-\cot\theta)\csc\theta\ d\theta$

220. $\int \frac{2y\ dy}{y^2-25}$

221. $\int \frac{\sec y\tan y}{2+\sec y}\,dy$

222. $\int \frac{5}{e^{-5x}+7}\,dx$

223. $\int \frac{e^{2x}}{4+e^{2x}}\,dx$

224. $\int \frac{dx}{x\ln x\ln(\ln x)}$

225. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

226. $\int \frac{e^{\sin x}}{\sec x}\,dx$

227. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

228. $\int \frac{4^{\cot x}}{\sin^2 x}\,dx$

229. $\int \frac{4x^2+2x+4}{x+1}\,dx$

230. $\int \frac{e^x}{4e^x+6}\,dx$

231. $\int \frac{x+4}{x^2+8x+25}\,dx$

232. $\int \frac{e^{2x}}{\sqrt{e^{2x}+4}}\,dx$

233. $\int \frac{x^2}{2x^3+1}\,dx$

234. $\int \frac{\sec^2 x}{\tan x}\,dx$

(235 – 373) Evaluate the integrals.

235. $\displaystyle\int_0^1 (2t+3)^3\,dt$

236. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

237. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$

238. $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\,dr$

239. $\displaystyle\int_0^{\pi/4} \tan x\,\sec^2 x\,dx$

240. $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x\,\sin x\,dx$

241. $\displaystyle\int_0^1 t^3\left(1+t^4\right)^3\,dt$

242. $\displaystyle\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

243. $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

244. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

245. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

246. $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\,\sin 3t\,dt$

247. $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

248. $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

249. $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw$

250. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)\,dt$

251. $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\,\left(1+\sqrt{y}\right)^2}$

252. $\displaystyle\int_0^1 \left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)\,dy$

253. $\displaystyle\int_0^5 |x-2|\,dx$

254. $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx$

255. $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

256. $\displaystyle\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

257. $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$

258. $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

259. $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

260. $\displaystyle\int_{\pi/4}^{\pi/2} \cot x \, dx$

261. $\displaystyle\int_{-\ln 2}^{0} e^{-x} \, dx$

262. $\displaystyle\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta \, d\theta$

263. $\displaystyle\int_{0}^{\sqrt{\ln\pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx$

264. $\displaystyle\int_{1}^{4} \frac{2^{\sqrt{x}}}{\sqrt{x}} \, dx$

265. $\displaystyle\int_{0}^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt$

266. $\displaystyle\int_{1}^{e} x^{(\ln 2)-1} \, dx$

267. $\displaystyle\int_{0}^{9} \frac{2\log_{10}(x+1)}{x+1} \, dx$

268. $\displaystyle\int_{1}^{e} \frac{2\ln 10 \, \log_{10} x}{x} \, dx$

269. $\displaystyle\int_{1}^{e^x} \frac{1}{t} \, dt$

270. $\displaystyle\frac{1}{\ln a}\int_{1}^{x} \frac{1}{t} \, dt \quad x>0$

271. $\displaystyle\int_{0}^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

272. $\displaystyle\int_{\pi/6}^{\pi 4} \frac{\csc^2 x}{1+(\cot x)^2} \, dx$

273. $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4}{t\left(1+\ln^2 t\right)} \, dt$

274. $\displaystyle\int_{1/2}^{1} \frac{6}{\sqrt{-4x^2+4x+3}} \, dx$

275. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2-1}} \, dx$

276. $\displaystyle\int_{0}^{3} \frac{x}{\sqrt{25-x^2}} dx$

277. $\displaystyle\int_{0}^{\pi} \sin^2 5\theta \, d\theta$

278. $\displaystyle\int_{0}^{\pi} \left(1-\cos^2 3\theta\right) d\theta$

279. $\displaystyle\int_{2}^{3} \frac{x^2+2x-2}{x^3+3x^2-6x} \, dx$

280. $\displaystyle\int_{0}^{\ln 2} \frac{e^x}{1+e^{2x}} \, dx$

281. $\displaystyle\int_{1}^{3} x \sqrt[3]{x^2-1} \, dx$

282. $\displaystyle\int_{0}^{2} (x+3)^3 \, dx$

283. $\displaystyle\int_{-2}^{2} e^{4x+8} \, dx$

284. $\displaystyle\int_{0}^{1} \sqrt{x}\left(\sqrt{x}+1\right) dx$

285. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{4-x^2}}$

286. $\displaystyle\int_0^2 \frac{2x}{\left(x^2+1\right)^2}\,dx$

287. $\displaystyle\int_0^{\pi/2} \sin^2\theta\,\cos\theta\,d\theta$

288. $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

289. $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}\,dx$

290. $\displaystyle\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

291. $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx$

292. $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\,d\theta$

293. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\,d\theta$

294. $\displaystyle\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x+2}\,dx$

295. $\displaystyle\int_0^{\pi/2} \sin^4\theta\,d\theta$

296. $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

297. $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx$

298. $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

299. $\displaystyle\int_0^{6/5} \frac{dx}{25x^2+36}$

300. $\displaystyle\int_0^2 x^3\sqrt{16-x^4}\,dx$

301. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

302. $\displaystyle\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7\,dx$

303. $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}\,dx$

304. $\displaystyle\int_0^1 \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\,dx$

305. $\displaystyle\int_1^2 \frac{4}{9x^2+6x+1}\,dx$

306. $\displaystyle\int_0^{\pi/4} e^{\sin^2 x}\sin 2x\,dx$

307. $\displaystyle\int_0^1 x\sqrt{x+a}\,dx \quad (a>0)$

308. $\displaystyle\int_0^1 x\sqrt[p]{x+a}\,dx \quad (a>0)$

309. $\displaystyle\int_0^1 x\sqrt{1-\sqrt{x}}\,dx$

310. $\displaystyle\int_0^1 \sqrt{x-x\sqrt{x}}\,dx$

311. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta\,\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta$

312. $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

313. $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$

314. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta$

315. $\displaystyle\int_0^1 2x\left(4-x^2\right)\,dx$

316. $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

317. $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$

318. $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$

319. $\displaystyle\int_0^2 \frac{x^2-1}{\sqrt{x^3-3x+4}}\,dx$

320. $\displaystyle\int_0^1 \frac{y^2+4y-4}{\sqrt{y^3+6y^2-12y+9}}\,dy$

321. $\displaystyle\int_{-1}^2 x^2e^{x^3+1}\,dx$

322. $\displaystyle\int_0^2 x^2e^{x^3}\,dx$

323. $\displaystyle\int_{\ln 2}^{\ln 3} \coth x\,dx$

324. $\displaystyle\int_0^1 \frac{x^2}{9-x^6}\,dx$

325. $\displaystyle\int_0^{\ln 2} \tanh x\,dx$

326. $\displaystyle\int_0^1 \cosh^2 x\,dx$

327. $\displaystyle\int_0^4 \frac{1}{25-x^2}\,dx$

328. $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx$

329. $\displaystyle\int_0^{\frac{\sqrt{2}}{4}} \frac{2}{\sqrt{1-4x^2}}\,dx$

330. $\displaystyle\int_0^{\ln 2} 2e^{-x}\cosh x\,dx$

331. $\displaystyle\int_0^{\ln 2} 2e^{x}\sinh x\,dx$

332. $\displaystyle\int_0^1 \cosh^3 3x\,\sinh 3x\,dx$

333. $\displaystyle\int_0^4 \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}}\,dx$

334. $\displaystyle\int_{\ln 2}^{\ln 3} \operatorname{csch} x\,dx$

335. $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\,dx$

336. $\displaystyle\int_0^{\pi/2} 2\sinh\left(\sin\theta\right)\cos\theta\,d\theta$

337. $\displaystyle\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx$

338. $\displaystyle\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right)dx$

339. $\displaystyle\int_{0}^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

340. $\displaystyle\int_{1}^{e^2} \frac{dx}{x\sqrt{\ln^2 x+1}}$

341. $\displaystyle\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}}$

342. $\displaystyle\int_{4}^{6} \frac{1}{\sqrt{x^2-9}}dx$

343. $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{16x^2+1}}dx$

344. $\displaystyle\int_{1}^{3} \frac{1}{x\sqrt{4+x^2}}dx$

345. $\displaystyle\int_{3}^{7} \frac{1}{\sqrt{x^2-4}}dx$

346. $\displaystyle\int_{-1}^{1} \frac{1}{16-9x^2}dx$

347. $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{25x^2+1}}dx$

348. $\displaystyle\int_{0}^{1} x^2\left(\operatorname{sech} x^3\right)^2 dx$

349. $\displaystyle\int_{1/2}^{1} \sin\left(\frac{\pi}{2}x-\frac{\pi}{4}\right)dx$

350. $\displaystyle\int_{-1}^{1} \frac{3dv}{4v^2+4v+4}$

351. $\displaystyle\int_{0}^{\pi} \left(1-\cos^2 3\theta\right)d\theta$

352. $\displaystyle\int_{0}^{1} \frac{6x}{\left(4-x^2\right)^{3/2}}dx$

353. $\displaystyle\int_{1}^{4} \frac{\left(1+\sqrt{u}\right)^{1/2}}{\sqrt{u}}du$

354. $\displaystyle\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1-\sin t}dt$

355. $\displaystyle\int_{0}^{1/2} x^3\left(1+9x^4\right)^{-3/2}dx$

356. $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t\sin\sqrt{t}}}dt$

357. $\displaystyle\int_{0}^{\ln 9} e^{\theta}\left(e^{\theta}-1\right)^{1/2}d\theta$

358. $\displaystyle\int_{-1/5}^{1/5} \frac{6dx}{\sqrt{4-25x^2}}$

359. $\displaystyle\int_{1}^{e} \frac{\sqrt{\ln x}}{x}dx$

360. $\displaystyle\int_{e^2}^{e^8} \frac{dx}{x\ln x}$

361. $\displaystyle\int_{1}^{4} \frac{10^{\sqrt{x}}}{\sqrt{x}}dx$

362. $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}dx$

363. $\displaystyle\int_{0}^{3} \frac{2x-1}{x+1}dx$

364. $\displaystyle\int_{e}^{e^2} \frac{dx}{x\ln^3 x}$

365. $\displaystyle\int_{0}^{\pi/2} \frac{\sin x}{1+\cos x}dx$

366. $\displaystyle\int_{3}^{4} \frac{dx}{2x\ln x\ln^3(\ln x)}$

367. $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)}$

368. $\displaystyle\int_{0}^{1} \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy$

369. $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx$

370. $\displaystyle\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}dz$

371. $\displaystyle\int_{0}^{\pi/2} 4^{\sin x}\cos x\, dx$

372. $\displaystyle\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2}dp$

373. $\displaystyle\int_{1}^{2} (1+\ln x)x^x dx$

374. Evaluate the integral $\displaystyle\int \frac{18\tan^2 x\ \sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx$

a) $u=\tan x$, followed by $v=u^3$ *then by* $w=2+v$

b) $u=\tan^3 x$, followed by $v=2+u$

c) $u=2+\tan^3 x$

375. Evaluate the integral $\displaystyle\int \sec^8\left(\tan x^2\right)\ \sin\left(\tan x^2\right)\ x\ \sec^2 x^2\ dx$

$(\mathbf{376-377})$ Solve the initial value problem

376. $\dfrac{dy}{dt}=e^t\sin\left(e^t-2\right),\quad y(\ln 2)=0$

377. $\dfrac{dy}{dt}=e^{-t}\sec^2\left(\pi e^{-t}\right),\quad y(\ln 4)=\dfrac{2}{\pi}$

378. Verify the integration formula: $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx=\ \ln x-\frac{1}{2}\ln\left(1+x^2\right)-\frac{\tan^{-1}x}{x}+C$

379. Verify the integration formula: $\displaystyle\int \ln\left(a^2+x^2\right)dx=\ x\ln\left(a^2+x^2\right)-2x+2a\tan^{-1}\frac{x}{a}+C$

380. Verify the integration $\displaystyle\int x\,\mathrm{sech}^{-1}x\,dx=\frac{x^2}{2}\mathrm{sech}^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$

381. Verify the integration $\displaystyle\int \tanh^{-1}x\,dx=x\tanh^{-1}x+\frac{1}{2}\ln\left(1-x^2\right)+C$

382. Use the Substitution Rule to prove $\displaystyle\int \sin^2 ax dx=\frac{x}{2}-\frac{\sin(2ax)}{4a}+C$

383. Use the Substitution Rule to prove $\displaystyle\int \cos^2 ax dx=\frac{x}{2}+\frac{\sin(2ax)}{4a}+C$

384. Find the area of the region bounded by the graphs of:

$x=3\sin y\sqrt{\cos y},\quad$ *and* $\quad x=0,\quad 0\le y\le\frac{\pi}{2}$

385. Find the area of the region bounded by the graph of $f(x)=\dfrac{x}{\sqrt{x^2-9}}$ *on* $3\le x\le 4$.

386. Find the area of the region bounded by the graph of $f(x)=\dfrac{x}{\sqrt{x^2-9}}$ and the *x-axis* between $x=4$ and $x=5$.

387. Find the area of the region bounded by the graph of $f(x)=x\ \sin x^2$ and the *x-axis* between $x=0$ and $x=\sqrt{\pi}$.

388. Find the area of the region bounded by the graph of $f(\theta)=\cos\theta\ \sin\theta$ and the θ*-axis* between $\theta=0$ and $\theta=\frac{\pi}{2}$.

389. Find the area of the region bounded by the graph of $f(x)=(x-4)^4$ and the *x-axis* between $x=2$ and $x=6$.

390. Perhaps the simplest change of variables is the shift or translation given by $u=x+c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that:

$$\int_a^b f(x+c)\,dx=\int_{a+c}^{b+c} f(u)\,du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x)=\sin x$, $a=0$, $b=\pi$, and $c=\frac{\pi}{2}$.

391. Another change of variables that can be interpreted geometrically is the scaling $u=cx$, where c is a real number. Prove and interpret the fact that

$$\int_a^b f(cx)\,dx=\frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

Draw a picture to illustrate this change of variables in the case that $f(x)=\sin x$, $a=0$, $b=\pi$, and $c=\frac{1}{2}$.

392. The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find f and check your answer by substitution.

393. Assume f' is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

394. The area of the shaded region under the curve $y = 2\sin 2x$ in

a) Equals the area on the shaded region under the curve $y = \sin x$

b) Explain why this is true without computation areas.

395. The area of the shaded region under the curve $y = \dfrac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$

a) Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1,\ 2]$

b) Explain why this is true without computation areas.

396. The family of parabolas $y=\frac{1}{a}-\frac{x^2}{a^3}$, where $a>0$, has the property that for $x\geq 0$, the *x*-intercept is $(a,\ 0)$ and the *y*-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the *x-axis*. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of *a*.

397. Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a>0$ and $b>0$. Show that the average vertical distance from points on the *x-axis* to the hypotenuse is $\frac{b}{2}$, for all $a>0$.

398. Consider the integral $I=\int \sin^2 x\ \cos^2 x\ dx$

a) Find *I* using the identity $\sin 2x=2\sin x\cos x$

b) Find *I* using the identity $\cos^2 x=1-\sin^2 x$

c) Confirm that the results in part (*a*) and (*b*) are consistent and compare the work involved in each method.

399. Let $H(x)=\int_0^x \sqrt{4-t^2}\ dt$, for $-2\leq x\leq 2$.

a) Evaluate $H(0)$

b) Evaluate $H'(1)$

c) Evaluate $H'(2)$

d) Use geometry to evaluate $H(2)$

e) Find the value of *s* such that $H(x)=sH(-x)$

$(\mathbf{400-401})$ Evaluate the limits

400. $\lim\limits_{x\to 2}\dfrac{\int_2^x e^{t^2}dt}{x-2}$

401. $\lim\limits_{x\to 1}\dfrac{\int_1^{x^2} e^{t^3}dt}{x-1}$

402. Prove that for nonzero constants *a* and *b*, $\int \frac{dx}{a^2x^2+b^2}=\frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C$

403. Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f, for a = 1, 2, 3.

b) Let $g(a)$ be the area of the region bounded by the graph of f and the *x-axis* on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

404. Explain why if a function u satisfies the equation $u(x) + 2\int_0^x u(t)\,dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that is u satisfies the second equation, then it satisfies the first equation?

405. Let $f(x) = \int_0^x (t-1)^{15}(t-2)^9\,dt$

a) Find the interval on which f is increasing and the intervals on which f is decreasing.

b) Find the intervals on which f is concave up and the intervals on which f is concave down.

c) For what values of x does f have local minima? Local maxima?

d) Where are the inflection points of f?

406. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function:

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \tfrac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

407. Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

408. An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y)\,dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

409. The population of a culture of bacteria has a growth rate given by $p'(t) = \frac{200}{(t+1)^r}$ bacteria per hour, for $t \geq 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (Note that the growth rate decreases in time, reflecting competition for space and food.)

a) Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?

b) Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?

c) Let ΔP be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does ΔP increase or decrease with the parameter *r*? Explain.

d) A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0,\ 10]$. Estimate the value of *r* that best fits this data point.

e) Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T > 0$. If the culture is allowed to grow indefinitely $(T \to \infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

410. Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t)\,dt$.

a) Graph f on the interval [0, 6].

b) Compute and graph *A* on the interval [0, 6].

c) Show that the local extrema of *A* occur at the zeros of f.

d) Give a geometric and analytical explanation for the observation in part (c).

e) Find the approximate zeros of *A*, other than 0, and call them x_1 and x_2.

f) Find *b* such that the area bounded by the graph of f and the *x*-axis on the interval

$\left[0,\ x_1\right]$ equals the area bounded by the graph of f and the x-axis on the interval $\left[x_1,\ b\right]$.

g) If f is an integrable function and $A(x)=\int_0^x f(t)\,dt$, is it always true that the local extrema of A occur at the zeros of f? Explain

411. Derive the formula $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$ for all real x. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

412. Find the linear approximation to $f(x)=\cosh x$ at $a=\ln 3$ and then use it to approximate the value of $\cosh 1$.

Summary

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ $\qquad$ $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\int_a^b f(x)dx = F(b) - F(a)$$

- ✓ If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- ✓ If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

$\int \frac{1}{u^2}\,du = -\frac{1}{u} + C$ $\qquad$ $\int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$ $\qquad$ $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$

$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C$ $\qquad$ $\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + C$

$\int \csc^2(ax)\,dx = -\frac{1}{a}\cot(ax) + C$ $\qquad$ $\int \csc(ax)\cot(ax)\,dx = -\frac{1}{a}\csc(ax) + C$

$\int \tan x\,dx = \ln|\sec x| + C$ $\qquad$ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \frac{du}{u} = \ln|u| + C$ $\qquad$ $\int e^u du = e^u + C$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$ $\qquad$ $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ $\qquad$ $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

$\int \sinh u\,du = \cosh u + C$ $\qquad$ $\int \cosh u\,du = \sinh u + C$ $\qquad$ $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

$\int \operatorname{sech}^2 u\,du = \tanh u + C$ $\qquad$ $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$ $\qquad$ $\int \operatorname{csch}^2 u\,du = -\coth u + C$

Exercises *Lecture* 4 – Review

$(\mathbf{1-42})$ Evaluate the integrals.

1. $\int \pi \, dt$

2. $\int \pi x \, dt$

3. $\int \pi t \, dt$

4. $\int 2x \, dx$

5. $\int \left(x^3 + 2x - 4\right) dx$

6. $\int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

7. $\int \sqrt{x}\left(\sqrt{x} - x\right) dx$

8. $\int x^3 \sqrt{x^4 + 2} \, dx$

9. $\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2} dx$

10. $\int \frac{1}{\sqrt{x+1}} dx$

11. $\int \frac{5x^2}{4x^3 - 1} dx$

12. $\int x^2 \left(1 + 4x^3\right)^{-2/3} dx$

13. $\int \frac{x-2}{\left(x^2 - 4x\right)^{1/3}} dx$

14. $\int \frac{dx}{2 - \sqrt{3x}}$

15. $\int x^2 \left(x^3 + 10\right)^{10} dx$

16. $\int \cos \pi x \, dx$

17. $\int \left(\tan x + \sec x\right) dx$

18. $\int x^3 \cos\left(x^4\right) dx$

19. $\int \frac{\sin x}{\sqrt{\cos x}} dx$

20. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

21. $\int \frac{dx}{\sqrt{9 - x^2}}$

22. $\int \frac{dx}{x^2 + 16}$

23. $\int \frac{x^2}{x^6 + 1} dx$

24. $\int \sec^2 x \sqrt[3]{\tan x} \, dx$

25. $\int \csc 2x \cot 2x \, dx$

26. $\int \sin\left(5x + 3\right) dx$

27. $\int \cos x \, e^{-\sin x} dx$

28. $\int \frac{e^{1/x}}{x^2} dx$

29. $\int \frac{dx}{x}$

30. $\int \frac{3x \, dx}{x^2 + 8}$

31. $\int \frac{dx}{2x + 1}$

32. $\int \frac{\sqrt[3]{\ln x}}{x} dx$

33. $\int \frac{1}{x\left(\ln x + 1\right)^2} dx$

34. $\int \frac{1}{\sqrt{1 + \sqrt{x}}} dx$

35. $\int 2x^2 e^{x^3 - 2} dx$

36. $\int e^{\pi x} dx$

37. $\int e^{-ax} dx$

38. $\int \left(e^x + 1\right) \sqrt{e^x + x} \, dx$

39. $\int \frac{e^x}{e^x + 2} dx$

40. $\int \frac{\left(\ln x\right)^2 + 2\ln x + 3}{x} dx$

41. $\int \sec x \tan x \sqrt[3]{\sec^2 x} \, dx$

42. $\int \left(x^{2/3} + 3x^{3/2} - 2x^{1/2}\right) dx$

$(\mathbf{43-63})$ Evaluate the definite integrals.

43. $\displaystyle\int_1^4 \pi^2\,dx$

44. $\displaystyle\int_{-2}^4 x\,dx$

45. $\displaystyle\int_0^3 \left(x^2+2x+1\right)dx$

46. $\displaystyle\int_{-2}^2 \left(x^4-x^2+1\right)dx$

47. $\displaystyle\int_{-3}^3 \left(x^5-x^3+x\right)dx$

48. $\displaystyle\int_1^2 \frac{3x^3-2x+2}{x}\,dx$

49. $\displaystyle\int_1^2 \left(\frac{2}{x^5}+\frac{1}{\sqrt{x}}\right)dx$

50. $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx$

51. $\displaystyle\int_0^{\sqrt{2}} \frac{x}{x^2+1}\,dx$

52. $\displaystyle\int_0^1 x\cos\left(\pi x^2\right)dx$

53. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\,dx$

54. $\displaystyle\int_1^4 \frac{2}{x}\,dx$

55. $\displaystyle\int_1^4 \frac{2x+1}{x^2+x-1}\,dx$

56. $\displaystyle\int_0^1 x\left(x^2+1\right)^6 dx$

57. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \csc^2 x\,dx$

58. $\displaystyle\int_0^1 \frac{e^x}{\left(e^x+1\right)^2}\,dx$

59. $\displaystyle\int_0^1 6x\sqrt[5]{1-x^2}\,dx$

60. $\displaystyle\int_0^1 \left(e^x+x\right)dx$

61. $\displaystyle\int_0^1 x\,e^{x^2}\,dx$

62. $\displaystyle\int_0^{\sqrt{2}} \frac{dx}{4+x^2}$

63. $\displaystyle\int_{-3}^3 \left|x^3+x^2-2x\right|dx$

64. $\displaystyle\int_0^{\ln 2} e^{-x}\sinh x\,dx$

65. $\displaystyle\int_0^{\ln 2} e^{2x}\sinh x\,dx$

66. $\displaystyle\int_0^{\ln 3} e^{-2x}\cosh x\,dx$

67. Find the area of the region between the x-axis and the graph of $f(x)=x^2-x-2,\quad -1\le x\le 2$.

68. Find the area of the region between the x-axis and the graph of $f(x)=x^2+x-2,\quad -1\le x\le 3$.

69. Find the area of the region above the x-axis and the graph of $f(x)=a^2-x^2,\quad -a\le x\le a$.

70. Find the area of the region below the *x-axis* bounded by $f(x)=x^4-81$.

Practice *Exam*

Evaluate the following integrals.

1. $\int 5\pi \, dx$

2. $\int x^2 \left(3x^3+1\right)^4 dx$

3. $\int \left(3z^2-4z+5\right) dz$

4. $\int \left(\sqrt{x}+\frac{1}{2\sqrt{x}}-\frac{1}{x^2}\right) dx$

5. $\int 2x\left(1-x^{-3}\right) dx$

6. $\int \frac{x^2+2x-2}{\left(x^3+3x^2-6x\right)^2} dx$

7. $\int \frac{x^2}{x^3+27} dx$

8. $\int \frac{2}{5}\sec\theta \tan\theta \, d\theta$

9. $\int x^9 \sin x^{10} \, dx$

10. $\int \left(2\cos 2x-3\sin 3x\right) dx$

11. $\int \sec\theta \tan\theta \sqrt{1+\sec\theta} \, d\theta$

12. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr$

13. $\int \frac{e^{2x}}{e^{2x}+1} dx$

14. $\int \sin^3 x \cos x \, dx$

15. $\int \tan x \, dx$

16. $\int 5e^{5x} \, dx$

17. $\int x^2 e^{x^3+1} \, dx$

18. $\int \frac{\ln x}{x} dx$

19. $\int_0^2 x(x-3) \, dx$

20. $\int_{-2}^2 \left(x^3-2x+3\right) dx$

21. $\int_0^{1/2} x^3\left(1+9x^4\right)^{-3/2} dx$

22. $\int_0^4 \frac{x}{x^2+1} dx$

23. $\int_0^{\ln 2} e^{-x}\sinh x \, dx$

24. Find the area of the region bounded by the graph of $f(x)=16-x^2$ *on* $[-4,\ 4]$

25. Find the area of the region bounded by the graph of $f(x)=x^2-4x+3$ *on* $0 \le x \le 3$

Lecture Five

Application of Integration

Section 5.1 – Velocity and Net Change

5.1−1 Velocity, Position, and Displacement

Definitions

1. ***Position*** of an object at time *t*, denoted $s(t)$, is the location of the object relative to the origin.

2. ***Velocity*** of an object at time *t* is $v(t) = s'(t)$

3. ***Displacement*** of the object between $t = a$ and $t = b > a$ is:

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

4. ***Distance traveled*** by the object between $t = a$ and $t = b > a$ is:

$$\int_a^b |v(t)|\,dt$$

where $|v(t)|$ is the speed of the object at time *t*.

$$Displacement = s(b) - s(a) > 0$$

$$Displacement = s(b) - s(a) < 0$$

$$Displacement = A_1 - A_2 = \int_a^b v(t)\,dt$$

$$Displacement = A_1 + A_2 = \int_a^b |v(t)|\,dt$$

Example 1

A cyclist pedals along a straight road with velocity $v(t) = 2t^2 - 8t + 6 \ \ (mi/hr)$ for $0 \le t \le 3$, where t is measured in *hours*.

a) Graph the velocity function over the interval [0, 3]. Determine when the cyclist moves in the position direction and when she moves in the negative direction.

b) Find the displacement of the cyclist (in miles) on the time intervals [0, 1], [1, 3], and [0, 3]. Interpret these results.

c) Find the distance traveled over the interval [0, 3]

Solution

a) $v(t) = 2t^2 - 8t + 6 = 0$

$t = 1,\ 3$

The velocity is zero at $t = 1$ and $t = 3$.

The velocity is positive on, which means the cyclist moves in the positive s direction.

The velocity is negative on $1 < t < 3$, which means the cyclist moves in the negative s direction.

b) Displacement over [0, 1]

$$s(1)-s(0)=\int_0^1 v(t)\,dt$$
$$=\int_0^1 \left(2t^2-8t+6\right)dt$$
$$=\frac{2}{3}t^3-4t^2+6t\Big|_0^1$$
$$=\frac{2}{3}-4+6$$
$$=\frac{8}{3}\quad mi$$

Displacement over [1, 3]

$$s(3)-s(1)=\int_1^3 \left(2t^2-8t+6\right)dt$$
$$=\frac{2}{3}t^3-4t^2+6t\Big|_1^3 \quad =-\frac{8}{3}\quad mi$$
$$=18-36+18-\frac{2}{3}+4-6$$

Displacement over [0, 3]

$$s(3)-s(0)=\int_0^3 \left(2t^2-8t+6\right)dt$$
$$=\frac{2}{3}t^3-4t^2+6t\Big|_0^3$$
$$=18-36+18$$
$$=0\quad mi$$

The cyclist returns to the starting point after 3 *hours*.

c) $$Distance=\int_0^3 \left|v(t)\right|\,dt$$
$$=\int_0^1 \left(2t^2-8t+6\right)dt-\int_1^3 \left(2t^2-8t+6\right)dt$$

$$= \frac{2}{3}t^3 - 4t^2 + 6t \Big|_0^1 - \left(\frac{2}{3}t^3 - 4t^2 + 6t\right)\Big|_1^3$$

$$= \frac{8}{3} + \frac{8}{3}$$

$$= \frac{16}{3} \quad mi$$

Displacement from $t = 0$ to $t = 3$ is 0

$$Distance = \int_0^3 |v(t)|\, dt$$

$$= \frac{8}{3} + \frac{8}{3}$$

$$= \frac{16}{3}$$

5.1–2 Future Value of the Position Function

Theorem

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underset{\substack{position \\ at\ time\ t}}{s(t)} = \underset{\substack{initial \\ position}}{s(0)} + \underbrace{\int_0^t v(x)\,dx}_{\substack{displacement \\ over\ [0,\ t]}}$$

Example 2

A block hangs at rest from a massless spring at the origin ($s = 0$). At $t = 0$, the block is pulled downward $\frac{1}{4}m$ to its initial position $s(0) = -\frac{1}{4}$ and released. Its velocity is given by $v(t) = \frac{1}{4}\sin t \ (m/s)$ for $t \geq 0$. Assume that the upward direction is positive.

$s(0) = -\frac{1}{4}$

a) Find the position of the block for $t \geq 0$
b) Graph the position function for $0 \leq t \leq 3\pi$.
c) When does the block move through the origin for the first time?
d) When does the block reach its highest point for the first time and what is its position at that time?
e) When does the block return to its lowest point?

Solution

a)

1st *method*	2nd *method*
$s(t) = \int v(t)\,dt$ $= \int \frac{1}{4}\sin t\,dt$ $= -\frac{1}{4}\cos t + C$ Since $s(0) = -\frac{1}{4}$, then $-\frac{1}{4} = -\frac{1}{4}\cos(0) + C$ $\rightarrow \underline{C = 0}$ $.\ \underline{s(t) = -\frac{1}{4}\cos t}$.	$s(t) = s(0) + \int_0^t v(x)\,dx$ $= -\frac{1}{4} + \int_0^t \frac{1}{4}\sin x\,dx$ $= -\frac{1}{4} - \frac{1}{4}(\cos x \Big\vert_0^t$ $= -\frac{1}{4} - \frac{1}{4}(\cos t - 1)$ $\underline{= -\frac{1}{4}\cos t}$

b)

c) The block moves through the origin for the first time when $s=0$

$$s(t)=-\frac{1}{4}\cos t=0$$

$$\rightarrow \quad \boxed{t=\frac{\pi}{2} \quad sec}$$

d) The block moves in the positive direction and reaches its high point for the first time when $t=\pi$

$$s(\pi)=-\frac{1}{4}\cos\pi$$

$$\boxed{=\frac{1}{4} \quad m}$$

e) The block returns to the lowest point at $t=2\pi$. This motion repeats every 2π *seconds*

Example 3

Suppose a skydiver leaps from a hovering helicopter and fall in a straight line. He falls at a terminal velocity of 80 *m/s* for 19 *sec*, at which time he opens his parachute.
The velocity decreases linearly to 6 *m/s* over two-second period and then remains constant until he reaches the ground at t = 40 *s*. The motion is described by the velocity function

$$v(t)=\begin{cases} 80 & if \;\; 0\le t<19 \\ 783-37t & if \;\; 19\le t<21 \\ 6 & if \;\; 21\le t\le 40 \end{cases}$$

Determine the altitude from which the skydiver jumper.

Solution

$$.d=\int_0^{40}|v(t)|\,dt\,.$$

$$=\int_0^{19}80\,dt+\int_{19}^{21}(783-37t)\,dt+\int_{21}^{40}6\,dt$$

$$= 80t \Big|_0^{19} + \left(783\,t - \frac{37}{2}t^2\right)\Big|_{19}^{21} + 6t\Big|_{21}^{40}$$

$$= 1{,}520 + 783(21) - \tfrac{1}{2}(37)(21)^2 - 783(19) + \tfrac{1}{2}(37)(19)^2 + 240 - 126$$

$$= 1{,}720 \ \ m$$

The skydiver jumped from 1720 *m* above the ground.

5.1–3 Acceleration

Theorem (velocity from Acceleration)

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

Example 4

An artillery shell is fired directly upward with an initial velocity of 300 *m/s* from a point 30 *m* above the ground. Assume that only the force of gravity acts on the shell and it produces an acceleration of 9.8 m/s^2 .

Find the velocity of the shell for $t \geq 0$

Solution

$$.\,v(t) = v(0) + \int_0^t a(x)\,dx\,.$$

$$= 300 + \int_0^t (-9.8)\,dx \qquad \textit{Upward}$$

$= 300 - 9.8t$

The velocity decreases from its initial value of 300 *m/s*, reaching zero at the high point of the trajectory when

$$v(t) = 300 - 9.8t = 0$$

$$t = \frac{300}{9.8}$$

$$= \frac{1500}{49} \quad sec$$

$$\approx 30.6 \quad sec$$

At this point the velocity becomes negative, and the shell begins its descent to Earth.

5.1–4 Net Change and Future Value

Theorem

Suppose a quantity Q changes over time at a known rate Q'. Then the net change in Q between $t = a$ and $t = b$ is

$$\int_a^b Q'(t)\,dt = Q(b) - Q(a)$$

$$= \textit{net change in } Q \textit{ over } [a,\ b]$$

$$\int_0^t Q'(t)\,dt = Q(t) - Q(0)$$

Given the ***initial value*** $Q(0)$, the ***future value*** Q at future times $t \geq 0$ is

$$\underset{\substack{\textit{future}\\ \textit{value}}}{Q(t)} = \underset{\substack{\textit{initial}\\ \textit{value}}}{Q(0)} + \underbrace{\int_0^t Q'(t)\,dt}_{\substack{\textit{net change}\\ \textit{over } [0,\ t]}}$$

Velocity–Displacement Problems	*General Problems*
Position $s(t)$	Quantity $Q(t)$ (such as volume or population size)
Velocity: $s'(t) = v(t)$	Rate of change $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$

Example 5

A culture of cells in a lab has a population of 100 cells when nutrients are added at time $t = 0$. Suppose the population $N(t)$ increases at a rate given by

$$N'(t) = 90e^{-0.1t} \quad \textit{cells / hr}$$

Find $N(t)$ for $t \geq 0$

Solution

$$N(t) = N(0) + \int_0^t N'(x)\,dx$$

$$= 100 + \int_0^t 90e^{-0.1x}\,dx$$

$$= 100 - \frac{90}{0.1}\left(e^{-0.1x}\right|_0^t$$

$$= 100 - 900\left(e^{-0.1t} - 1\right)$$

$$= 1000 - 900e^{-0.1t}$$

The graph of the population function shows that the population increases, but at a decreasing rate. Note that the initial condition $N(0) = 100$ cells is satisfied, and that population size approaches 1000 cells as $t \to \infty$.

Example 6

A book publisher estimates that the marginal cost of a particular title (in dollars/book) is given by

$$C'(x) = 12 - 0.0002x$$

Where $0 \le x \le 50{,}000$ is the number of books printed. What is the cost of producing the 12,001st through the 15,000 books?

Solution

$$C(15{,}000) - C(12{,}000) = \int_{12{,}000}^{15{,}000} C'(x)\,dx$$

$$= \int_{12{,}000}^{15{,}000} (12 - 0.0002x)\,dx$$

$$= 12x - 0.0001x^2 \Big|_{12{,}000}^{15{,}000}$$

$$= 180 \times 10^3 - 225 \times 10^2 - 144 \times 10^3 + 144 \times 10^2$$

$$= \$27{,}900.00$$

5.1–5 Resource Consumption

The rate at which energy is conssumed is called ***power***.
The basic unit power is the ***watt*** (***W***).
The basic unit energy is the ***joule*** (***J***).

$$1\,W = 1\,J/s$$

$$\text{Total energy used} = \int_a^b E'(t) = \int_a^b P(t)\,dt$$

$E(t)$: the total energy used

$P(t)$: Power is the rate at which energy used

Example 7

At the beginning of 2010, the rate energy consumption for the city of Denver was 7,000 megawatts (MW), where $1\,MW = 10^6\ W$. That rate is expected to increase at an annual growth rate of 2% per year.

a) Find the function that gives the power or rate of energy consumption for all times after the beginning of 2010.
b) Find the total amount of energy used during 2014.
c) Find the function that gives the total (cumulative) amount of energy used by the city between 2010 and any time $t \geq 0$.

Solution

a) Let $t \geq 0$, be the number of years after the brginning of 2010.

$$k = \frac{1}{T}\ln\left(\frac{P(1)}{P_0}\right)$$
$$= \ln 1.02$$
$$\approx 0.0198$$

$$P(t) = 7{,}000e^{0.0198t}, \quad t \geq 0$$

b) Entire year 2014 $\rightarrow$ $4 \leq t \leq 5$

$$Total\ energy = \int_4^5 P(t)\,dt$$
$$= \int_4^5 7{,}000\,e^{0.0198t}\,dt$$

$$= \frac{7000}{0.0198} e^{0.0198t} \Big|_4^5$$

$$\approx 7652 \quad MW - yr$$

$$\approx 7652 \ (MW \cdot yr) \times 8760 \frac{hr}{yr}$$

$$\approx 6.7 \times 10^7 \quad MWh$$

c) The total (cumulative) amount of energy used $t \geq 0$ is given by

$$E(t) = E(0) + \int_0^t E'(s)\, ds$$

$$= E(0) + \int_0^t P(s)\, ds$$

$$= 0 + \int_0^t 7000\, e^{0.0198s}\, ds$$

$$\approx 353{,}535\left(e^{0.0198t} - 1\right)$$

The total amount of energy consumed increases expotentially.

Exercises *Section* 5.1 – Velocity and Net Change

$(\mathbf{1-3})$ Assume t is time measured in seconds and velocities have units of *m/s*.

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

1. $v(t) = 6 - 2t$; $0 \le t \le 6$ **2.** $v(t) = 10\sin 2t$; $0 \le t \le 2\pi$ **3.** $v(t) = 50e^{-2t}$; $0 \le t \le 4$

$(\mathbf{4-5})$ Consider an object moving along a line with the following velocities and initial positions

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.
b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.
c) Graph the position function on the given interval.

4. $v(t) = 6 - 2t$ *on* $[0,\ 5]$ $s(0) = 0$ **5.** $v(t) = 9 - t^2$ *on* $[0,\ 4]$ $s(0) = -2$

$(\mathbf{6-7})$ Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

6. $a(t) = -9.8,\quad v(0) = 20,\quad s(0) = 0$ **7.** $a(t) = e^{-t},\quad v(0) = 60,\quad s(0) = 40$

8. A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t) = 2\pi\cos\pi t$ for $t \ge 0$. Assume that the position direction is upward and $s(0) = 0$.

a) Determine the position function for $t \ge 0$.
b) Graph the position function on the interval [0, 3].
c) At what times does the mass reach its lowest point the first three times?
d) At what times does the mass reach its highest point the first three times?

9. The velocity of an airplane flying into a headwind is given by $v(t) = 30\left(16 - t^2\right)$ *mi/hr* for $0 \le t \le 3$ *hr*. Assume that $s(0) = 0$

a) Determine and graph the position function for $0 \le t \le 3$.
b) How far does the airplane travel in the first 2 *hr*.?
c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

10. A car slows down with an acceleration of $a(t) = -15\ ft/s^2$. Assume that $v(0) = 60\ ft/s$ and $s(0) = 0$

a) Determine and graph the position function for $t \geq 0$.
b) How far does the car travel in the time it takes to come to rest?

11. The owners of an oil reserve begin extracting oil at $t = 0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t) = 3t^2(40-t)^2$, where $0 \leq t \leq 40$, Q is measured in millions of barrels, and t is measured in *years*.

a) When does the peak extraction rate occur?
b) How much oil is extracted in the first 10, 20, and 30 *years*?
c) What is the total amount of oil extracted in 40 *years*?
d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

12. Starting with an initial value of $P(0) = 55$, the population of a prairie dog community grows at a rate of $P'(t) = 20 - \frac{t}{5}$ (in units of prairie *dogs/month*), for $0 \leq t \leq 200$.

a) What is the population 6 months later?
b) Find the population $P(t)$ for $0 \leq t \leq 200$.

13. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?
b) Find the population $P(t)$ at any time $t \geq 0$.

14. A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr*.) given by $u(\varphi) = 3 - 2\cos\varphi$ and Bess runs with a speed given by $v(\theta) = 3 + 2\cos\theta$, where φ and θ are the central angles of the runners

a) Graph the speed functions *u* and *v*, explain why they describe the runners' speed (considering the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}\ mi$, how long does it take each runner to complete one lap and who wins the race?

15. A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty, and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time *t*, the flow rate of water into reservoir (in m^3 / hr) oscillates on 24-*hr* cycle and is given by

$$Q'(t) = 20\left(1 + \cos\frac{\pi t}{12}\right)$$

a) How much water flows into the reservoir in the first 2 *hrs*.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0,\ t]$ where $t \geq 0$.

c) When is the reservoir full?

16. The velocity of an object moving along a line is given by $v(t) = 20\cos\pi t\ \ (ft / s)$. What is the displacement of the object after 1.5 *sec*?

17. A projectile is launched vertically from the ground at $t = 0$, and its velocity in flight (in *m/s*) is given by $v(t) = 20 - 10t$. Find the position, displacement, and distance traveled after *t* seconds, for $0 \leq t \leq 4$

18. At $t = 0$, a car begins decelerating from a velocity of 80 *ft/s* at a constant rate of $5\ ft / s^2$. Find its position function assuming $s(0) = 0$.

19. The acceleration of an object moving along a line is given by $a(t) = 2\sin\left(\frac{\pi t}{4}\right)$. The initial velocity and position are $v(0) = -\frac{8}{\pi}$ and $s(0) = 0$

a) Find the velocity and position for $t \geq 0$

b) What are the minimum and maximum values of *s*?

c) Find the average velocity and average position over the interval $[0,\ 8]$

20. Starting at the same point on a straight road, Anna and Benny begin running with velocities (in *mi/hr*) given by $v_A(t) = 2t + 1$ and $v_B(t) = 4 - t$, respectively.

a) Graph the velocity functions, for $0 \leq t \leq 4$.

b) If the runners run for 1 *hr*, who runs farther? Interpret your conclusion geometrically using the graph in part (*a*).

c) If the runners run for 6 *mi* who wins the race? Interpret your conclusion geometrically using the graph in part (*a*).

21. A small plane in flight consumes fuel at a rate (in *gal/min*) given by

$$R'(t) = \begin{cases} 4t^{1/3} & if \quad 0 \leq t \leq 8 \quad (take-off) \\ 2 & if \quad t > 8 \quad (cruising) \end{cases}$$

a) Find a function *R* that gives the total fuel consumed, for $0 \leq t \leq 8$

b) Find a function *R* that gives the total fuel consumed, for $t \geq 0$

c) If the fuel tank capacity is 150 *gal*, when does the fuel run out?

22. Water flows out of a tank at a rate $\left(in\ m^3 / hr\right)$ given by $V'(t) = \frac{15}{t+1}$. If the tank initially holds 75 m^3 of water, when will the tank be empty?

23. A projectile is fired upward, and its velocity in *m/s* is given by $v(t) = 200e^{-t/10}, \quad for \quad t \geq 0$.

a) Graph the velocity function, for $t \geq 0$.

b) When does the velocity reach 50 *m/s*?

c) Find and graph the position function for the projectile for $t \geq 0$, assuming $s(0) = 0$.

d) Given unlimited time, can the projectile travel 2500 *m*? If so, at what time does the distance traveled equal 2500 *m*?

24. A projectile is fired upward, and its velocity in *m/s* is given by $v(t) = \frac{200}{\sqrt{t+1}}, \quad for \quad t \geq 0$.

a) Graph the velocity function, for $t \geq 0$.

b) Find and graph the position function for the projectile for $t \geq 0$, assuming $s(0) = 0$.

c) Given unlimited time, can the projectile travel 2500 *m*? If so, at what time does the distance traveled equal 2500 *m*?

25. *Jeff* and *Mel* took a bike ride, both starting at the same time and position. Jeff started riding at 20 *mi/hr*, and his velocity decreased according to the function $v(t) = 20e^{-2t}, \quad for \;\; t \geq 0$. Mel started riding at 15 *mi/hr*, and her velocity decreased according to the function $u(t) = 15e^{-t}, \quad for \;\; t \geq 0$

a) Find and graph the positions of *Jeff* and *Mel*.
b) Find the times at which the riders have the same position at the same time.
c) Who ultimate took the lead and remained in the lead?

26. Skydivers ***A*** and ***B*** are in a helicopter hovering at 6,400 *ft*. Skydiver ***A*** jumps and descends for 4 *sec* before opening her parachute. The helicopter then climbs to 7,000 *ft.* and hovers there. Forty-five seconds after ***A*** leaves the aircraft, ***B*** jumps and descends for 13 *sec* before opening his parachute. Both skydivers descend at 16 *ft/sec* with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a) At what altitude does ***A***'s parachute open?

b) At what altitude does ***B***'s parachute open?

c) Which skydiver lands first?

27. Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

28. On the first day of the year $(t = 0)$, a city uses electricity at a rate of 2000 *MW*. That rate is projected to increase at a rate of 1.3% per *year*.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.
b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t = 0$
c) Find a function that gives the total energy used (in MW-*yr*) between $t = 0$ and any future time $t > 0$

Section 5.2 – Area Between Curves

5.2–1 *Areas* between Curves

Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout [*a*, *b*], then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Area of $f(x)$

$$A_f = \int_a^b f(x)\, dx$$

Area of $g(x)$

$$A_g = \int_a^b g(x)\, dx$$

Area between $f(x)$ & $g(x)$

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Example 1

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

The limits of integrations are found by letting:

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$\boxed{x = -1,\ 2}$

Testing $0 \rightarrow \begin{cases} (1) = 2 \\ (2) = 0 \end{cases}$

$$Area = \int_{-1}^{2} \left(f(x) - g(x) \right) dx$$

$$= \int_{-1}^{2} \left(2 - x^2 - (-x) \right) dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x \right) dx$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$

$$= 4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2} \right)$$

$$= 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2}$$

$$= 5 - \frac{1}{2}$$

$$= \frac{9}{2} \quad unit^2$$

Example 2

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution

$$\left(y = \sqrt{x} \right) \cap \left(y = 0 \right) \quad \rightarrow \quad (0,\ 0)$$

$$\left(y = \sqrt{x} \right) \cap \left(y = x - 2 \right) \quad \rightarrow \quad \sqrt{x} = x - 2$$

$$\left(\sqrt{x} \right)^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\rightarrow \quad x = \cancel{1},\ 4$$

$$(y = 0) \cap (y = x - 2) \quad \rightarrow \quad x = 2$$

$$\text{Total Area} = \int_{0}^{2} \left(\sqrt{x} - 0 \right) dx + \int_{2}^{4} \left(\sqrt{x} - (-x + 2) \right) dx$$

$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big|_2^4$$

$$= \frac{2}{3}\left(2^{3/2}\right) - 0 + \left(\frac{2}{3}4^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3}2^{3/2} - \frac{2^2}{2} + 2(2)\right)$$

$$= \frac{2}{3}\left(2^{3/2}\right) + \frac{2}{3}4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3}2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3}\ unit^2$$

5.2–1 Integration with Respect to *y*

$$A = \int_c^d \left(f(y) - g(y)\right) dy$$

Example 3

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$.

Solution

$$y = \sqrt{x} \quad \rightarrow \quad x = y^2$$
$$y = x - 2 \quad \rightarrow \quad x = y + 2$$

$$\left(x = y^2\right) \cap (y = 0) \quad \rightarrow \quad (0,\ 0)$$

$$\left(x = y^2\right) \cap (x = y + 2)$$

$$y^2 = y + 2$$

$$y^2 - y - 2 = 0 \quad \rightarrow \quad y = -1,\ 2$$

$$(y = 0) \cap (x = y + 2) \quad \rightarrow \quad y = 0$$

$$Area = \int_0^2 \left(y + 2 - y^2\right) dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3} \ unit^2$$

Example 4

Find the area of the region bounded between by the graphs of $x = 2y^2 - 4$, $x = y^2$

Solution

$$2y^2 - 4 = y^2$$

$$y^2 = 4$$

$$y = \pm 2$$

$$Testing \quad 0 \quad \rightarrow \begin{cases} (1) = -4 \\ (2) = 0 \end{cases}$$

$$Area = \int_{-2}^{2} \left(y^2 - 2y^2 + 4\right) dy$$

$$= 2\int_0^2 \left(-y^2 + 4\right) dy$$

$$= 2\left(-\frac{1}{3}y^3 + 4y\right. \Big|_0^2$$

$$= 2\left(-\frac{8}{3} + 8\right)$$

$$= \frac{32}{3} \ unit^2$$

Example 5

Find the area of the region bounded between by the graphs of $y = \ln x$, $y = 0$, $x = e$

Solution

$\ln x = 0$

$x = 1$

$$Area = \int_1^e \ln x \, dx \quad ?$$

$y = \ln x \quad \rightarrow \quad x = e^y$

$x = e \quad \rightarrow \quad y = 1$

$$Area = \int_0^1 \left(e - e^y\right) dy$$

$$= ey - e^y \Big|_0^1$$

$$= e - e + 1$$

$$= 1 \ unit^2$$

Example 6

Find the area of the region bounded by the graphs of $y = \cos x$, $y = -\sin x$, $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

Solution

$y = \cos x = -\sin x \qquad \rightarrow QII, IV$

$x = \frac{3\pi}{4} + n\pi$

$n = 0 \rightarrow \quad x = \frac{3\pi}{4}$

$n = -1 \rightarrow \quad x = -\frac{\pi}{4}$

Testing $0 \rightarrow \begin{cases} (1) = 1 \\ (2) = 0 \end{cases}$

$$Area = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\cos x + \sin x\right) dx$$

$$= \sin x - \cos x \Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= 2\sqrt{2} \quad unit^2$$

Example 7

Find the area of the region bounded by the graphs of $y = x^3$, $y - x = 6$, $2y + x = 0$

Solution

$$y = x^3 \quad \cap \quad y - x = 6$$

$$x^3 - x = 6$$

$$x = 2$$

$$y = x^3 \quad \cap \quad 2y + x = 0$$

$$2x^3 + x = 0$$

$$x\left(2x^2 + 1\right) = 0$$

$$x = 0$$

$$y - x = 6 \quad \cap \quad 2y + x = 0$$

$$x + 6 = -\frac{1}{2}x$$

$$\frac{3}{2}x = -6$$

$$x = -4$$

$$Area = \int_{-4}^{0} \left(x + 6 + \frac{1}{2}x\right) dx + \int_{0}^{2} \left(x + 6 - x^3\right) dx$$

$$= \frac{3}{4}x^2 + 6x \Big|_{-4}^{0} + \left(\frac{1}{2}x^2 + 6x - \frac{1}{4}x^4\right. \Big|_{0}^{2}$$

$$= -12 + 24 + 2 + 12 - 4$$

$$= 22 \quad unit^2$$

Exercises *Section* 5.2 – Area Between Curves

$(\mathbf{1}-\mathbf{79})$ Find the area of the region bounded by the graphs of:

1. $y=2x-x^2$ and $y=-3$

2. $y=7-2x^2$ and $y=x^2+4$

3. $y=x^4-4x^2+4$ and $y=x^2$

4. $x=2y^2,\quad x=0,\quad and\quad y=3$

5. $x=y^3-y^2\quad and\quad x=2y$

6. $4x^2+y=4\quad and\quad x^4-y=1$

7. $y=\sin\frac{\pi x}{2}\quad and\quad y=x$

8. $y=3-x^2\ and\ y=2x$

9. $y=x^2-x-2\ and\ x-$axis

10. $y=\sqrt{x},\quad y=x\sqrt{x}$

11. $y=x^{1/2}\quad and\quad y=x^3$

12. $x+4y^2=4,\quad x+y^4=1,\quad x\ge 0$

13. $y=2\sin x,\quad y=\sin 2x,\quad 0\le x\le\pi$

14. $y=x^2+1\ and\ y=x$ for $0\le x\le 2$

15. $y=x^2-2x\ and\ y=x$ on $0\le x\le 4$

16. $x=1,\quad x=2,\quad y=x^3+2,\quad y=0$

17. $y=x^2-18,\quad y=x-6$

18. $y=-x^2+3x+1,\quad y=-x+1$

19. $y=x,\quad y=2-x,\quad y=0$

20. $y=\frac{4}{x^2},\quad y=0,\quad x=1,\quad x=4$

21. $f(y)=y^2,\quad g(y)=y+2$

22. $f(x)=2^x,\quad g(x)=\frac{3}{2}x+1$

23. $x=\sqrt[3]{y}\quad and\quad x=\sqrt[5]{y}$

24. $f(x)=\sqrt[3]{x-1},\quad g(x)=x-1$

25. $f(y)=y(2-y),\quad g(y)=-y$

26. $f(x)=x^3+2x^2-3x,\quad g(x)=x^2+3x$

27. $y=\sec^2 x,\quad y=\tan^2 x,\quad x=-\frac{\pi}{4},\quad x=\frac{\pi}{4}$

28. $f(x)=\sqrt{x}+3,\quad g(x)=\frac{1}{2}x+3$

29. $f(y)=\frac{y}{\sqrt{16-y^2}},\quad g(y)=0,\quad y=3$

30. $f(y)=y^2+1,\quad g(y)=0,\quad y=-1,\quad y=2$

31. $f(x)=\frac{10}{x},\quad x=0,\quad y=2,\quad y=10$

32. $g(x)=\frac{4}{2-x},\quad y=4,\quad x=0$

33. $f(x)=\cos x,\quad g(x)=2-\cos x,\quad 0\le x\le 2\pi$

34. $f(x)=\sin x,\quad g(x)=\cos 2x,\quad -\frac{\pi}{2}\le x\le\frac{\pi}{6}$

35. $f(x)=2\sin x,\quad g(x)=\tan x,\quad -\frac{\pi}{3}\le x\le\frac{\pi}{3}$

36. $f(x)=xe^{-x^2},\quad y=0,\quad 0\le x\le 1$

37. $y=\sin x\ and\ y=x\quad 0\le x\le 2\pi$

38. $y=x^2,\ y=2x^2-4x$ and $y=0$

39. $y=8\cos x,\quad y=\sec^2 x,\quad -\frac{\pi}{3}\le x\le\frac{\pi}{3}$

40. $y^2=4x+4,\quad y=4x-16$

41. $x=2y^2,\quad x=0,\quad y=3$

42. $x=y^3\quad and\quad y=x$

43. $x=y^3\quad and\quad y=x^2$

44. $y=x^3\quad and\quad x=y^2$

45. $y=x^2\quad and\quad y=2x$

46. $y=x^2\quad and\quad y=x^4$

47. $y=4-x^2\quad and\quad y=-2$

48. $y=\sqrt{x}\quad and\quad y=x$

49. $y=\sqrt{x}\quad and\quad y=-x$

50. $y=\sqrt{x}$, $y=-x$ $1\le x\le 4$

51. $y=\sqrt{x}$ *and* $y=x^2$

52. $y=\sqrt{x}$ *and* $y=x^3$

53. $y=\cos x$, $y=\frac{1}{2}$ $0\le x\le \frac{\pi}{3}$

54. $y=\sin x$, $y=\frac{1}{2}$ $0\le x\le \frac{\pi}{3}$

55. $y=3-x$ *and* $y=3-x^2$

56. $y=x^2-4$ *and* $y=4-x^2$

57. $y=3x-8$ *and* $y=10x-x^2$

58. $x=y^2$ *and* $x=y+2$

59. $y=x^3-x^2-6x$ *and* $y=0$

60. $y=\sqrt{4-4x}$ *and* $y=\sqrt{4-x}$

61. $y=\operatorname{sech}\frac{x}{2}$, $-4\le x\le 4$

62. $y=\tanh 2x$, $0\le x\le 2$

63. $y=\sinh 3x$, $0\le x\le 1$

64. $y=\dfrac{5x}{\sqrt{x^4+1}}$, $0\le x\le 2$

65. $y=\dfrac{6}{\sqrt{x^2-4}}$, $3\le x\le 5$

66. $f(x)=8\operatorname{sech}^2 x$ *and* $g(x)=\cosh x$

67. $y=x\sqrt{x^2-9}$, $y=0$, $x=5$

68. $y=e^x$, $y=e^{-x}$, $x=\ln 2$

69. $y=e^x$, $y=1-x$, $x=1$

70. $y=x$, $y=3x$, $y=4-x$

71. $y=x^2-4$, $y=x+2$, $y=2-x$

72. $y=4-x^2$, $y=x$, $y=-x$

73. $y=x+1$, $y=7x-17$, $y=-2x-2$

74. $y=x-2$, $y=x+2$, $y=2-x$, $y=-x-2$

75. $f(x)=\sec\frac{\pi x}{4}\tan\frac{\pi x}{4}$, $g(x)=\left(\sqrt{2}-4\right)x+4$, $x=0$

76. $f(x)=-x^2+1$, $g(x)=2x+4$, $x=-1$, $x=2$

77. $x=y^2-y$ & $x=2y^2-2y-6$

78. $y=x^2-4$, $y=-x^2-2x$, $-3\le x\le 1$

79. $y=x^2$, $y=2x^2-4x$, $y=0$

80. Find the area of the region bounded by $y=\operatorname{sech} x$, $x=1$, and the unit circle.

81. Find the area of the region in the first quadrant bounded by $y=4x$ *and* $y=x\sqrt{25-x^2}$.

82. Find the area of the region in the first quadrant bounded by the curve $\sqrt{x}+\sqrt{y}=1$.

83. Find the area of the region in the first quadrant bounded by $y=\frac{x}{6}$ and $y=1-\left|\frac{x}{2}-1\right|$.

84. Find the area of the region in the first quadrant bounded by $y=x^p$ and $y=\sqrt[p]{x}$ where $p=100$ and $p=1000$.

85. Consider the functions $y=\frac{x^2}{a}$ and $y=\sqrt{\frac{x}{a}}$, where $a>0$. Find $A(a)$, the area of the region between the curves.

86. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

87. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and lines $x = y$ and $y = -1$.

88. Find the area of the "triangular region in the first quadrant bounded on the left by the *y-axis* and on the right by the curves $\sin x$ *and* $\cos x$.

89. Find the area of the "triangular region in the first quadrant bounded above by the curve $y = e^{2x}$, below by the curve $y = e^{x}$, and on the right by the line $x = \ln 3$.

90. Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$

91. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of *f* and the *x*-axis.

(92 – 109) Determine the area of the shaded region in the following:

92.

93.

94.

95.

96.

97.

98. $y = \sin x$ *and* $y = \sin 2x$, $0 \le x \le \pi$

99. Bounded by $x^2 = y^4\left(1 - y^3\right)$

100. bounded by $x = y^2$ and $x = \left(2 - y^2\right)^2$

101. $x^3 + \sqrt{y} = 1$, $x = 0$, $y = 0$, $0 \le x \le 1$

102. $y = x^2 - 4$, $y = -x^2 - 2x$, $-3 \le x \le 1$

103. $y = \frac{1}{4}x^2$, $y = x$, $y = 1$

104. $y = -x^2 + 3x, \quad y = 2x^3 - x^2 - 5x, \quad -2 \le x \le 2$

105. $y = 4 - x^2, \quad y = -x + 2, \quad -2 \le x \le 3$

106. $y = \frac{1}{3}x^3 - x, \quad y = \frac{1}{3}x, \quad -2 \le x \le 3$

107. $y = \frac{\pi}{2}\cos x \ \sin(\pi + \pi \sin x) \quad -\pi \le x \le 0$

108. $y = x^2, \quad y = -2x^4, \quad -1 \le x \le 1$

109. $y = 2x^2, \quad y = x^4 - 2x^2, \quad -2 \le x \le 2$

110. Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.

111. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$

a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

112. Find the area of the regions R_1 and R_2 (separately) shown in the figure, which are formed by the graphs of $y = 16 - x^2$ and $y = 5x - 8$

113. Find the area of the regions R_1, R_2 and R_3 (separately) shown in the figure, which are formed by the graphs of $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x-3)$

114. Concrete sections for a new building have the dimensions (in *meters*) and shape shown in figure

a) Find the area of the face of the section superimposed on the rectangularcoordinate system.

b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.

c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

115. A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions

$$L(0) = 0,\ L(1) = 1,\ \textit{and}\ L'(x) \ge 0 \quad \textit{on}\ [0,\ 1]$$

c) Graph the Lorenz curves $L(x) = x^p$ corresponding to p = 1.1, 1.5, 2, 3, 4. Which value of p corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)?
Which value of p corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let A be the area of the region between $y = x$ and

$y = L(x)$. Let B be the area of the region between $y = L(x)$ and the x-axis. Then the Gini index is $G = \frac{A}{A+B}$.

Show that $G = 2A = 1 - 2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and p = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [a, b] on which values of the Gini index lie, for $L(x) = x^p$ with $p \geq 1$? Which endpoints of [a, b] correspond to the least and most equitable distribution of wealth?

g) Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \geq 0 \quad on\ [0,\ 1]$. Find the Gini index for this function.

116. Let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic function and choose two points $x = p$ and $x = q$. Let L_1 be the line tangent to the graph of f at the point $(p,\ f(p))$ and let L_2 be the line tangent to the graph at the point $(q,\ f(q))$. Let $x = s$ be the vertical line through the intersection point of L_1 and L_2. Finally, let R_1 be the region bounded by $y = f(x)$, L_1, and the vertical line $x = s$, and let R_2 be the region bounded by $y = f(x)$, L_2, and the vertical line $x = s$.

Prove that the area of R_1 equals the area of R_2

117. Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.

What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^{1} \frac{1}{2x}\,dx$ and evaluate the integral.

118. Consider the parabola $y = x^2$. Let P, Q, and R be points on the parabola with R between P and Q on the curve. Let ℓ_P, ℓ_Q, and ℓ_R be the lines tangent to the parabola at P, Q, and R, respectively. Let P' be the intersection point of ℓ_Q and ℓ_R; let Q' be the intersection point of ℓ_P and ℓ_R; and let R' be the intersection point of ℓ_P and ℓ_Q. Prove that Area $\triangle PQR = 2 \bullet Area\ \triangle P'Q'R'$ in the following cases.

a) $P\left(-a,\ a^2\right)$, $Q\left(a,\ a^2\right)$, and $R(0,\ 0)$, where a is a positive real number

b) $P\left(-a,\ a^2\right)$, $Q\left(b,\ b^2\right)$, and $R(0,\ 0)$, where a and b are positive real numbers

c) $P\left(-a,\ a^2\right)$, $Q\left(b,\ b^2\right)$, and R is any point between P and Q on the curve.

Section 5.3 – Volume

5.3–1 Volume by Slicing

If we want to find a volume of a solid S and if the cylindrical solid has known base area A and height h, then the volume of the cylindrical solid is

$$Volume = area \times height = A \bullet h$$

5.3–2 *Definition*

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b

$$V = \int_a^b A(x)\,dx$$

Calculating the volume of a solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

Example 1

A pyramid 3*m* high has a square base that is 3*m* on a side. The cross-section of the pyramid perpendicular to the altitude ***x*** *m* down from the vertex is a square ***x*** *m* on a side. Find the volume of the pyramid.

Solution

The area of the square is given by the formula: $A(x) = x^2$

The volume: $V = \int_0^3 A(x)\,dx$

$$= \int_0^3 x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_0^3$$

$$= \frac{1}{3}\left(3^3 - 0\right)$$

$$= 9 \ m^3$$

Example 2

A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2 + y^2 = 9$.

Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is semi-circle.

When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at *x* which is a rectangle of height *x*.

The area of this cross-section is: $A(x) = height \times width$

$$= x\left(2\sqrt{9-x^2}\right)$$

$$= 2x\sqrt{9-x^2}$$

The rectangles run from $x = 0$ to $x = 3$, so

$$V = \int_0^3 A(x)\,dx$$

$$= \int_0^3 2x\sqrt{9-x^2}\,dx$$

$$= -\int_0^3 \left(9-x^2\right)^{1/2} d\left(9-x^2\right)$$

$$= -\frac{2}{3}\left(9-x^2\right)^{3/2}\Big|_0^3$$

$$= -\frac{2}{3}\left[\left(9-3^2\right)^{3/2} - \left(9-0^2\right)^{3/2}\right]$$

$$= 18 \ \ unit^3$$

5.3–3 Solids of Revolution: The *Disk* Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a ***solid of revolution***.

The cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution.

The area then

$$A(x) = \pi(radius)^2$$

$$= \pi\left[R(x)\right]^2$$

And the volume
$$V = \int_a^b A(x)\,dx$$
$$= \int_a^b \pi\left[R(x)\right]^2 dx$$

Example 3

The region between the curve $y = \sqrt{x}, \quad 0 \le x \le 4$, and the *x*-axis is revolved about the *x*-axis to generate a solid. Find its volume.

Solution

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx \qquad\qquad V = \int_a^b \pi\left[R(x)\right]^2 dx$$
$$= \pi \int_0^4 x\,dx$$
$$= \pi \left.\frac{x^2}{2}\right|_0^4$$
$$= \frac{\pi}{2}\left(4^2 - 0\right)$$
$$= 8\pi \ \ unit^3$$

Example 4

The circle $x^2 + y^2 = a^2$ is rotated about *x*-axis to generate a sphere. Find its volume.

Solution

$$A(x) = \pi y^2$$
$$= \pi\left(a^2 - x^2\right)$$

$$V = \int_{-a}^{a} \pi\left(a^2 - x^2\right) dx$$

$$= 2\pi \left(a^2 x - \frac{1}{3}x^3 \right|_0^a$$

$$= 2\pi\left(a^2(a) - \frac{1}{3}a^3\right)$$

$$= 2\pi\left(\frac{2}{3}a^3\right)$$

$$= \frac{4}{3}\pi a^3 \ \ unit^3$$

5.3–4 Volume by Disks for Rotation about the *y*-axis

$$V = \int_c^d A(y)\,dy$$

$$= \pi \int_c^d \left(R(y)\right)^2 dy$$

Example 5

Find the volume of the solid generated by revolving the region between the *y*-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the *y*-axis.

Solution

$$V = \pi \int_1^4 \left[R(y)\right]^2 dy$$

$$= \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$$

$$= \pi \int_1^4 \frac{4}{y^2} dy$$

$$= 4\pi \left(-\frac{1}{y} \right|_1^4$$

$$= 4\pi \left[-\frac{1}{4} - \left(-\frac{1}{1} \right) \right]$$

$$= 4\pi \left(\frac{3}{4} \right)$$

$$= 3\pi \ \ unit^3$$

Example 6

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution

$$x = y^2 + 1$$

$$y^2 = x - 1$$

When $x = 3 \quad \Rightarrow \quad y^2 = 2$

$$y = \pm\sqrt{2}$$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[R(y) \right]^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[3 - \left(y^2 + 1\right)\right]^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(2 - y^2\right)^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - 4y^2 + y^4\right) dy \qquad \textit{Even Function}$$

$$= 2\pi \left(4y - \tfrac{4}{3}y^3 + \tfrac{1}{5}y^5\right|_0^{\sqrt{2}}$$

$$= 2\pi\left(4\sqrt{2} - \tfrac{8}{3}\sqrt{2} + \tfrac{4}{5}\sqrt{2}\right)$$

$$= \frac{64\pi\sqrt{2}}{15} \ unit^3$$

5.3–5 Solids of Revolution: The *Washer Method*

If the region we revolved to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are ***washers*** instead of disks.

The dimensions of a typical washer are:

Outer radius: $R(x)$

Inner radius: $r(x)$

The washer's area is:

$$A(x) = \pi\left[R(x)\right]^2 - \pi\left[r(x)\right]^2$$

$$= \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)$$

The washer's volume:

$$V = \int_a^b A(x)\,dx$$

$$= \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx$$

Example 7

The R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is the greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution

About the ***x-axis***

$$x = y^3 = \left(\frac{x}{4}\right)^3$$

$$4^3 x - x^3 = 0$$

$$x\left(64 - x^2\right) = 0$$

$$\Rightarrow \quad \underline{x = 0,\ 8}\,,\quad \cancel{-8}\ (\notin QI)$$

$$V = \pi\int_0^8 \left(f(x)^2 - g(x)^2\right) dx$$

$$= \pi\int_0^8 \left(x^{2/3} - \frac{x^2}{16}\right) dx$$

$$= \pi\left(\frac{3}{5}x^{5/3} - \frac{1}{48}x^3\right)\Bigg|_0^8$$

$$= \pi\left(\frac{96}{5} - \frac{32}{3}\right)$$

$$= \frac{128\pi}{15} \quad unit^3$$

$$\approx 26.81 \ \text{unit}^3$$

About the ***y-axis***

$$x = y^3 = 4y$$

$$y\left(y^2 - 4\right) = 0$$

$$y = 0,\ 2, \quad \cancel{-2} \ (\notin QI)$$

$$V = \pi \int_0^2 \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \int_0^2 \left(16y^2 - y^6\right) dy$$

$$= \pi \left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right) \Big|_0^2$$

$$= \pi\left(\frac{128}{3} - \frac{128}{7}\right)$$

$$= \frac{512\pi}{21} \quad unit^3 \qquad \approx 76.60 \quad unit^3$$

The region that is revolving about the ***y-axis*** produces a solid of greater volume.

Example 8

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x–axis to generate a solid.

Find the volume of the solid.

Solution

$$y = x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0 \qquad \textit{Solve for x}$$

$$x = -2,\ 1$$

$$V = \int_{-2}^{1} \pi\left((-x+3)^2 - \left(x^2+1\right)^2\right)dx$$

$$= \pi\int_{-2}^{1}\left(x^2 - 6x + 9 - x^4 - 2x^2 - 1\right)dx$$

$$= \pi\int_{-2}^{1}\left(-x^4 - x^2 - 6x + 8\right)dx$$

$$= \pi\left(-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x\right|_{-2}^{1}$$

$$= \pi\left[\left(-\frac{1}{5} - \frac{1}{3} - 3 + 8\right) - \left(-\frac{(-2)^5}{5} - \frac{(-2)^3}{3} - 3(-2)^2 + 8(-2)\right)\right]$$

$$= \pi\left(5 - \frac{8}{15} - \frac{32}{5} - \frac{8}{3} + 28\right)$$

$$= \pi\left(33 - \frac{144}{15}\right)$$

$$= \pi\left(\frac{351}{15}\right)$$

$$= \frac{117\pi}{5} \quad unit^3$$

Example 9

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y–axis to generate a solid.

Find the volume of the solid.

Solution

$$y = x^2 \quad \rightarrow \quad x = \sqrt{y} = R(y)$$

$$y = 2x \quad \rightarrow \quad x = \frac{1}{2}y = r(y)$$

$$\sqrt{y} = \frac{1}{2}y \quad \rightarrow \quad 4y = y^2$$

$$y^2 - 4y = 0$$

$$\underline{y = 0,\ 4}$$

$$V = \int_0^4 \pi \left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{2}\right)^2 \right) dy$$

$$= \pi \int_0^4 \left(y - \frac{1}{4}y^2 \right) dy$$

$$= \pi \left(\frac{y^2}{2} - \frac{1}{12}y^3 \right|_0^4$$

$$= \pi \left(\frac{4^2}{2} - \frac{1}{12}(4)^3 \right)$$

$$\underline{= \frac{8\pi}{3} \quad unit^3}$$

Example 10

The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results when R is revolved about the x-axis?

Solution

$$V = \pi \int_0^1 \left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2 \right) dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_0^1 \left(x - x^4\right) dx$$

$$= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{5} \right)$$

$$= \frac{3\pi}{10} \quad unit^3$$

5.3−6 Slicing with Cylinders

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid

If we rotate a vertical strip of thickness Δx, this rotation produces a cylindrical shell of height y_k above a point x_k within the base of the vertical strip.

$$\Delta V_k = circumference \times height \times thickness$$

$$= 2\pi\left(1 + x_k\right) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x_k$$

The Riemann sum:

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot \Delta x_k$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot \Delta x_k$$

$$= 2\pi \int_0^3 (x+1)\left(3x - x^2\right)dx$$

$$= 2\pi \int_0^3 \left(3x^2 + 3x - x^2 - x^3\right)dx$$

$$= 2\pi \int_0^3 \left(2x^2 + 3x - x^3\right)dx$$

$$= 2\pi \left(\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4 \right|_0^3$$

$$= 2\pi \left(\frac{2}{3}(3)^3 + \frac{3}{2}(3)^2 - \frac{1}{4}(3)^4\right)$$

$$= \frac{45\pi}{2} \quad unit^3$$

5.3–7 *Shell* Method

The ***shell method*** is used to find the volume of a solid of revolution along an axis perpendicular to the axis of rotation using the surface area of successive cylinders within the solid.

The shell method is a method of finding volumes by decomposing a solid of revolution into cylindrical shells

$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

Example 11

Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\frac{\pi}{2}}$.

Find the volume of the solid generated when R is revolved about the y-axis.

Solution

$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

$$= 2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \, dx$$

$$= \pi \int_0^{\sqrt{\pi/2}} \sin x^2 \, d\left(x^2\right)$$

$$= -\pi \cos\left(x^2\right) \Big|_0^{\sqrt{\pi/2}}$$

$$= -\pi\left(\cos\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$= \pi \quad unit^3$$

Example 12

Let R be the region in the first region bounded by the graph $y=\sqrt{x-2}$ and the line $y=2$.

a) Find the volume of the solid generated when R is revolved about the x-axis.
b) Find the volume of the solid generated when R is revolved about the line $y=-2$.

Solution

a) $y=\sqrt{x-2}$

$y^2=x-2$

$x=y^2+2$

$0\le y\le 2$

$$V=2\pi\int_c^d \begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix} dy$$

$$=2\pi\int_0^2 y\left(y^2+2\right)dy$$

$$=2\pi\int_0^2 \left(y^3+2y\right)dy$$

$$=2\pi\left(\frac{y^4}{4}+y^2\right|_0^2$$

$$=16\pi \quad unit^3$$

b) Revolved R about the line $y=-2$.

$$V=2\pi\int_c^d \begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix} dy$$

$$=2\pi\int_0^2 (y+2)\left(y^2+2\right)dy$$

$$=2\pi\left(\frac{1}{4}y^4+\frac{2}{3}y^3+y^2+4y\right|_0^2$$

$$=2\pi\left(4+\frac{16}{3}+4+8\right)$$

$$=\frac{128\pi}{3} \quad unit^3$$

Example 13

The region R is bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x$ on the interval $[0,\ 1]$. Use the washer method and the shell method to find the volume of the solid formed when R is revolved about the x-axis.

Solution

$$f(x) = g(x)$$

$$2x - x^2 = x$$

$$x^2 - x = 0 \ \Rightarrow \ \underline{x = 0,\ 1}$$

Washer Method:

$$V = \pi \int_0^1 \left[\left(2x - x^2\right)^2 - x^2 \right] dx$$

$$= \pi \int_0^1 \left(3x^2 - 4x^3 + x^4\right) dx$$

$$= \pi \left(x^3 - x^4 + \frac{x^5}{5} \right|_0^1$$

$$= \pi \left(1 - 1 + \frac{1}{5}\right)$$

$$\underline{= \frac{\pi}{5} \quad unit^3}$$

Shell Method:

$$x = y$$

$$y = 2x - x^2$$

$$x^2 - 2x + y = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4y}}{2}$$

$$= \begin{cases} 1 - \sqrt{1 - y} \\ \cancel{1 + \sqrt{1 - y}} \end{cases}$$

$$x = 0 \quad \rightarrow y = 0$$

$$x = 1 \quad \rightarrow y = 1$$

$$V = 2\pi \int_0^1 y \left[y - \left(1 - \sqrt{1 - y}\right) \right] dy$$

$$= 2\pi \int_0^1 y\left(y - 1 + \sqrt{1-y}\right) dy$$

$$= 2\pi \int_0^1 \left(y^2 - y + y(1-y)^{1/2}\right) dy$$

$$Let \quad u = 1 - y \quad \rightarrow \quad y = 1 - u$$

$$dy = -du$$

$$\int y(1-y)^{1/2}\, dy = -\int (1-u)u^{1/2} du$$

$$= -\int \left(u^{1/2} - u^{3/2}\right) du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2}$$

$$V = 2\pi \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2}\right|_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{2} - \left(\frac{2}{5} - \frac{2}{3}\right)\right)$$

$$= 2\pi\left(-\frac{1}{6} + \frac{4}{15}\right)$$

$$= 2\pi\left(\frac{9}{90}\right)$$

$$= \frac{\pi}{5} \quad unit^3$$

Summary of the Shell Method

1. Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (*shell height*) and distance from the axis of revolution (*shell radius*)
2. Find the limits of integration for the thickness variable.
3. Integrate the product 2π (*shell radius*) (*shell height*) with respect to the thickness variable (x or y) to find the volume.

Note:

Using ***Washer*** or ***Shell*** method:

Function in $\boldsymbol{x}$	Revolved	$\begin{cases} x-axis & \textit{Washer Method} \\ y-axis & \textit{Shell Method} \end{cases}$
Function in $\boldsymbol{y}$	Revolved	$\begin{cases} x-axis & \textit{Shell Method} \\ y-axis & \textit{Washer Method} \end{cases}$

Example 14

The region R is bounded by the graphs of $f(x)=3x^3-2$ $g(x)=x$ and $x=2$

Find the volume of the solid formed when R is revolved about the y-axis.

Solution

$$y=3x^3-2=x$$

$$3x^3-x-2=0$$

$$x=1$$

Since the function are in x and revolved about y-axis. The best way is to use Shell method.

$$V=2\pi\int_1^2 x\left(3x^3-2-x\right)dx \qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=2\pi\int_1^2\left(3x^4-x^2-2x\right)dx$$

$$=2\pi\left(\frac{3}{5}x^5-\frac{1}{3}x^3-x^2\right|_1^2$$

$$= 2\pi\left(\frac{96}{5}-\frac{8}{3}-4-\frac{3}{5}+\frac{1}{3}+1\right)$$

$$= 2\pi\left(\frac{93}{5}-\frac{7}{3}-3\right)$$

$$= 2\pi\left(\frac{199}{15}\right)$$

$$= \frac{398\pi}{15} \quad unit^3$$

Example 15

The region R is bounded by the graphs of $f(x)=3x^3-2$ $g(x)=x$ and $x=2$

Find the volume of the solid formed when R is revolved about the x-axis.

Solution

$$y = 3x^3 - 2 = x$$

$$3x^3 - x - 2 = 0$$

$$x = 1$$

Since the function are in x and revolved about x-axis. The best way is to use Washer method.

$$V = \pi\int_1^2 \left(\left(3x^3-2\right)^2 - x^2\right) dx$$

$$V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right) dx$$

$$= \pi\int_1^2 \left(9x^6 - 12x^3 + 4 - x^2\right) dx$$

$$= \pi\left(\frac{9}{7}x^7 - 3x^4 + 4x - \frac{1}{3}x^3\right|_1^2$$

$$= \pi\left(\frac{1{,}152}{7} - 48 + 8 - \frac{8}{3} - \frac{9}{7} + 3 - 4 + \frac{1}{3}\right)$$

$$= \pi\left(\frac{1{,}143}{7} - \frac{7}{3} - 41\right)$$

$$= \frac{2{,}519\pi}{21} \quad unit^3$$

Integration With respect to x 	***Disk/washer method about the x-axis*** Disks/washers are ***perpendicular*** to the x-axis $$V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$
 	Shell method about the y-axis Shells are ***parallel*** to the y-axis $$V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$
Integration With respect to y 	***Disk/washer method about the y-axis*** Disks/washers are ***perpendicular*** to the y-axis $$V=\pi\int_c^d\left(p(y)^2-q(y)^2\right)dy$$
 	Shells are ***parallel*** to the x-axis $$V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

Exercises *Section* 5.3 – Volume

$(\mathbf{1-6})$ Find the volume of the solid formed by revolving the region about the $x-axis$

1. $y=-x+1$

2. $y=4-x^2$

3. $y=\sqrt{x}$

4. $y=\sqrt{9-x^2}$

5. $y=x^2, \quad y=x^5$

6. $y=2, \quad y=4-\dfrac{x^2}{4}$

$(\mathbf{7-10})$ Find the volume of the solid formed by revolving the region about the $y-axis$

7. $y=x^2$

8. $y=\sqrt{16-x^2}$

9. $y = x^{2/3}$

10. $x = -y^2 + 4y$

(11 – 23) Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

11. $y = x$

12. $y = 1 - x$

13. $y = \sqrt{x}$

14. $y = \frac{1}{2}x^2 + 1$

15. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$

16. $y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$

17. $y = x^2, \quad y = 4x - x^2$

18. $y = 9 - x^2, \quad y = 0$

19. $y = 4x - x^2, \quad x = 0, \quad y = 4$

20. $y = x^{3/2}, \quad y = 8, \quad x = 0$

21. $y = \sqrt{x - 2}, \quad y = 0, \quad x = 4$

22. $y = -x^2 + 1, \quad y = 0$

23. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

(24 – 25) Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

24. $y = 2 - \frac{1}{4}x^2$

25. $x = y^2$

(26 – 60) Use any method to find the volume of the solid of revolution formed by rotating the area under the given function around the *x-axis* and *y-axis*.

26. $y = 4$, $1 \le x \le 4$

27. $y = x$, $0 \le x \le 2$

28. $y = 3x$, $1 \le x \le 3$

29. $y = x + 1$, $-1 \le x \le 2$

30. $y = e^x$, $-2 \le x \le 2$

31. $y = \frac{1}{x}$, $1 \le x \le 3$

32. $y = \frac{1}{x+1}$, $0 \le x \le 2$

33. $y = \frac{1}{x^2+1}$, $1 \le x \le 7$

34. $x = \frac{1}{y^2+1}$, $1 \le y \le 4$

35. $y = \frac{1}{x^2}$, $2 \le x \le 4$

36. $y = \frac{1}{\sqrt{x}}$, $1 \le x \le 4$

37. $y = \frac{2}{\sqrt{x}}$, $4 \le x \le 9$

38. $y = \sqrt{x+1}$, $-1 \le x \le 2$

39. $y = 2\sqrt{x}$, $1 \le x \le 2$

40. $y = x^2$, $0 \le x \le 3$

41. $y = x^2 + 1$, $0 \le x \le 6$

42. $y = 1 - x^2$, $0 \le x \le 1$

43. $y = x^2 + 2$, $1 \le x \le 3$

44. $y = x^3$, $0 \le x \le 3$

45. $y = x^3 + 1$, $2 \le x \le 4$

46. $x = y^3 - 4y^2$, $-1 \le x \le 2$

47. $y = \left(x^2 + 1\right)^3$, $0 \le x \le 1$

48. $y = \sqrt{4 - x^2}$, $-2 \le x \le 2$

49. $y = \sqrt{1 - x^2}$, $-1 \le x \le 1$

50. $y = \sqrt{1 + x^2}$, $0 \le x \le 3$

51. $x = \cos y$, $0 \le y \le \pi$

52. $y = 3x^2$, $y = 1$, $0 \le x \le 1$

53. $y = 4 - x$, $y = x$, $x = 0$

54. $y = x + 2$, $y = 2x - 1$, $x = 0$

55. $y = \sqrt{x}$, $y = 0$, $x = 4$

56. $y = x^3$, $x = 2$

57. $y = 2 - x^2$, $y = x^2$

58. $y^2 = x^4\left(1 - x^2\right)$

59. $4x^2 + 9y^2 = 36$

60. $y = x^2$, $y = 4x - x^2$

61. Find the volume of the solid obtained by the region in the first quadrant bounded between the parabola $y = 4x^2$ and the lines $y = 16$ and $x = 0$, by revolving about the

$a)\ y = -2 \qquad b)\ x = -1 \qquad c)\ x - axis \qquad d)\ y - axis$

62. Find the volume of the solid of the region bounded by the curve $y^2 = 8x$, and the line $x = 2$ by revolving about the

$a)\ x - axis \qquad b)\ y - axis \qquad c)\ x = 2$

63. Find the volume of the solid of the region bounded by the curve $y = -x^2 - 3x + 6$, and the line $x + y = 3$ by revolving about the

$a)\ x - axis \qquad b)\ y - axis \qquad c)\ x = 3$

64. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = \sqrt{x}, \quad y = 0, \quad x = 3$

a) the *x-axis* *b)* the *y-axis* *c)* the line $x = 3$ *d)* the line $x = 6$

65. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = 2x^2, \quad y = 0, \quad x = 2$

a) the *x-axis* *b)* the *y-axis* *c)* the line $y = 8$ *d)* the line $x = 2$

66. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = x^2, \quad y = 4x - x^2$

a) the *x-axis* *b)* the line $y = 6$

67. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = -x^2 + 2x + 4, \quad y = 4 - x$

a) the *x-axis* *b)* the line $y = 1$

68. Use the shell method to find the volume of the solid generated by revolving the shaded region about the *y*-axis.

69. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = x^2, \quad y = 2 - x, \quad x = 0, \quad for\ x \geq 0$ about the y-axis.

70. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = 2 - x^2, \quad y = x^2, \quad x = 0$ about the y-axis.

71. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \dfrac{3}{2\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 4$ about the y-axis.

72. Let $g(x) = \begin{cases} \dfrac{(\tan x)^2}{x} & 0 < x \leq \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \leq x \leq \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the y-axis.

73. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = \sqrt{y}, \quad x = -y, \quad y = 2$ about the x-axis.

74. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$ about the x-axis.

75. Compute the volume of the solid generated by revolving the region bounded by the lines $y = x \quad and \quad y = x^2$ about each coordinate axis using:

a) The *shell* method.

b) The *washer* method.

76. Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis.

b) The line $y = 1$.

c) The line $y = \frac{8}{5}$.

d) The line $y = -\frac{2}{5}$.

77. Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about:

a) the x-axis.

b) the y-axis.

c) the line $x = 4$.

d) the line $y = 1$.

78. Find the volume of the solid generated the region bounded by the curve by revolving the region bounded by $y = \frac{4}{x^3}$ and the lines $x = 1, \quad and \quad y = \frac{1}{2}$ about:

a) the x-axis.

b) the y-axis.

c) the line $x = 2$.

d) the line $y = 4$.

79. The region in the first quadrant that is bounded by the curve $y = \frac{1}{\sqrt{x}}$, on the left by the line $x = \frac{1}{4}$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by:

a) The *shell* method.

b) The *washer* method.

80. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

81. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

82. Find the volume of the solid generated by revolving the region bounded by $y = \sin x$ and the lines $x = 0, \quad x = \pi, \quad and \quad y = 2$ about the line $y = 2$.

83. A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

84. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2 - x, \quad y = 0$ about the x-axis.

85. Find the volume of the region bounded by $y = \dfrac{\ln x}{x^2}, \quad y = 0, \quad x = 1, \quad and \ x = 3$ revolved about the $y\text{-}axis$.

86. Find the volume of the region bounded by $y = \dfrac{e^x}{x}, \quad y = 0, \quad x = 1, \quad and \ x = 2$ revolved about the $y\text{-}axis$.

87. Find the volume of the region bounded by $y^2 = \ln x, \quad y^2 = \ln x^3, \quad and \ y = 2$ revolved about the $x\text{-}axis$.

88. The profile of a football resembles the ellipse. Find the football's volume to the nearest *cubic inch.*

$(\mathbf{89-95})$ Find the volume using both the *disk/washer* and *shell* methods of

89. $y = (x-2)^3 - 2, \quad x = 0, \quad y = 25$; revolved about the $y\text{-}axis$.

90. $y = \sqrt{\ln x}, \quad y = \sqrt{\ln x^2}, \quad y = 1$; revolved about the $x\text{-}axis$.

91. $y = \dfrac{6}{x+3}, \quad y = 2 - x$; revolved about the $x\text{-}axis$.

92. $y = 3 - x, \quad y = 0, \quad x = 0, \quad and \ x = 2$ revolved about the $y\text{-}axis$.

93. $y = x^3, \quad y = 0, \quad y = 8$; revolved about the $y\text{-}axis$.

94. $y = x^2 \quad and \quad y = x$; revolved about the $y\text{-}axis$.

95. $y = x^2 \quad and \quad y = \sqrt{x}$; revolved about the $y\text{-}axis$.

$(\mathbf{96-102})$ Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line.

96. $y = 2x - x^2$, $y = 0$, *about the line* $x = 4$

97. $y = \sqrt{x}$, $y = 0$, $x = 4$, *about the line* $x = 6$

98. $y = x^2$, $y = 4x - x^2$, *about the line* $x = 4$

99. $y = \frac{1}{3}x^3$, $y = 6x - x^2$, *about the line* $x = 3$

100. $y = \sqrt{x}$, $x = 0$, $x = 1$, *about the line* $x = 2$

101. $y = \frac{1}{4-x}$, $x = 1$, $x = 2$, *about the line* $x = 4$

102. $y = x^2$ *and* $y = \sqrt{x}$; *about the line* $x = 2$

103. Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines

$$y = x^3, \quad y = 0, \quad x = 2$$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $x = 4$

104. Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines

$$y = \frac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $y = 10$

105. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = \frac{1}{x}$, $y = 0$, $x = \frac{1}{4}$, and $x = c$ $\left(\textit{where } c > \frac{1}{4}\right)$ is revolved about the *x-axis* and the *y-axis*, respectively. Find the value of c for which $V_1 = V_2$

106. The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring

a) By integrating with respect to x.
b) By integrating with respect to y.

107. The region R in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes is revolved about the y-axis to produce a dome-shaped solid. Find the volume of the solid in the following ways.

a) Apply the disk method and integrate with respect to y.

b) Apply the shell method and integrate with respect to x.

108. The region bounded by the curves $y = 1 + \sqrt{x}$, $y = 1 - \sqrt{x}$, and the line $x = 1$ is revolved about the y-axis. Find the volume of the resulting solid by

a) Integrating with respect to x and

b) Integrating with respect to y.

109. The region bounded by the graphs of $x = 0$, $x = \sqrt{\ln y}$, and $x = \sqrt{2 - \ln y}$ in the first quadrant is revolved about the y-axis. What is the volume of the resulting solid?

110. The region bounded by $y = \left(1 - x^2\right)^{-1/2}$ and the x-axis over the interval $\left[0, \frac{\sqrt{3}}{2}\right]$ is revolved about the y-axis. What is the volume of the solid that is generated?

111. The region bounded by the graph $y = 4 - x^2$ and the x-axis over the interval $[-2, 2]$ is revolved about the line $x = -2$. What is the volume of the solid that is generated?

112. The region bounded by the graph $y = 6x$ and $y = x^2 + 5$ is revolved about the line $y = -1$ and the line $x = -1$. Find the volumes of the resulting solids. Which one is greater?

113. The region bounded by the graph $y = 2x$, $y = 6 - x$ and $y = 0$ is revolved about the line $y = -2$ and the line $x = -2$. Find the volumes of the resulting solids. Which one is greater?

114. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

115. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

116. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve. Find the volume of the solid.

117. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$. Find the volume of the solid.

118. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

119. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. Find the volume of the solid.

120. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

a) Equilateral triangles with bases running from the x-axis to the curve as shown.
b) Squares with bases running from the x-axis to the curve. Find the volume of the solid.

121. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

122. Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges).

123. Find the volume in function of height of a pyramid (tetrahedron shape) with an equilateral triangle base with side of 4 units.

124. Find the volume of the solid generated by rotating an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a) About $x - axis$
b) About $y - axis$

125. Find the volume of the solid of a circle sector with endpoint in first quadrant with value $(a,\ b)$ $a, b > 0$ and radius r, revolved:

a) About $x - axis$

b) About $y - axis$

126. Find the volume of the watermelon, given the shape of sphere $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

127. Consider a pyramid with height $12\ ft$. Find the volume if the base is:

a) Rectangular base $4\times 6\ ft$.
b) Square base $4\ ft$.
c) Rectangular base, where one side twice the other.

128. Find the volume of the solid of revolution $y=\dfrac{1}{\sqrt{1+x^2}}$; $-\dfrac{\sqrt{3}}{3}\le x\le\sqrt{3}$

129. Find the volume of the solid generated by revolving the region bounded by $y=x^2$ and the lines $y=0$, $x=2$ about the x-axis.

130. Find the volume of the solid generated by revolving the region bounded by $y=x-x^2$ and the line $y=0$ about the x-axis.

131. Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{\cos x}$ and the lines $0\le x\le\dfrac{\pi}{2}$, $y=0$, $x=0$ about the x-axis.

132. Find the volume of the solid generated by revolving the region bounded by $y=\sec x$ and the lines $y=0$, $x=-\dfrac{\pi}{4}$, $x=\dfrac{\pi}{4}$ about the x-axis.

133. Find the volume of the solid generated by revolving the region bounded by $x=\sqrt{5}\,y^2$ and the lines $x=0$, $y=-1$, $y=1$ about the y-axis.

134. Find the volume of the solid generated by revolving the region bounded by $y=2\sqrt{x}$ and the lines $y=2$, $x=0$ about the x-axis.

135. Find the volume of the solid generated by revolving the region bounded by $y=\sec x$, $y=\tan x$ and the lines $x=0$, $x=1$ about the x-axis.

136. Find the volume of the solid generated by revolving the region bounded by $x=\sqrt{2\sin 2y}$ and the lines $0\le y\le\dfrac{\pi}{2}$, $x=0$ about the y-axis.

137. What is the volume of the solid whose base is the region in the first quadrant bounded by $y=\sqrt{x}$, $y=2-x$, and the x-axis, and whose cross sections perpendicular to the base and parallel to the y-axis are squares?

138. What is the volume of the solid whose base is the region in the first quadrant bounded by $y=\sqrt{x}$, $y=2-x$, and the *x*-axis, and whose cross sections perpendicular to the base and parallel to the *y*-axis are semicircles?

139. What is the volume of the solid whose base is the region in the first quadrant bounded by $y=\sqrt{x}$, $y=2-x$, and the *y*-axis, and whose cross sections perpendicular to the base and parallel to the *x*-axis are square?

140. The region bounded by the curves $y=-x^2+2x+2$ and $y=2x^2-4x+2$ is revolved about the *x*-axis. What is the volume of the solid that is generated?

141. The region bounded by the curves $y=2e^{-x}$, $y=e^x$, and the *y*-axis is revolved about the *x*-axis. What is the volume of the solid that is generated?

142. The region bounded by the curves $y=\sec x$, $y=2$, for $0\le x\le\frac{\pi}{3}$ is revolved around the *x*-axis. What is the volume of the solid that is generated?

143. The region bounded by the graph $y=(x-2)^2$ and $y=4$ is revolved about the line $y=4$. What is the volume of the resulting solid?

144. Find the volume of a solid ball having radius *a*.

145. You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 *L* if you make it 9 *cm* deep and give the sphere a radius of 16 *cm*. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\,L=1{,}000\ cm^3\right)$

146. Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.

147. The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the *x*-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.

148. The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the *x*-axis squares. Use the general slicing method to find the volume of the solid.

149. A circular cylinder of radius *r* and height *h* whose curved surface is at an angle of $\frac{\pi}{4}\ rad$. Use the general slicing method to find the volume of the solid

150. Let *R* be the region bounded by $y = e^{-x}, \quad y = 0, \quad x = 0, \quad x = \ln 4$. Use the disk method to find the volume of the solid generated when *R* is revolved about *x*-axis.

151. Let R be the region bounded by $y = \cos x, \quad y = 0, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,

152. Let R be the region bounded by $y = e^{x/2}, \quad y = e^{-x}, \quad x = \ln 2, \quad x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

153. Let R be the region bounded by $y = 0, \quad y = \ln x, \quad y = 2, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

154. Let R be the region bounded by $y = \sin^{-1} x, \quad x = 0, \quad y = \frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

155. Find the volume of the solid of revolution bounded by $y = \frac{\ln x}{\sqrt{x}}, \quad y = 0, \quad \textit{and} \quad x = 2$ revolved about the x-axis. Sketch the region.

156. Find the volume of the solid of revolution bounded by $y = e^{-x}, \quad y = e^{x}, \quad x = 0, \quad x = \ln 4$ revolved about the x-axis. Sketch the region.

(157 – 160) Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

157. $y = x, \quad y = 3, \quad x = 0$

158. $y = \frac{1}{2}x^3, \quad y = 4, \quad x = 0$

159. $y = \frac{3}{1+x}, \quad y = 0, \quad x = 0, \quad x = 3$

160. $y = \sec x, \quad y = 0, \quad 0 \le x \le \frac{\pi}{3}$

(161 – 164) Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

161. $y = x, \quad y = 0, \quad y = 4, \quad x = 5$

162. $y = 3 - x, \quad y = 0, \quad y = 2, \quad x = 0$

163. $x = y^2, \quad x = 4$

164. $xy = 3, \quad y = 1, \quad y = 4, \quad x = 5$

165. Find the volume generated by rotating the given region $y = x^2$ *and* $y = x$ about the specified line.

a) R_1 *about* $x = 0$

b) R_1 *about* $x = 1$

c) R_2 *about* $y = 0$

d) R_2 *about* $y = 1$

e) R_3 *about* $x = 0$

f) R_3 *about* $x = 1$

g) R_2 *about* $x = 0$

h) R_2 *about* $x = 1$

166. The region R is bounded by the curves $x = y^2 + 2, \quad y = x - 4, \quad$ *and* $y = 0$

a) Write a single integral that gives the area of R.

b) Write a single integral that gives the volume of the solid generated when R is revolved about the x-axis.

c) Write a single integral that gives the volume of the solid generated when R is revolved about the y-axis.

d) Suppose S is a solid whose base is R and whose cross sections perpendicular to R and parallel to the x-axis are semicircles. Write a single integral that gives the volume of S.

167. The region R is bounded by $y = \frac{1}{x^p}$ and the x-axis on the interval $[1, a]$, where $p > 0$ and $a > 1$.

Let V_x and V_y be the volumes of the solids generated when R is revolved about the x- and y-axes, respectively.

a) With $a = 2$ and $p = 1$, which is greater, V_x or V_y?

b) With $a = 4$ and $p = 3$, which is greater, V_x or V_y?

c) Find a general expression for V_x in terms of a and p. Note that $p = \frac{1}{2}$ is a special case, what is V_x when $p = \frac{1}{2}$?

d) Find a general expression for V_y in terms of a and p. Note that $p = 2$ is a special case, what is V_y when $p = 2$?

e) Explain how parts (*c*) and (*d*) demonstrate that $\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a$

f) Find any values of a and p for which $V_x > V_y$

168. Let R be the region bounded by the graph of $f(x) = cx(1-x)$ and the *x-axis* on $[0, 1]$. Find the positive value of c such that the volume of the solid generated by revolving R about the *x-axis* equals the volume of the solid generated by revolving R about the *y-axis*.

169. A region in the first quadrant is bounded above the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

170. Find the volume of the torus (doughnut formed when the circle of radius 2 centered at (3, 0) is revolved about the *y*-axis.

a) Use geometry to evaluate the integral
b) Use Shell method (use integral table)

171. The nose of a rocket is a solid of revolution of base radius *r* and height *h* that must join smoothly to the cylindrical body of the rocket. Taking the origin at the tip of the nose and the *x*-axis along the central axis of the rocket, various nose shapes can be obtained by revolving the cubic curve about *x*-axis.

The cubic curve must have slope 0 at $x = h$, and its slope must be positive for $0 < x < h$. Find the particular cubic curve that maximizes the volume of the nose. Also show that his choice of the cubic makes the slope $\frac{dy}{dx}$ at the origin as large as possible and, hence, corresponds to the bluntest nose.

172. A landscaper wants to create on level ground a ring-shaped pool having an outside radius of 10 *m* and a maximum depth of 1 *m* surrounding a hill that will be built up using all the earth excavated from the pool. She decided to use a fourth-degree polynomial to determine the cross-sectional shape of the hill and pool bottom: at distance ***r*** *m* from the center of the development the height above or below normal ground level will be

$$h(r) = a\left(r^2 - 100\right)\left(r^2 - k^2\right) \quad m$$

For some $a > 0$, where k is the inner radius of the pool.

Find k and a so that the requirements given above are all satisfied.
How much earth must be moved from the pool to build the hill?

173. Let $f(x)=\begin{cases} x & if \quad 0\le x\le 2 \\ 2x-2 & if \quad 2<x\le 5 \\ -2x+18 & if \quad 5<x\le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of *f*, the *x*-axis, and the line $x=6$ is revolved about the *x*-axis.

174. Consider the solid formed by revolving the region bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about the *x-axis*

a) Find the value of *x* in the interval $[0,\ 4]$ that divides the solids into two parts of equal volume.

b) Find the values of *x* in the interval $[0,\ 4]$ that divide the solids into three parts of equal volume.

175. The arc of $y=4-\frac{1}{4}x^2$ on the interval $[0,\ 4]$ is revolved about the line $y=b$

a) Find the volume of the resulting solid as a function of *b*.

b) Graph the function in part (*a*), and approximate the value of *b* that minimizes the volume of the solid.

c) Find the value of *b* that minimizes the volume of the solid, and compare the result with the answer in part (*b*).

176. Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids hace the same volume.

$$Area\ of\ R_1 = Area\ of\ R_2$$

177. Cavalieri's principle states that if two solids of equal altitudes have the same cross-sectional areas at every height, then they have equal volumes.

a) Use the theory to justify Cavalieri's principle

b) Find the radius of a circular cylinder of height 10 *m* that has the same volume as a 2-*m* by 2-*m* by 10-*m* box.

178. Find the volumes of the solids whose bases are bounded by the graph of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

b) Rectangles of height 1

179. Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

b) Equilateral triangles

c) Semicircles

d) Iscosceles right triangles

180. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius *r* whose axes meet at right angles.

Two intersecting cylinders

Solid of intersection

181. The solid shown in the figure has cross sections bounded by the graph $|x|^a + |y|^a = 1$ where $1 \le a \le 2$.

a) Describe the cross section when $a = 1$ and $a = 2$.

b) Describe a procedure for approximating the volume of the solid.

$|x|^1 + |y|^1 = 1$

$|x|^a + |y|^a = 1$

$|x|^2 + |y|^2 = 1$

182. Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first.

a) Find the volume of the wedge if $\theta = 45°$.

b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?

183. For the given torus (donut).

a) Show that the volume of the torus is given by the integral

$$8\pi R \int_0^r \sqrt{r^2 - y^2}\,dy \quad \textit{where} \quad R > r > 0$$

b) Find the volume of the torus

184. Find the volume of a donut when circle $x^2 + y^2 = 1$ is rotated around the line $x = 3$

185. Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder. Find the volume of the wedge.

186. A martini glass in the shape of a right-circular cone of height h and semi-vertical angle α is filled with liquid. Slowly a ball is lowered into the glass, displacing liquid and causing it to overflow.

Find the radius R of the ball that causes the greatest volume of liquid to overflow out the glass.

187. A 45° notch is cut to the center of a cylindrical log having radius 20 *cm*. One plane faces the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?

188. A solid wooden object turned on a lathe has a length of 50 *cm* and diameters (measured in *cm*) shown in the figure. (A lathe is a tool that spins and cuts a block of wood so that it has circular cross sections.). Use ***left Riemann sums*** to estimate the volume of the object.

189. Consider the region R in the first quadrant bounded by $y = x^{1/n}$ and $y = x^n$, where n is a positive number.

a) Find the volume $V(n)$ of the solid generated when R is revolved about the x-axis. Express your answer in terms of n.

b) Evaluate $\lim_{n\to\infty} V(n)$. Interpret this limit geometrically.

190. A hole of radius a is bored through the center of a sphere of radius r $(a < r)$. Find the volume that had been removed from the sphere.

191. A hole of radius a is bored along the axis of a cone of height h and base of radius r. Find the volume of the remaining part of the sphere $(a < r)$.

192. A hole of radius r is cut by a horizontal plane at a distance c above the center of the sphere. Find the volume of the part of the sphere above the plane $(c < r)$.

193. Derive the formula $V = \frac{2}{3}\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been moved, as in the figure.

194. Find the volume interior to the inverted catenary *kiln* (an oven used to fire pottery).

195. Yogurt container can be shaped like frustum by rotating the line $y = \frac{1}{m}x$ around the $y-axis$ to find the volume between $y = a$ and $y = b$

196. To approximate the volume of a football, the better way is by rotating $y = \sin x \quad 0 \le x \le \pi$. What is the volume of this football?

197. A typical heart pumps 20 *ml*. of blood with each stroke (stroke volume). Assuming a heart rate of 60 *beats/min* (1 *beat/sec*), a reasonable model for the outflow rate of the heart is $V'(t) = 20(1 + \sin(2\pi t))$, where $V(t)$ is the total volume of blood pumped at time *t* measured in milliliters. Assume that $V(0) = 0$, *t* in seconds.

a) Graph the outflow function.
b) Verify that the amount of blood pumped over one-second interval is 20 *ml*.
c) Find the function that gives the total blood pumped between $t = 0$ and future time $t > 0$.
d) What is the cardiac output over a period of 1 *min*?

198. Solve the following problems with and without calculus

a) A cube with side length *r* is inscribed in a sphere, which is inscribed in a right circular cone, which is inscribed in a right circular cylinder. The side length (slant height) of the cone is equal to its diameter. What is the volume of the cylinder?
b) A cube is inscribed in a right circular cone with a radius of 1 and a height of 3. What is the volume of the cube?
c) A cylindrical hole 10 *in* long is drilled symmetrically through the center of a sphere. How much material is left in the sphere?

199. Consider the cap thickness h that has been sliced from a sphere of radius r. Verify that the volume of the cap is $\dfrac{\pi h^2(3r-h)}{3}$ using:

a) The washer method
b) The shell method
c) The general slicing method.
d) What is the volume of the sphere with cap removed from the top?

Check for consistency among the three methods and check the special cases $h=r$ and $h=0$.

Section 5.4 – Length of Curves and Surface Area

5.4–1 Length of a curve

We assume that f has a continuous derivative at every point of $[a, b]$. Such function is called ***smooth***, and its graph is a ***smooth curve*** because it doesn't have any breaks, corners, or cusps.

The idea of finding the length of a graph had its beginning with Archimedes (287–212 BC), was a Greek astronomer, engineer, mathematician, and physicist). Ancient people knew that the ratio of the circumference of any circle to its diameter is π.

5.4–2 *Definition*

If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1

Find the length of the curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1$

Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3}\frac{3}{2}x^{1/2}$$

$$= 2\sqrt{2}\,x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2$$

$$= 8x$$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 (1+8x)^{1/2}\, dx$$

$$= \frac{1}{8}\int_0^1 (1+8x)^{1/2}\, d(1+8x)$$

$$= \frac{1}{8}\left(\frac{2}{3}(1+8x)^{3/2}\right|_0^1$$

$$= \frac{1}{12}\left[(1+8(1))^{3/2} - (1+8(0))^{3/2}\right]$$

$$= \frac{1}{12}\left((9)^{3/2} - (1)^{3/2}\right)$$

$$= \frac{1}{12}[27-1]$$

$$= \frac{1}{12}(26)$$

$$= \frac{13}{6} \quad unit$$

5.4–3 *Formula*

***Length* of a curve** $y = f(x)$:

If $f(x) = ax^m + bx^n$, then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

$$= \left(ax^m - bx^n \right|_c^d$$

For proof check **Appendix**

Example 2

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4$

Solution

$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$

1. $m + n = 3 - 1 = 2$ ✓

2. $abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(\frac{x^3}{12} - \frac{1}{x}\right|_1^4$$

$$= \left(\frac{4^3}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - \frac{1}{1}\right)$$

$$= \frac{72}{12}$$

$$= 6 \quad unit$$

5.4–4 *Discontinuities* in $\frac{dy}{dx}$

Formula for the length of $x = g(y), \quad c \le y \le d$

If g' is continuous on [*c*, *d*], the length of the curve $x = g(y)$ from the point $A = (g(c), c)$ to the point $B = (g(d), d)$ is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 3

Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

Solution

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$= \frac{1}{3}\left(\frac{2}{x}\right)^{1/3} \qquad \boxed{x \ne 0} \ (CP)$$

$$y = \left(\frac{x}{2}\right)^{2/3} \qquad \textit{Raised both sides to the power 3/2}$$

$$y^{3/2} = \frac{x}{2}$$

$$x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2}$$

$$= 3y^{1/2}$$

$$\rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1+\left(3y^{1/2}\right)^2}\ dy$$

$$= \int_0^1 \sqrt{1+9y}\ dy$$

$$= \int_0^1 (1+9y)^{1/2}\ dy$$

$$= \frac{1}{9}\frac{2}{3}\ (1+9y)^{3/2}\ \Big|_0^1$$

$$= \frac{2}{27}\left[(1+9)^{3/2}-(1+0)^{3/2}\right]$$

$$= \frac{2}{27}\left(10^{3/2}-1\right)$$

$$= \frac{2}{27}\left(10\sqrt{10}-1\right)\ \ unit \qquad \approx 2.27\ \ unit$$

5.4–5 Exponential *Formula*

If $f(x) = ae^{mx} + be^{nx}$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\ dx$$

$$= \left(ae^{mx} - be^{nx}\ \Big|_c^d\right.$$

For proof check ***Appendix***

Example 4

Find the arc length function for the curve $f(x) = \ln\left(x+\sqrt{x^2-1}\right)$ on the interval $\left[1,\ \sqrt{2}\right]$

Solution

$$f'(x) = \frac{1+x\left(x^2-1\right)^{-1/2}}{x+\sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1}+x}{x\sqrt{x^2-1}+x^2-1}$$

$$y = \ln\left(x+\sqrt{x^2-1}\right)$$

$$x+\sqrt{x^2-1} = e^y$$

$$\left(\sqrt{x^2-1}\right)^2 = \left(e^y - x\right)^2$$

$$x^2-1 = e^{2y} - 2xe^y + x^2$$

$$2xe^y = e^{2y}+1$$

$$x = \frac{e^{2y}+1}{2e^y}\left(\frac{e^y}{e^y}\right)$$

$$= \frac{e^y+e^{-y}}{2}$$

$$y = \ln\left(x+\sqrt{x^2-1}\right)$$

$$x = \frac{e^y+e^{-y}}{2} = g(y)$$

$$x = 1 \quad \rightarrow \quad y = 0$$

$$x = \sqrt{2} \quad \rightarrow \quad y = \ln\left(\sqrt{2}+1\right)$$

$$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$$

1. $m = -n$ ✓

2. $abmn = -\frac{1}{4}$ ✓

$$L = \frac{1}{2}\left(e^y - e^{-y}\right|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$= \frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$= \frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$= \frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$= 1 \quad unit$$

OR

$$L = \int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1+g'(y)^2}\, dy$$

$$= \int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1+\left(\frac{e^y - e^{-y}}{2}\right)^2}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{4+e^{2y}-2+e^{-2y}}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{e^{2y}+2+e^{-2y}}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{\left(e^y+e^{-y}\right)^2}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \left(e^y+e^{-y}\right) dy$$

$$= \frac{1}{2}\left(e^y - e^{-y}\right|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$= \frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$= \frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$= \frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$= 1 \quad unit$$

5.4–6 The differential Formula for Arc length

If $y = f(x)$ and if f' is continuous on [*a*, *b*], then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x) = \int_a^x \sqrt{1+\left[f'(t)\right]^2}\, dt$$

$$\frac{ds}{dx} = \sqrt{1+\left[f'(t)\right]^2}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{(dx)^2 + (dx)^2 \frac{(dy)^2}{(dx)^2}}$$

$$\boxed{ds = \sqrt{dx^2 + dy^2}}$$

Example 5

Find the arc length function for the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$ taking $A = \left(1, \frac{13}{12}\right)$ as the starting point

Solution

$$1+\left[f'(x)\right]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$s(x) = \int_1^x \sqrt{1+\left[f'(t)\right]^2}\, dt$$

$$= \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left(\frac{t^3}{12} - \frac{1}{t}\right|_1^x$$

$$= \left(\frac{x^3}{12} - \frac{1}{x}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$$

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}$$

$$= 6 \quad unit$$

5.4–7 Surface Area

Consider a curve $y = f(x)$ on an interval $[a, b]$, where *f* is a nonnegative function with a continuous first derivative on $[a, b]$. Revolving the curve about the *x-axis* to generate a surface of revolution.

Consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0, h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the *x-axis*, it generates the surface of a cone of radius *r* and height *h*,

The surface area of a right circular cone, excluding the base, is $\pi r\sqrt{r^2 + h^2} = \pi r\ell$

One way to derive the formula for the surface area of a cone to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi\ell^2 \frac{r}{\ell} = \pi r\ell$

Curved length = $2\pi r$

Curved length = $2\pi r$

Curved edge length = $2\pi r$

Surface area of large cone
S_b

Surface area of small cone
S_a

Surface area of frustrum
$Surface = S_a - S_b$

5.4–8 *Definition*

If the function $f(x) \geq 0$ is continuously differentiable on [*a*, *b*], the area of the surface generated by revolving the graph of $y = f(x)$ about the *x-axis* is

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example 6

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 3$, about the *x-axis* .

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \quad a = 1, \quad b = 3$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1+\frac{1}{x}}$$

$$= \sqrt{\frac{x+1}{x}}$$

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_1^3 \left(\sqrt{x}\right) \frac{\sqrt{x+1}}{\sqrt{x}}\, dx$$

$$= 4\pi \int_1^3 (x+1)^{1/2}\, dx$$

$$= \frac{8\pi}{3}(x+1)^{3/2}\Big|_1^3$$

$$= \frac{8\pi}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{8\pi}{3}\left(8 - 2\sqrt{2}\right)$$

$$= \frac{16\pi}{3}\left(4-\sqrt{2}\right) \quad unit^2$$

Example 7

A spherical cap is produced when a sphere of radius a is sliced by a horizontal plane that is a vertical distance h below the north pole of the sphere, where $0 \le h \le 2a$. We take the spherical cap to be that part of the sphere above the plane, so that h is the depth of the cap.

Show that the area of a spherical cap of depth h cut from sphere of radius a is $2\pi ah$.

Solution

To generate the spherical surface, we revolved the curve $f(x)=\sqrt{a^2-x^2}$ on the interval $[-a,\ a]$ about the x-*axis* .

The spherical cap of height h corresponds to that part of the sphere on the interval $[-a+h,\ a]$ for $0 \le h \le 2a$

$$f'(x)=-x\left(a^2-x^2\right)^{-1/2}$$

$$1+f'(x)^2=1+\frac{x^2}{a^2-x^2}$$

$$=\frac{a^2}{a^2-x^2}$$

$$S=2\pi\int_{a-h}^{a}\sqrt{a^2-x^2}\ \frac{a}{\sqrt{a^2-x^2}}\ dx$$

$$S=2\pi\int_{a}^{b}f(x)\ \sqrt{1+f'(x)^2}\,dx$$

$$=2\pi\int_{a-h}^{a}a\ dx$$

$$=2\pi ax\Big|_{a-h}^{a}$$

$$=2\pi ah\quad unit^2$$

5.4–9 Surface Area for revolution about the y-axis

If $x = g(y) \geq 0$ is continuously differentiable on [*c*, *d*], the area of the surface generated by revolving the graph of $x = g(y)$ about the *y*-axis is

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_c^d g(y) \sqrt{1 + \left(g'(y)\right)^2}\, dy$$

Example 8

The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the *y*-axis to generate the cone. Find its lateral surface area (which excludes the base area)

Solution

$$Lateral\ Surface\ Area = \frac{base\ circumference}{2} \times slant\ height$$

$$= \pi\sqrt{2}$$

$x = 1 - y$

$\frac{dx}{dy} = -1, \quad c = 0, \quad d = 1$

$$S = 2\pi \int_0^1 (1 - y) \sqrt{1 + (-1)^2}\, dy$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^1 (1 - y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2} \left(y - \frac{y^2}{2} \right|_0^1$$

$$= 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2} \quad unit^2$$

5.4–10 *Formula*

Surface **of a curve** $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx$$

If $f(x) = ax^m + bx^n$, then $\sqrt{1+\left(f'(x)\right)^2} = \overline{f'(x)}$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

For proof check ***Appendix***

Example 9

Consider the function $y = \ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$

Find the area of the surface generated when the part of the curve between the points $\left(\frac{5}{4},\ 0\right)$ and $\left(\frac{17}{8},\ \ln 2\right)$ is revolved about y-*axis*.

Solution

$$y = \ln\left(\frac{x+\sqrt{x^2-1}}{2}\right)$$

$$e^y = \frac{x+\sqrt{x^2-1}}{2}$$

$$\left(2e^y - x\right)^2 = \left(\sqrt{x^2-1}\right)^2$$

$$4e^{2y} - 4xe^y + x^2 = x^2 - 1$$

$$4xe^y = 4e^{2y} + 1$$

$$x = e^y + \frac{1}{4}e^{-y} = g(y)$$

$$g'(y) = e^y - \frac{1}{4}e^{-y}$$

$$a = \frac{1}{2},\quad m = 1,\quad b = \frac{1}{2},\quad n = -1$$

1. $m = -n$ ✓
2. $abmn = -\frac{1}{4}$ ✓

$$S = 2\pi \int_0^{\ln 2} \left(e^y + \frac{1}{4}e^{-y}\right)\left(e^y + \frac{1}{4}e^{-y}\right) dy$$

$$= 2\pi \int_0^{\ln 2} \left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right) dy$$

$$= 2\pi \left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right|_0^{\ln 2}$$

$$= 2\pi\left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$= \pi\left(\frac{195}{64} + \ln 2\right) \quad unit^2$$

$$\sqrt{1 + g'(y)^2} = \sqrt{1 + \left(e^y - \frac{1}{4}e^{-y}\right)^2}$$

$$= \sqrt{1 + e^{2y} - \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{\left(e^y + \frac{1}{4}e^{-y}\right)^2}$$

$$= e^y + \frac{1}{4}e^{-y}$$

$$S = 2\pi \int_0^{\ln 2} \left(e^y + \frac{1}{4}e^{-y}\right)^2 dy$$

$$= 2\pi \int_0^{\ln 2} \left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right) dy$$

$$= 2\pi \left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right)\Big|_0^{\ln 2}$$

$$= 2\pi\left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$= \pi\left(\frac{195}{64} + \ln 2\right) \quad unit^2$$

Exercises *Section* 5.4 – Length of Curves

(1 – 41) Find the length of the curve of

1. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$, $0\le x\le 3$

2. $y=(x)^{3/2}$, $0\le x\le 4$

3. $x=\frac{y^{3/2}}{3}-y^{1/2}$ from $y=1$ to $y=9$

4. $x=\frac{y^3}{6}+\frac{1}{2y}$ from $y=2$ to $y=3$

5. $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$

6. $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3}$, $1\le x\le 2$

7. $y=\frac{1}{3}x^{1/2}-x^{3/2}$, $0\le x\le \frac{1}{3}$

8. $y=\frac{1}{3}x^3+\frac{1}{4x}$, $1\le x\le 2$

9. $y=2e^x+\frac{1}{8}e^{-x}$, $0\le x\le \ln 2$

10. $y=e^{2x}+\frac{1}{16}e^{-2x}$, $0\le x\le \ln 3$

11. $y=\ln(\cos x)$, $0\le x\le \frac{\pi}{4}$

12. $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}$ $0\le y\le \frac{\ln 2}{\sqrt{2}}$

13. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 2$

14. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 4$

15. $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)$ $\ln 2\le x\le \ln 3$

16. $f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2}$ $1\le x\le 4$

17. $f(x)=x^3+\frac{1}{12x}$ $1\le x\le 4$

18. $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}$, $1\le x\le 10$

19. $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}$, $1\le x\le 10$

20. $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2}$, $3\le x\le 8$

21. $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3}$, $1\le x\le 7$

22. $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3}$ $0\le x\le 12$

23. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$, $2\le x\le 9$

24. $y=x^{1/2}-\frac{1}{3}x^{3/2}$, $1\le x\le 4$

25. $x=y^{2/3}$, $1\le y\le 8$

26. $y=2x+4$, $-2\le x\le 2$

27. $y=\frac{x^3}{6}+\frac{1}{2x}$, $x\in[1,\ 2]$

28. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$, $1\le x\le 3$

29. $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5$, $1\le x\le 8$

30. $y=\ln x-\frac{1}{8}x^2$; $1\le x\le 2$

31. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x$; $1\le x\le 3$

32. $y=x^2-\frac{1}{8}\ln x$; $1\le x\le e$

33. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x$, $1\le x\le e$

34. $y=\cosh^{-1}x$, $\sqrt{2}\le x\le \sqrt{5}$

35. $y=3+\frac{1}{2}\cosh 2x$, $0\le x\le 1$

36. $y^3=8x^2$, $1\le x\le 8$

37. $6xy=x^4+3$, $1\le x\le 2$

38. $24xy=x^4+48$, $2\le x\le 4$

39. $x=3y^{3/2}-1$, $0\le y\le 4$

40. $y=\frac{2}{3}x^{3/2}-\frac{1}{2}\sqrt{x}$, $1\le x\le 4$

41. $x=\frac{1}{8}y^4+\frac{1}{4y^2}$, $1\le y\le 2$

42. Find the length of the curve $y = \int_{-2}^{x} \sqrt{2t^4 - 1}\, dt \quad -2 \le x \le -1$

43. Find the length of the curve $x = \int_{0}^{y} \sqrt{\sec^4 t - 1}\, dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

44. The portion of the curve $y = \frac{17}{15} - \cosh x$ that lies above the x-axis forms a catenary arch. Find the average height of the arch above the x-axis.

45. Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

46. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area} = \frac{1}{2} \times \textit{base circumference} \times \textit{slant height}$$

47. Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the y-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area} = \frac{1}{2} \times \textit{base circumference} \times \textit{slant height}$$

48. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the x-axis. Check your answer with the geometry formula

$$\textit{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \textit{slant height}$$

49. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the y-axis. Check your answer with the geometry formula

$$\textit{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \textit{slant height}$$

$(\mathbf{50-80})$ Find the area of the surface generated by revolving the curve about the *x-axis*

50. $y=\frac{1}{3}x^3$

51. $y=2\sqrt{x}$

52. $y=\frac{x^3}{9}$, $0\le x\le 2$

53. $y=\sqrt{x+1}$, $1\le x\le 5$

54. $y=\sqrt{5-x}$, $2\le x\le 5$

55. $y=\sqrt{2x-x^2}$, $\frac{1}{2}\le x\le \frac{3}{2}$

56. $y^2=x-1$, $1\le x\le 5$

57. $y=3x+4$, $0\le x\le 6$

58. $y=12-3x$, $1\le x\le 3$

59. $y=x^{3/2}-\frac{1}{3}x^{1/2}$, $1\le x\le 2$

60. $y=\sqrt{4x+6}$, $0\le x\le 5$

61. $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right)$, $-2\le x\le 2$

62. $y=\frac{1}{8}x^4+\frac{1}{4x^2}$, $1\le x\le 2$

63. $y=8\sqrt{x}$, $9\le x\le 20$

64. $y=x^3$, $0\le x\le 1$

65. $y=\frac{1}{3}x^3+\frac{1}{4x}$, $\frac{1}{2}\le x\le 2$

66. $y=\sqrt{5x-x^2}$, $1\le x\le 4$

67. $y=\frac{1}{6}x^3+\frac{1}{2x}$, $1\le x\le 2$

68. $y=\sqrt{4-x^2}$, $-1\le x\le 1$

69. $y=\sqrt{9-x^2}$, $-2\le x\le 2$

70. $y=2e^x+\frac{1}{8}e^{-x}$, $0\le x\le \ln 2$

71. $y=\frac{1}{10}x^5+\frac{1}{6x^3}$, $1\le x\le 2$

72. $x=1+2y^2$, $1\le y\le 2$

73. $y=\sqrt{1+e^x}$, $0\le x\le 1$

74. $y^{2/3}+x^{2/3}=1$, $0\le y\le 1$

75. $y=\frac{1}{4}x^2-\frac{1}{2}\ln x$, $1\le x\le 2$

76. $y=x^2-\frac{1}{8}\ln x$, $e\le x\le e^2$

77. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x$, $1\le x\le e^2$

78. $y=-\frac{1}{4}x^2+\frac{1}{2}\ln x$, $e\le x\le e^2$

79. $y=e^x+\frac{1}{4}e^{-x}$, $0\le x\le \ln 2$

80. $y=\cosh x$, $-2\le x\le 2$

$(\mathbf{81-90})$ Find the area of the surface generated by revolving the curve about the y-*axis*

81. $y=\sqrt[3]{x}+2$

82. $y=9-x^2$

83. $y=(3x)^{1/3}$; $0\le x\le \frac{8}{3}$

84. $x=\sqrt{12y-y^2}$; $2\le y\le 10$

85. $x=4y^{3/2}-\frac{1}{12}y^{1/2}$; $1\le y\le 4$

86. $y=1-\frac{1}{4}x^2$, $0\le x\le 2$

87. $x=\frac{1}{8}y^4+\frac{1}{4y^2}$, $1\le y\le 2$

88. $y=\frac{1}{2}x+3$, $1\le x\le 5$

89. $x=\frac{1}{2}y^3+\frac{1}{6y}$, $1\le y\le 2$

90. $x=\frac{1}{2}e^{-2y}+\frac{1}{8}e^{2y}$, $0\le y\le 2$

91. A right circular cone is generated by revolving the region bounded by $y=\frac{3}{4}x$, $y=3$, and $x=0$ about the y-*axis*. Find the lateral surface area of the cone.

92. A right circular cone is generated by revolving the region bounded by $y=\frac{h}{r}x$, $y=h$, and $x=0$ about the y-*axis*. Verify that the lateral surface area of the cone is $S=\pi r\sqrt{r^2+h^2}$

93. Find the area of the zone of a sphere formed by revolving the graph of $y=\sqrt{9-x^2}$, $0\le x\le 2$, about the y-*axis*

94. Find the area of the zone of a sphere formed by revolving the graph of $y=\sqrt{r^2-x^2}$, $0\le x\le a$, about the y-*axis*. Assume that $a<r$.

95. Find the area of the surface generated by part of the curve $y=4x-1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

96. Find the area of the surface generated by part of the curve $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis* .

97. Find the area of the surface generated by $y = 1 + \sqrt{1 - x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

98. Find the area of the surface generated by $y = \frac{1}{3}x^3$, $0 \le x \le 1$, $x - axis$

99. Find the area of the surface generated by $x = \sqrt{4y - y^2}$, $1 \le y \le 2$, $y - axis$

100. Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$$

101. Confirm that the circumference of a circle of radius *a* is $2\pi a$

102. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and *x* and *y* are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.

103. Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

104. A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of square feet of roofing on the barn.

105. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy-plane. Find the area of the surface formed by these perpendiculars from $(0,\ 0)$ *to* $\left(3,\ 2\sqrt{3}\right)$

106. Find the area of the surface generated by $x = 2\sqrt{4 - y} \quad 0 \le y \le \frac{15}{4}, \quad y - axis$

107. $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y - axis$ (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of *dy*, and evaluate the integral $S = \int 2\pi y \, ds$ with appropriate limits.)

108. Find the area of the surface generated by $x = \sqrt{2y - 1} \quad \frac{5}{8} \le y \le 1, \quad y - axis$

109. Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y*-axis

110. Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the *x*-axis to generate a sphere. Let *AB* be an arc of the semicircle that lies above an interval of length *h* on the *x*-axis. Show that the area swept out by *AB* does not depend on the location of the interval. (It does depend on the length of the interval.)

111. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the $x\text{-}axis$. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

112. When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r,\ r]$ is revolved about the $x\text{-}axis$, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

113. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the $x\text{-}axis$. Assume *x* and *y* are in *meters*.

114. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the $x\text{-}axis$. Assume *x* and *y* are in *meters*.

115. Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

116. Let $f(x) = \frac{1}{3}x^3$ and let *R* be the region bounded by the graph of *f* and the $x\text{-}axis$ on the interval $[0,\ 2]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 2]$ is revolved about the $x\text{-}axis$.

b) Find the volume of the solid generated when *R* is revolved about the $y\text{-}axis$.

c) Find the volume of the solid generated when *R* is revolved about the $x\text{-}axis$.

117. Let $f(x) = \sqrt{3x - x^2}$ and let *R* be the region bounded by the graph of *f* and the $x\text{-}axis$ on the interval $[0,\ 3]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 3]$ is revolved about the $x\text{-}axis$.

b) Find the volume of the solid generated when *R* is revolved about the $x\text{-}axis$.

118. Let $f(x)=\frac{1}{2}x^4+\frac{1}{16x^2}$ and let R be the region bounded by the graph of f and the x-*axis* on the interval $[1,\ 2]$

a) Find the area of the surface generated when the graph of f on $[1,\ 2]$ is revolved about the x-*axis* .

b) Find the length of the curve $y=f(x)$ on $[1,\ 2]$

c) Find the volume of the solid generated when R is revolved about the y-*axis* .

d) Find the volume of the solid generated when R is revolved about the x-*axis* .

119. Suppose a sphere of radius r is sliced by two horizontal planes h units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.

120. An ornamental light bulb is designed by revolving the graph of $y=\frac{1}{3}x^{1/2}-x^{3/2}$, $0\le x\le\frac{1}{3}$ about the x-*axis* , where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.

(Assume that the glass is 0.015 *inch* thick)

121. The shaded band is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$

122. A drawing of a 90-*ft* dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?

123. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1+\frac{4h^2}{w^4}x^2}\,dx$

124. A climber anchors a rope at 2 points of equal height, separated by 100 *feet*. to perform a Tyrolean traverse. The rope follows the catenary $f(x) = 200\cosh\frac{x}{200}$ over the interval $[-50,\ 50]$.

Find the length of the rope between the two anchor points.

125. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

126. Find the arc length from $(0,\ 3)$ clockwise to $\left(2,\ \sqrt{5}\right)$ along the circle $x^2 + y^2 = 9$

127. Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

128. Given $y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\ \ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.

129. Imagine a cable, like a telephone line or *TV* cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = \frac{H}{w}$ on the y-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

a) Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying displays the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\left(\frac{w}{H}x\right)$$

b) Using the result in part (*a*) and the fact that the horizontal tension at *P* must equal *H* (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of *y* units of cable.

c) The length of arc *AP* is $s = \frac{1}{a}\sinh ax$, where $a = \frac{w}{H}$. Show that the coordinates of *P* may be expressed in terms of *s* as

$$x = \frac{1}{a}\sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}$$

130. A power line is attached at the same height to two utility poles that are separated by a distance of 100 *ft*; the power line follows the curve $f(x) = a\cosh\left(\frac{x}{a}\right)$. Use the following steps to find the value of *a* that produces a sag of 10 *ft*. midway between the poles. Use the coordinate system that places the poles at $x = \pm 50$

a) Show that *a* satisfies the equation $\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$

b) Let $t = \frac{10}{a}$, confirm that the equation in part (*a*) reduces to $\cosh 5t - 1 = t$, and solve for t using a graphing utility. (2 decimal places)

c) Use the answer in part (*b*) to find a and then compute the length of the power line.

131. The shape of the Gateway Arch in St. Louis (with a height and a base length of 630 *ft*) is modeled by the function $y = -630\cosh\left(\frac{x}{239.2}\right) + 1260$, where $|x| \le 315$, and *x* and *y* measured in *feet*.

Estimate the length of the Gateway Arch

132. A cone of radius *r* and height *h* has a smaller cone of radius $\frac{r}{2}$ and height $\frac{h}{2}$ removed from the top. The result solid called a frustum.

a) Find the volume of the frustum.

b) Find the surface area of the frustum.

133. Johann Bernoulli evaluated the arc length of curves of the form $y = x^{(2n+1)/2n}$, where n is a positive integer, on the interval $[0,\ a]$.

a) Write the arc length integral.

b) Make the change of variables $u^2 = 1 + \left(\frac{2n+1}{2n}\right)^2 x^{1/n}$ to obtain a new integral with respect to u.

c) Use the Binomial Theorem to expand this integrand and evaluate the integral.

d) The case $n = 1$ $\left(y = x^{3/2}\right)$, with $a = 1$, evaluate the arc length in the cases $n = 2$ *and* $n = 3$. Does the arc length increase or decrease with n?

e) Graph the arc length of the parabolas for $a = 1$ as a function of n.

134. A solid is revolving about $y - axis$ the region in the first quadrant inside the circle $x^2 + y^2 = r^2$, and between $y = a$ & $y = r$ where $0 < a < r$.

a) Find the volume of the solid.

b) Find the surface of the solid

135. A solid with a circular base of radius 4 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles.

a) Find the volume of the solid.

b) Find the surface.

136. A cross-section of a cooling tower is a hyperbola with equation

$$\frac{x^2}{90^2} - \frac{y^2}{130^2} = 1$$

The tower is 450 *feet.* tall and the distance from the top of the tower to the center of the hyperbola is half the distance from the base of the tower to the center of the hyperbola.

a) Find the diameter of the ***top*** and the ***base*** of the tower.

b) What is the volume of the tower, disregard the thickness?

c) What is the surface of the tower, disregard the thickness?

d) If the thickness of the tower is 2 *feet*, what is the volume?

e) What is the surface inside the tower?

f) How much material needed in ft^3?

137. If the top of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is revolved about the $x-axis$, the result is an ellipsoid whose axis along the $x-axis$ has length $2a$, whose axis along the $y-axis$ has length $2b$, and whose axis perpendicular to the $xy-plane$ has length $2b$. We assume that $0<b<a$. Use the following steps to find the surface area S if this ellipsoid.

a) Use the surface area formula to show that

$$S=\frac{4\pi b}{a}\int_0^a \sqrt{a^2-c^2x^2}\,dx, \quad where \quad c^2=1-\frac{b^2}{a^2}$$

b) Use the change of variables $u=cx$ to show that $S=\frac{4\pi b}{\sqrt{a^2-b^2}}\int_0^{\sqrt{a^2-b^2}}\sqrt{a^2-u^2}\,du$

c) A table of integrals shows that $\int\sqrt{a^2-u^2}\,du=\frac{1}{2}\left(u\sqrt{a^2-u^2}+a^2\sin^{-1}\frac{u}{a}\right)+C$

Use the fact to show that the surface area of the ellipsoid is

$$S=2\pi b\left(b+\frac{a^2}{\sqrt{a^2-u^2}}\sin^{-1}\frac{\sqrt{a^2-u^2}}{a}\right)$$

d) If a and b have units of length (*feet*), what are the units of S according to the formula?

e) Use part (*a*) to show that if $a = b$, then $S = 4\pi a^2$, which is the surface area of a sphere of radius *a*.

Section 5.5 – Physical Applications

5.5–1 Density and Mass

Density is the concentration of mass in an object and is usually measured in units of mass per volume. An object with uniform density satisfies the basic relationship

$$mass = density \cdot volume$$

When density of an object varies, this formula no longer holds, and we must appeal to calculus.

5.5–2 *Definition*

Suppose a thin bar or wire can be represented as a line segment on the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The mass of the object is

$$m = \int_a^b \rho(x)\, dx$$

Example 1

A thin 2-*m* bar, represented by the interval $0 \le x \le 2$, is made of any alloy whose density in units of *kg/m* is given by $\rho(x) = 1 + x^2$. What is the mass of the bar?

Solution

$$m = \int_0^2 \left(1 + x^2\right) dx$$

$$= x + \frac{1}{3}x^3 \Big|_0^2$$

$$= 2 + \frac{8}{3}$$

$$= \frac{14}{3} \quad kg$$

5.5–3 *Work* Done by a Constant Force

When a body moves a distance d along a straight line because of being acted on by a force of constant magnitude $\boldsymbol{F}$ in the direction of motion, we define the ***work*** W done by the force on the body with the formula

$$Work = force \cdot distance$$

$$W = Fd \quad \textbf{(}\textit{Constant-force formula for work}\textbf{)}$$

The unit of work is a newton-meter (***N.m***), also called ***joule***.

5.5–4 *Definition*

The work done by a variable force $F(x)$ in the direction of motion along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

5.5–5 Hooke's Law for Springs: $F = kx$

Hooke's Law says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x. In symbols

$$F = kx$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.

- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.

Example 2

Find the work required to compress a spring from its natural length of 1 *ft* to a length of 0.75 *ft* if the force constant is $k = 16$ *lb./ft*.

Solution

$$F = kx = 16x$$

$$F(0) = 16 \cdot 0 = 0 \ lb$$

$$F\left(\frac{1}{4}\right) = 16 \cdot \left(\frac{1}{4}\right)$$

$$= 4 \ lb$$

$$W = \int_a^b F(x)\,dx$$

$$= \int_0^{0.25} 16x \ dx$$

$$= 8x^2 \Big|_0^{\frac{1}{4}}$$

$$= 8\left(\frac{1}{16}\right)$$

$$= \frac{1}{2} \ ft - lb$$

Example 3

A spring has a natural length of 1 *m*. A force of 24 *N* holds the spring stretched to a total length of 1.8 *m*.

a) Find the force constant *k*.
b) How much work will it take to stretch the spring 2 *m* beyond its natural length?
c) How far will a 45-*N* force stretch the spring?

Solution

a) $F = kx$

$$24 = k(1.8 - 1)$$

$$24 = k(0.8)$$

$$k = \frac{24}{0.8}$$

$$= 30 \quad N/m$$

b) $F(x) = 30x$

$$W = \int_0^2 30x\,dx$$

$$= 15x^2 \Big|_0^2$$

$$= 15\left(2^2 - 0\right)$$

$$= 60 \quad J$$

c) $45 = 30x$

$$x = \frac{45}{30}$$

$$= \frac{3}{2} \quad m$$

Example 4

A 5-*lb* bucket is lifted from the ground into the air by pulling in 20 *feet* of rope at a constant speed. The rope weighs 0.08 *lb/ft*.

How much work was spent lifting the bucket and rope?

Solution

Work done on lifting the bucket only is:

weight x *distance* = 5 (20)

= 100 *ft-lb*

Work on the rope:

$$W = \int_0^{20} 0.08(20 - x)\,dx$$

$$= \frac{8}{100}\left(20x - \frac{x^2}{2}\right|_0^{20}$$

$$= \frac{2}{25}\left[\left(20(20) - \frac{20^2}{2}\right) - 0\right]$$

$$= \frac{2}{25}(400 - 200)$$

$$= \frac{2}{25}(200)$$

$$= 16 \quad ft - lb$$

The total work for the bucket and the rope combined is:

$$100 + 16 = 116 \quad ft - lb$$

5.5–6 Gravity

As an object moves away from the earth, its weight decreases.

Newton's law of universal gravitation:

> *The force of earth's gravity on an object is inversely proportional to the square of the distance from the object to the center of the earth.*

Therefore, the force required to lift an object is given by:

$$F(x) = \frac{k}{x^2}$$

Where k is the proportionality constant

x is the distance of the object from the center of the earth

$(radius \approx 6{,}378\ km \approx 3{,}963\ miles)$

Example 5

A satellite orbits the earth at altitudes of about 15×10^3 *miles* with weight around 10^3 *lbs*. How much work done against earth's gravity in lifting such a satellite into orbit?

Assume the earth radius is 4×10^3 *miles*

Solution

By *Newton's law* of universal gravitation: $F(x)=\dfrac{k}{x^2}$

$$10^3=\frac{k}{\left(4\times10^3\right)^2}$$

$$\boxed{k\approx16\times10^9}$$

$$\boxed{F(x)=\frac{16\times10^9}{x^2}}$$

$$4\times10^3\le x\le15\times10^3+4\times10^3$$

$$4\times10^3\le x\le19\times10^3$$

$$W=\int_{4\times10^3}^{19\times10^3}\frac{16\times10^9}{x^2}\,dx$$

$$=-16\times10^9\left.\frac{1}{x}\right|_{4\times10^3}^{19\times10^3}$$

$$=-16\times10^9\left(\frac{1}{19\times10^3}-\frac{1}{4\times10^3}\right)$$

$$=-16\times10^6\left(\frac{1}{19}-\frac{1}{4}\right)$$

$$=-16\times10^6\left(-\frac{15}{76}\right)$$

$$\boxed{=\frac{60}{19}\times10^6\quad \textit{mile-lb}}$$

5.5–7 Lifting

Another common work problem arises when the motion is vertical, and the force is the gravitational force.

The gravitational force exerted on an object with a mass of m is $F = mg$, where

$g \approx 9.8\ m/s^2 \approx 32.2\ ft/s^2$ is the acceleration due the gravity near the surface of Earth.

The work in joules required to lift an object of mass m a vertical distance of y meters is

$$\begin{aligned} work &= force \cdot distance \\ &= mgy \end{aligned}$$

This type of problem leads to 3 key observations to the solution:

- ✓ Water from different levels of the tank is lifted different vertical distances, requiring different amounts of work,
- ✓ Water from the same horizontal plane is lifted the same distance, requiring the same amount of work.
- ✓ A volume V of water has mass ρV, where $\rho = 1\ g/cm^3 = 1000\ kg/m^3$ is the density of water.

$$\begin{aligned} F_k &= mg \\ &\approx \underbrace{\overbrace{A(\bar{y}_k)\Delta y}^{mass} \cdot \rho}_{volume} \cdot g \end{aligned}$$

$$W_k = \underbrace{A(\bar{y}_k)\,\Delta y\,\rho\,g}_{force} \cdot \underbrace{(h - \bar{y}_i)}_{distance}$$

$$W \approx \sum_{k=1}^{n} W_k$$

$$= \rho\, g \sum_{k=1}^{n} A\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y$$

$$W = \rho\, g \lim_{n\to\infty} \sum_{k=1}^{n} A\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y$$

$$= \rho\, g \int_0^b A(y)\,\underbrace{(h-y)}_{D(y)}\,dy$$

5.5–8 *Solving* Lifting Problems

1. Draw a *y-axis* in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a,\ b]$ corresponds to the vertical extent of the fluid.

2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is

$$W = \rho\, g \int_a^b A(y)\,D(y)\,dy$$

Example 6

The conical tank is filled to within 2 *feet* of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution

The volume of a slab between the planes y and Δy: $\frac{1}{2}y$

$$\Delta V = \pi(radius)^2(thickness)$$

$$= \pi\left(\frac{1}{2}y\right)^2 \Delta y$$

$$= \frac{\pi}{4}y^2\Delta y \quad ft^3$$

The force $F(y)$ required to lift this slab is equal to its weight

$$F(y) = 57\Delta V$$

$$= 57\frac{\pi}{4}y^2\Delta y$$

Distance to lift to the level of the rim of the cone is about $(10 - y)$ *feet*, so the work done lifting the slab

$$W = \frac{57\pi}{4}\int_0^8 y^2(10-y)\,dy$$

$$= \frac{57\pi}{4}\int_0^8 \left(10y^2 - y^3\right)dy$$

$$= \frac{57\pi}{4}\left(\frac{10}{3}y^3 - \frac{1}{4}y^4\right|_0^8$$

$$= \frac{57\pi}{4}\left[\left(\frac{10}{3}(8)^3 - \frac{1}{4}(8)^4\right) - 0\right]$$

$$= \frac{57\pi}{4}\left(\frac{5120}{3} - 1{,}024\right)$$

$$= \frac{57\pi}{4}\left(\frac{2{,}048}{3}\right)$$

$$= 9{,}728\pi \quad ft - lb$$

5.5–9 Pressure and Force

Dams are built thicker at the bottom than at the top because the pressure against them increases with depth

Pressure is a force per unit area, measured in units such as N / m^2.

For example, the pressure of the atmosphere on the surface of Earth is about 14 lb / in^2 $\left(\approx 100 \text{ kilopascals, or } 10^5 \ N / m^2\right)$

Another example, if you stood on the bottom of a swimming pool, you would feel pressure due to the weight (force) of the column of water above your head. If your head is flat and has surface area $A \ m^2$ and it is h meters below the surface, then the column of water above your head has volume $Ah \ m^3$. That column of water exerts a force:

$$F = mass \cdot acceleration$$

$$= \underbrace{volume \cdot density}_{mass} \cdot g$$

$$= Ah\rho g$$

Where ρ is the density
g is the acceleration due to gravity.

Therefore, the pressure on your head is the force divided by the surface area of your head

$$pressure = \frac{force}{A}$$

$$= \frac{Ah\rho g}{A}$$

$$= \rho g h$$

This pressure is called ***hydrostatic pressure*** (meaning the pressure of water at rest), and it has the following important property: it has the same magnitude in all directions.

Suppose that a plate submerge vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis.

Let $L(y)$ be the ***length*** (or ***width***) of the horizontal strip measured from left to right along the surface of the plate at level y.

Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (strip\ depth) \cdot L(y)\,dy$$

Solving Force / Pressure Problems

1. Draw a y-*axis* on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).

2. Find the width function $w(y)$ for each value of y on the face of the dam.

3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \rho g \int_0^a \underbrace{(a - y)}_{depth}\ \underbrace{w(y)}_{width}\ dy$$

Example 7

A flat isosceles right-triangular plate with base 6 *feet* and height 3 *feet* is submerged vertically, base up, 2 *feet* below the surface of a swimming pool. Find the force exerted by the water against on side of the plate. (Freshwater Weight density: 62.4 lb/ft^3)

Solution

The width of a thin strip at level y is: $L(y) = 2x = 2y$

The depth of the strip beneath the surface is: $(5-y)$

$$F = \int_a^b w \cdot (strip\ depth) \cdot L(y)\,dy$$

$$= 62.4 \int_0^3 (5-y)\cdot(2y)\,dy$$

$$= 124.8 \int_0^3 \left(5y - y^2\right) dy$$

$$= \frac{1,248}{10} \left(\frac{5}{2}y^2 - \frac{1}{3}y^3\right|_0^3$$

$$= \frac{624}{5}\left[\left(\frac{5}{2}(3)^2 - \frac{1}{3}(3)^3\right) - 0\right]$$

$$= \frac{624}{5}\left(\frac{45}{2} - 9\right)$$

$$= \frac{624}{5}\left(\frac{27}{2}\right)$$

$$= \frac{8,424}{5}\ lb$$

Example 8

A cylindrical tank with a length of 10 *m* and radius of 5 *m* is on its side and half-full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank?

The density of gasoline is $\rho \approx 737\ kg/m^3$.

Solution

$$x^2 + y^2 = 5^2$$

$$x = \pm\sqrt{25 - y^2}$$

$$A(y) = 2(10)\sqrt{25 - x^2}$$

$$W = 737(9.8)\int_{-5}^{0} 20\sqrt{25-y^2}\,(5-y)\,dy$$

$$= 144{,}452\underbrace{\int_{-5}^{0} 5\sqrt{25-y^2}\,dy}_{\text{area of } \frac{1}{4} \text{ circle}} - 144{,}452\int_{-5}^{0} y\sqrt{25-y^2}\,dy$$

$$= 144{,}452\left(5\cdot\frac{25\pi}{4} + \frac{1}{2}\int_{-5}^{0}\sqrt{25-y^2}\,d\left(25-y^2\right)\right)$$

$$= 144{,}452\left(\frac{125\pi}{4} + \frac{1}{3}\left(25-y^2\right)^{3/2}\Big|_{-5}^{0}\right)$$

$$= 144{,}452\left(\frac{125\pi}{4} + \frac{125}{3}\right)$$

$$= 18{,}056{,}500\left(\frac{3\pi+4}{12}\right)$$

$$\approx 20.2\times10^6 \quad joules$$

Material	***Weight Density***	
	$\left(kg / m^3\right)$	$\left(lb / ft^3\right)$
Aluminum	2700	169
Copper	8940	558
Freshwater	1000	62.4
Gasoline	720	42 – 45
Gold	19320	1206
Iron	7870	491
Lead	11.34×10^3	708
Magnesium	1740	109
Mercury	13546	849
Milk	1030	64.5
Molasses	1600	100
Olive Oil	913	57
Platinum	21.45×10^3	1340
Seawater	1030	64

Exercises *Section* 5.5 – Physical Applications

(**1 – 8**) Find the mass of a thin bar with the given density function

1. $\rho(x) = 1 + \sin x$; $0 \le x \le \pi$

2. $\rho(x) = 1 + x^3$; $0 \le x \le 1$

3. $\rho(x) = 2 - \frac{x}{2}$; $0 \le x \le 2$

4. $\rho(x) = 5e^{-2x}$; $0 \le x \le 4$

5. $\rho(x) = x\sqrt{2 - x^2}$; $0 \le x \le 1$

6. $\rho(x) = \begin{cases} 1 & if \ 0 \le x \le 2 \\ 2 & if \ 2 < x \le 3 \end{cases}$

7. $\rho(x) = \begin{cases} 1 & if \ 0 \le x \le 2 \\ 1 + x & if \ 2 < x \le 4 \end{cases}$

8. $\rho(x) = \begin{cases} x^2 & if \ 0 \le x \le 1 \\ x(2 - x) & if \ 1 < x \le 2 \end{cases}$

9. Find the mass of a bar on the interval $0 \le x \le 9$ with a density (in *g/cm*) given by

$\rho(x) = 3 + 2\sqrt{x}$

10. Find the mass of a 3-*m* bar on the interval $0 \le x \le 3$ with a density (in *g/m*) given by

$\rho(x) = 150e^{-x/3}$

11. Find the mass of a bar on the interval $0 \le x \le 6$ with a density

$$\rho(x) = \begin{cases} 1 & if \quad 0 \le x < 2 \\ 2 & if \quad 2 \le x < 4 \\ 4 & if \quad 4 \le x \le 6 \end{cases}$$

12. It takes 50 *J* of work to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

13. It takes 50 *N* of force to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

14. A cylindrical water tank has a height of 6 *m* and a radius of 4 *m*. how much work is required to empty the full tank by pumping the water to an outflow pipe at the top of the tank?

15. Find the total force on the face of a semicircular dam with a radius of 20 *m* when its reservoir is full of water. The diameter of the semicircle is the top of the dam.

16. A rock climber is about to haul up 100 *N* (about 22.5 *lb.*) of equipment that has been hanging beneath her on 40 *m* rope that weighs 0.8 *N/m*. How much work will it take? (*Hint*: Solve for the rope and equipment separately, then add)

17. A 2-*oz* tennis ball was served at $160\ \ ft/sec$. How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by $32\ \ ft/sec^2$, the acceleration of gravity.)

18. How many foot-pounds of work does it take to throw a baseball 90 *mph*? A baseball weights 5 *oz*.

19. A 1.6-*oz* golf ball is driven off the tee at a speed of $280\ \ ft/sec$. How many foot-pounds of work are done on the ball getting it into the air?

20. You drove an 800-*gal* tank truck of water from the base of a mountain to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *min*. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 *lb./gal*.

21. A force of 200 *N* will stretch a garage door spring 0.8 *m* beyond its unstressed strength. How far will a 300-*N* force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

22. A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

23. A spring has a restoring force given by $F(x) = 25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x = 0)$ to a variable distance *x*. Graph the work function. Compare the work required to stretch the spring *x* units from equilibrium to the work required to compress the spring *x* units from equilibrium.

24. A particle free to move on the *x*-axis is attracted towards the origin by a force of magnitude equal to kx^2 (*k* constant). Find the work done when the paticle moves from the position $x = 4$ to $x = 2$.

25. A ship's anchor weighs $2,000\ lbs.$ and the anchor weighs $50\ lbs.$ what is the work done in pulling up (vertically) the anchor if 100 feet of chain are out ?

26. A 60-*lb* bucket is raised from the ground into the air by pulling in 60 *feet* of rope at a constant speed. The rope weighs $\frac{1}{4}$ *lb/ft*. Determine the work needed to raise the weight 60 *feet*.

27. A 20-*lb* is raised from the ground into the air by pulling in 100 *feet* of rope at a constant speed. The rope weighs $\frac{1}{2}$ *lb/ft*.

Determine the work needed to raise the weight 50 *feet*.

28. A satellite orbits the earth at altitudes of about 15×10^3 *miles* with weight around 10^3 *lbs*. How much additional work is required to lift a satellite to an altitude 10^5 *miles* ?

29. A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

30. Find the fluid force on a rectangular metal sheet measuring 3 *feet* by 4 *feet* that is submerged in 6 *feet* of water.

31. It took 1800 *J* of work to stretch a spring from its natural length of 2 *m* to a length of 5 *m*. Find the spring's force constant.

32. How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence.

33. How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in *N*) is given by $F(x)=\frac{2}{x^2}$ acting along the *x-axis* .

34. A force of 200 *N* will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-*N*-force stretch the spring?
b) How much work does it take to stretch the spring this far?

35. A spring on a horizontal surface can be stretched and held 0.5 *m* from its equilibrium position with a force of 50 *N*.

a) How much work is done in stretching the spring 1.5 *m* from its equilibrium position?
b) How much work is done in compressing the spring 0.5 *m* from its equilibrium position?

36. Suppose a force of 10 *N* is required to stretch a spring 0.1 *m* from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant *k*.
b) How much work is needed to ***compress*** the spring 0.5 *m* from its equilibrium position?
c) How much work is needed to ***stretch*** the spring 0.25 *m* from its equilibrium position?
d) How much additional work is required to stretch the spring 0.25 *m* if it has already been stretched 0.1 *m* from its equilibrium position?

37. A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.
b) How much work is done in stretching the spring from 10 *in* to 12 *in*?
c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

38. It takes a force of 21,714 *lb*. to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in-lb*.

39. A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 10 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

40. A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

41. An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *ft* of cable are paid out, and effectively 0 *ft* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

42. The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

a) How much work will it take to empty the cistern?
b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?
c) How long will it take the pump in part (b) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)
d) What are the answers to parts (a) through (c) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?

43. When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

44. The strength of Earth's gravitation field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975 \times 10^{24}\ kg$ is Earth's mass, $G = 6.6720 \times 10^{-11}\ N \cdot m^2 kg^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \quad joules$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not consider energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

45. You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 minutes.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8- *lb./gal*.

46. A cylindrical water tank has height 8 *m* and radius 2 *m*

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

47. A cylindrical tank has a radius $r = 5$ *feet* and 15 *feet* high and contains 10 *feet* of water.

What is the work done in pumping the water to a height of 15 *feet* above the top of the tank?

48. A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain

49. A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.

a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

50. A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000\frac{kg}{m^3},\ g = 9.8\frac{m}{s^2}\right)$

51. A vertical gate in a dam has the shape of an isosceles trapezoid 8 *feet* across the top and 6 *feet* across the bottom. With a height of 5 *feet*. What is the fluid force on the gate if the top of the gate is 4 *feet* below the surface of the water?

52. Pumping water from a lake 15-*feet* below the bottom of the tank can fill the cylindrical tank shown here.

There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer.

53. A tank truck hauls milk in a 6-*feet* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

54. A triangular plate, base 5 *feet*, height 6 *feet*, is submerged in water, vertex down, plane vertical, and 2 *feet* below the surface. Find the total force on one face of the plate.

55. The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?

56. A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***half full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

57. A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

58. A swimming pool is 20 *feet* wide, 40 *feet* long, 4 *feet* deep at one end, and 8 *feet* deep at the other end. The bottom is an inclined plane. Find the fluid force on each vertical wall.

59. A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 *m* at the other end.

Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

(60 – 69) Find the total force on the face of the given dam

60.

61.

62.

63.

64.

65. $y = -\frac{1}{2}\sqrt{36 - 9x^2}$ (ft)

66.

67.

68.

69.

$(\mathbf{70}-\mathbf{77})$ Find the fluid force on the vertical plate submerged in water

70.

71.

72.

73.

74.

75.

76.

77.

78. A rectangular plate of height ***h*** *feet* and base ***b*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center is ***k*** *feet* below the surface of the fluid, where $h \le \frac{k}{2}$. Show that the fluid force on the surface of the plate is $F = wkhb$.

79. A circular plate of radius ***r*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center of the circle is $k\ (k > r)$ *feet* below the surface of the fluid. Show that the fluid force on the surface of the plate is $F = \pi wkr^2$.

80. A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

81. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

82. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

83. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

84. A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t) = 4t^2$, where *x* is measured in *meters* and *t* is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

a) Note that $x''(t) = 8$; then use Newton's second law, $(F = ma = mx''(t))$ to evaluate the work integral $W = \int_{x_0}^{x_f} F(x)\,dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to *t*.

85. A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater

86. A square plate 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater

87. Water fills a tank in the shape of a right-circular cone with top radius 3 *m* and depth 4 *m*. How much work must be done (against gravity) to pump all the water out of the tank over the top edge of the tank?

88. You oversee the evacuation and repair of the storage tank.

The tank is a hemisphere of radius 10 *feet* and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for $\frac{1}{2}$ ¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 *feet* above the top of the tank. If you have $5,000 budget for the job, can you afford to hire the firm?

89. You decided to drill a well to increase a water supply. You have determined that a water tower will be necessary to provide the pressure needed for distribution

The water is to be pumped from a 300-*ft* well through a vertical 4-*in*. pipe into the base of a cylindrical tank 20 *feet* in diameter and 25 *feet* high. The base of the tank will be 60 *feet* above ground. The pump is a 3-*hp* pump, rated at $1,650\ \ ft \cdot lb/sec$. How long will it take to fill the tank the first time? (Include the time it takes to fill the pipe). Assume that water weighs $62.4\ \ lb/ft^3$.

90. Calculate the fluid force on one side of a right-triangular plate with edges 3 *feet*, 4 *feet*, and 5 *feet* if the plate sits at the bottom of the pool filled with water to a depth of 6 *feet* on its 3-*feet* edge and titlted at $60°$ to the bottom of the pool.

91. Two electrons *r meters* apart repel each other with a force of $F = \dfrac{23 \times 10^{29}}{r^2}\ \ newtons$

a) Suppose one electron is held fixed at the point $(1,\ 0)$ on the *x*-axis (units in *meters*). How much work does it take to move a second electron along *x*-axis from the point $(-1,\ 0)$ to the origin?

b) Suppose one electron is held fixed at the point $(-1,\ 0)$ and $(1,\ 0)$. How much work does it take to move a third electron along *x*-axis from the point $(5,\ 0)$ to $(3,\ 0)$?

92. The isosceles triangular plate is submerged vertically 1 *foot* below the surface of a freshwater lake.

a) Find the fluid force against one face of the plate.
b) What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?

93. The isosceles triangular plate is submerged vertically 3 *feet* above the surface of a freshwater lake. What force does the water exert on one face of the plate now?

94. The cubical metal tank has parabolic gate held in place by bolts and designed to withstand a fluid force of 160 *lb*. without rupturing. The liquid you plan to store has a weight-density of $50\ lb/ft^3$.

a) What is the fluid force on the gate when the liquid is 2 *feet* deep?
b) What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

95. The end plates of the trough were designed to withstand a fluid force of 6,667 *lb*.

a) What is the value of *h*?

b) How many cubic feet of water can the tank hold without exceeding this limitation?

96. A circular observation window on a marine science ship has a radius of 1 *foot*, and the center of the window is 8 *feet* below water level. What is the fluid force on the window?

97. Water pours into the tank at the rate of $4\ ft^3/min$. The tank's cross-sections are 4-*ft*-diameter semicircles. One end of the tank is movable but moving it to increase the volume compresses a spring. The spring constant is $k = 100\ lb/ft$. If the end of the tank moves 5 feet against the spring, the water will drain out of a safety hole in the bottom at the rate of $5\ ft^3/min$. Will the movable end reach the hole before the tank overflows?

98. A rectangular tank has a hole at the top for the output water. The tank dimension is given by $a = 8\ m$, $b = 4\ m$, *and* $c = 5\ m$.

The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full tank.

99. A rectangular tank has a spout (***d***–height) at the top for the water exist. The tank dimension is given by $a = 10\ m$, $b = 4\ m$, $c = 4\ m$, *and* $d = 1\ m$.

The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full tank.

100. Water exists a circular cone from the top. The cone dimension is given by $r = 4\ m$, *and* $h = 8\ m$.

The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full cone.
c) How much work, *W*, required to pump full cone over until the height of the water remaining in the cone is 2 *m*.

101. Water exists a spout located at the top of the cone. The cone dimension is given by $r = 6\ m$, $h = 18\ m$, *and* $\ell = 2\ m$.

The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full cone.

102. Water flows thru an open semi-sphere, with a radius $R = 8\ m$, from the top. The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full hemispheric.

103. Water exits an open semi-sphere, with a radius $R = 8\ m$, through spout $h = 1\ m$. The density of water is $\rho = 10^3\ kg/m^3$ *and* $g = 9.8\ m/s^2$

a) How much work, *W*, required to pump all the water out of half-filled.
b) How much work, *W*, required to pump all the water out of full hemispheric.

Summary

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ $\qquad$ $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

Solid: $\quad V = \int_a^b A(x)\,dx$

Disk: $\quad V = \pi\int_a^b f(x)^2\,dx \quad (about\ \ x-axis)$

$V = \pi\int_c^d [R(y)]^2\,dy \quad (about\ \ y-axis)$

Washer: $\quad V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx \quad (about\ \ x-axis)$

$V = \pi\int_c^d \left([R(y)]^2 - [r(y)]^2\right)dy \quad (about\ \ y-axis)$

Shell: $\quad V = 2\pi\int_a^b y\cdot g(y)\,dy \quad (about\ \ x-axis)$

$V = 2\pi\int_a^b x\cdot f(x)\,dx \quad (about\ \ y-axis)$

Length $\quad y = f(x) \quad L = \int_a^b \sqrt{1+f'(x)^2}\,dx$ $\qquad$ $x = g(y) \quad L = \int_c^d \sqrt{1+g'(y)^2}\,dy$

Area of the Surface $\quad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad x\text{-axis}$

$S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad y\text{-axis}$

Using ***Washer*** or ***Shell*** method:

Function in ***x***	Revolved	$\begin{cases} x-axis & \textit{Washer Method} \\ y-axis & \textit{Shell Method} \end{cases}$
Function in ***y***	Revolved	$\begin{cases} x-axis & \textit{Shell Method} \\ y-axis & \textit{Washer Method} \end{cases}$

Length	
$f(x)=ax^m+bx^n$ **1.** $m+n=2$ ✓ **2.** $abmn=-\frac{1}{4}$ ✓ $L=ax^m-bx^n\Big\vert_c^d$	$f(x)=a\,e^{mx}+b\,e^{nx}$ **3.** $m=-n$ ✓ **4.** $abmn=-\frac{1}{4}$ ✓ $L=a\,e^{mx}-b\,e^{nx}\Big\vert_c^d$

$$S=2\pi\int_c^d f(x)\,\overline{f'(x)}\,dx \quad \textit{iff} \quad \begin{cases} m+n=2 \\ abmn=-\frac{1}{4} \end{cases}$$

Force: $$F=\rho g\int_0^a \underbrace{(a-y)}_{depth}\ \underbrace{w(y)}_{width}\,dy$$

Work: $$W=\rho\,g\int_a^b A(y)\,D(y)\,dy$$

Exercises *Lecture* 5 – Review

$(\mathbf{1-10})$ Find the area of the region bounded by the graphs of:

1. $y=\cos x$, $y=\sin x$, $0\le x\le \frac{\pi}{2}$

2. $y^2=2x+6$, $y=x-1$

3. $y^4=x$, $y=\sqrt{2-x}$, $y=0$

4. $y=x^2-x-6$, $y=-4$

5. $y=x^3$, $y=-x$, $y=1$

6. $x+y+y^2=2$, $y\ge 0$, $x\ge 0$

7. $x=y^2$, $y=x^2$

8. $y=4x^3+3x^2+2$, $y=2$

9. $y=x^3-3x$, $y=x$

10. $y=1-\frac{1}{x^2}$, $y=1$, $x=1$, $x=4$

$(\mathbf{11-29})$ Find the volume of the solid obtained by rotating the region bounded by:

11. $y=x^3$, $y=8$, *and* $x=0$ *about* $y-axis$

12. $y=x^2$, $y=x$ about the $x-axis$.

13. $y=x^2$, $y=x$ *about* $y=2$.

14. Under the curve $y=\sqrt{x}$ from 0 to 1.

15. $y=x-x^2$, $y=0$ *about* $x=2$.

16. $y=-x^2+6x-8$, $y=0$, *about* $y-axis$.

17. $y=-x^2+6x-8$, $y=0$, *about* $x-axis$.

18. $x=(y-3)^2$, $x=4$, and rotate about $y=1$.

19. Between $x-axis$ and the parabola $y=4x-x^2$ about the line $y=6$

20. $y=x^2-5x+6$, $y=0$ $y-axis$

21. $y=2x^2$, $y=0$, $x=0$, $x=5$ *about* $x-axis$

22. $y=2x^2$, $y=0$, $x=0$, $x=5$ *about* $y=6$

23. $x^2-y^2=16$, $y=0$, $x=8$ *about* $x-axis$

24. $x^2-y^2=16$, $y=0$, $x=8$ *about* $y-axis$

25. $y^2=x^3$, $y=0$, $x=2$, about $x-axis$

26. $y=x^3$, $y=0$, $x=2$, about $x=2$

27. $y=x^3$, $y=8$, $x=0$, about $x=2$

28. $y=4x^2$, $y=16$, $x=0$ *about* $y=16$

29. $x=9-y^2$, $y=x-7$, $x=0$ *about* $y=3$

$(\mathbf{30-37})$ Find the length generated by:

30. $x=\frac{1}{4}y^4+\frac{1}{8y^2}$, $1\le y\le 2$

31. $y=\frac{1}{3}x^3+\frac{1}{4x}$, $1\le x\le 3$

32. $y=\frac{1}{4}x^3+\frac{1}{3x}$, $1\le x\le 2$

33. $y=\frac{1}{4}e^{2x}+\frac{1}{4}e^{-2x}$, $1\le x\le \ln 3$

34. $y=\frac{1}{4}e^{x}+e^{-x}$, $0\le x\le \ln 2$

35. $y=\frac{1}{4}x^2-\frac{1}{2}\ln x$, $1\le x\le e$

36. $y=x^{2/3}$, $1\le x\le 8$

37. $y=\frac{2}{3}\left(1+x^2\right)^{3/2}$, $0\le x\le 3$

$(\mathbf{38-45})$ Find the surface generated by:

38. $x=\frac{1}{4}y^4+\frac{1}{8y^2}$, $1\le y\le 2$ *about* $y-axis$

39. $y=\frac{1}{8}x^4+\frac{1}{4x^2}$, $1\le x\le 2$ *about* $x-axis$

40. $y=\frac{1}{3}x^3+\frac{1}{4x}$, $1\le x\le 3$ *about* $y=-2$

41. $y=\frac{1}{4}e^{x}+e^{-x}$, $0\le x\le \ln 2$ *about* $x-axis$

42. $y=e^{2x}+\frac{1}{16}e^{-2x}$, $1\le x\le \ln 2$ *about* $x-axis$

43. $y=x^3$, $0\le x\le 1$ *about* $x-axis$

44. $y=\frac{1}{3}\sqrt{x}\,(3-x)$, $1\le x\le 3$ *about* $x-axis$

45. $y^2+4x=2\ln y$, $1\le y\le 3$ *about* $x-axis$

46. Find the surface area if a right circular cone if height h and radius r.

47. Find the surface are of a cap of a sphere with radius a dertermined by a distance b from the center.

48. A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Find its volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6.

49. Find the volume of a solid that has a base a circle of radius r. Each cross section perpendicular to a fixed diameter of the circle is an isosceles triangle with altitude equal to one-half of its base.

50. A solid has a circular base. Find the volume of the solid if the semicircular is perpendicular the major axis of an ellipse.

51. Find the volume of a wedge, cut from a perfectly round of radius r by two planes, one perpendicular to the axis of the tree and the other intersecting the first plane at an angle of $30°$ along a diameter.

52. Find the volume of the tetrahedron formed by three perpendicular sides and three perpendicular edges of lenths a, b, c.

53. A spring with a natural length of 1 *foot* is stretches $\frac{1}{2}$ *inch* by a 12-*lbs*. force. Find the work done stretching the spring from 12 to 18 *inches*.

54. A spring supporting a railroad car has a natural length of 1 *foot*, and a force of 8,000 *lbs*. compresses it $\frac{1}{2}$ *inch*. Find the work done in compressing it from 12 to 9 *inches*.

55. An empty 5-*lbs* bucket filled with 60 lbs. of sand, then raised from the ground 10 *feet* at a constant speed. Sand leaks out of a hole in the bucket at a uniform rate, and a third of the sand is lost by the end of the lifting. Find the work done in the lifting process.

56. A conical tank, 10 *feet* deep and 8 *feet* wide, is filled with water to a depth of 5 *feet*. The tank is emptied by pumping the water over the top edge.

How much work is done?

57. Find the fluid force on the vertical plate submerged in water (in *meters*).

Practice *Exam A*

1. Find the ***area*** between the curves $y = x^2 - 18$ and $y = x - 6$

2. Find the ***area*** of the region bounded by the graphs of $y^2 = 2x + 6, \quad y = x - 1$

3. Find the ***volume*** of the solid obtained by rotating about the $x - axis$ and the region under the curve $y = \sqrt{x}$ from 0 to 1.

4. Find the ***volume*** of the solid generated by the revolving $y = x^2 - 5x + 6$, $y = 0$, revolves about $y - axis$

5. Find the ***volume*** of the solid generated by the revolving the plane region about the given line

$$y = x^2, \quad y = 4x - x^2, \quad \textit{about the line} \quad x = 4$$

6. Find the ***length*** of the curve $f(x) = \frac{1}{4}x^3 + \frac{1}{3x}$ from $x = 1$ to $x = 3$.

7. Find the ***length*** of the curve $y = e^{2x} + \frac{1}{16}e^{-2x} \quad 0 \le x \le \ln 2$.

8. Find the area of the ***surface*** generated by

$$y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3} \quad \textit{revolved about } x - axis$$

9. Find the ***mass*** of a thin bar with given density function $\rho(x) = x^2\sqrt{3 - x^3}$, $0 \le x \le 1$

10. A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total ***force*** on the face of the dam when the reservoir is full? $\left(\rho = 10^3 \frac{kg}{m^3}, \ g = 9.8\frac{m}{s^2}\right)$ $\left(use\ 10^*\right)$

Practice *Exam B*

1. Find the ***area*** between the curves $y = 4 - x^2$ and $y = x^2$

2. Find the ***area*** of the region bounded by the graphs of $y = \cos x$, $y = \frac{3x}{\pi}$, $x \geq 0$

3. Find the ***volume*** of the solid obtained by rotating about the $x - axis$ and the region under the curve $y = x^2$, $x - axis$, $x = 1$ from 0 to 1.

4. Find the ***volume*** of the solid generated by the revolving $y = x^2$, $y = 4x - x^2$, revolves about $y - axis$

5. Find the ***volume*** of the solid generated by the revolving the plane region about the given line

$$y = 2x - 2x^2, \quad about\ the\ line \quad x = 2$$

6. Find the ***length*** of the curve $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2}$ from $x = 1$ to $x = 3$.

7. Find the ***length*** of the curve $y = e^x + \frac{1}{4}e^{-x}$ $0 \leq x \leq \ln 2$.

8. Find the area of the ***surface*** generated by $y = \sqrt{x}\,(3 - x)$, $1 \leq x \leq 4$ *revolved about* $x - axis$

9. Find the ***mass*** of a thin bar with given density function $\rho(x) = \frac{x}{x^2 + 2}$, $0 \leq x \leq 2$

10. A cylindrical tank, 20 *feet* deep and 5 *feet* wide, is filled with water to a depth of 6 *feet*. The tank is emptied by pumping the water over the top edge.

How much work is done?

Lecture Six

Integration Techniques

Section 6.1 – Integration by Parts

In calculus, and more generally in mathematical analysis, integration by parts or partial integration is a process that finds the integral of a product of functions in terms of the integral of the product of their derivative and antiderivative.

The rule can be thought of as an integral version of the product rule of differentiation.

Integration by parts is a technique for simplifying integrals of the form:

$$\int f(x)g(x)dx$$

Example: $\int x \cos x \, dx, \quad \int x^2 e^x \, dx, \quad and \quad \int x \ln x \, dx$

6.1–1 Integration by Parts – *Formula*

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)dx$$

Let u and v be differentiable functions of x.

$$\int u \, dv = uv - \int v \, du$$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example 1

Evaluate: $\int x \cos x \, dx$

Solution

Let:

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \int dv = \int \cos x \, dx = \sin x$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx \qquad \int u dv = uv - \int v du$$

$$= x \sin x + \cos x + C$$

Example 2

Evaluate: $\int \ln x \, dx$

Solution

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = \int dx = x$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx \qquad \int u dv = uv - \int v du$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

6.1–2 Tabular Integration

Example 3

Evaluate the integral $\int x^2 e^x \, dx$

Solution

$f(x) = x^2 \quad and \quad g(x) = e^x$

$f(x)$ & derivatives		$\int g(x) = \int e^x dx$
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x

It is called ***tabular integration.***

$$\int x^2 e^x \, dx = \left(x^2 - 2x + 2\right) e^x + C$$

$u = x^2 \qquad dv = e^x dx$

$du = 2x dx \qquad v = \int e^x dx = e^x$

$$\int x^2 \, e^x \, dx = x^2 e^x - 2 \int x \, e^x \, dx$$

Let: $u = x \qquad dv = e^x dx$

$du = dx \qquad v = \int e^x dx = e^x$

$$\int x^2 \, e^x \, dx = x^2 e^x - 2\left[xe^x - \int e^x dx\right]$$

$$= x^2 e^x - 2\left(xe^x - e^x\right) + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Example 4

Evaluate the integral $\int x^3 e^{-2x}\, dx$

Solution

$$\int e^{-2x}\, dx$$

+	x^3	$-\frac{1}{2} e^{-2x}$
–	$3x^2$	$\frac{1}{4} e^{-2x}$
+	$6x$	$-\frac{1}{8} e^{-2x}$
–	6	$\frac{1}{16} e^{-2x}$

$$\int x^3 e^{2x}\, dx = \left(-\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8}\right)e^{2x} + C$$

Example 5

Evaluate the integral $\int x^3 \sin x\, dx$

Solution

$$\int \sin x$$

+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

$$\int x^3 \sin x\, dx = \left(-x^3 + 6x\right)\cos x + \left(3x^2 - 6\right)\sin x + C$$

Example 6

Evaluate the integral $\int x^4 \cos 2x \, dx$

Solution

$$\int \cos 2x \, dx$$

+	x^4	$\frac{1}{2}\sin 2x$
–	$4x^3$	$-\frac{1}{4}\cos 2x$
+	$12x^2$	$-\frac{1}{8}\sin 2x$
–	$24x$	$\frac{1}{16}\cos 2x$
+	24	$\frac{1}{32}\sin 2x$

$$\int x^4 \cos 2x \, dx = \left(\frac{1}{2}x^4 - \frac{3}{2}x^2 + \frac{3}{4}\right)\sin 2x + \left(x^3 - \frac{3}{2}x\right)\cos 2x + C$$

Example 7

Evaluate the integral $\int e^x \cos x \, dx$

Solution

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x(\sin x + \cos x)$$

$$\int e^x \cos x \, dx = \frac{1}{2}e^x(\sin x + \cos x) + C$$

$$\int \cos x \, dx$$

+	e^x	$\sin x$
–	e^x	$-\cos x$
+	e^x	$-\int \cos x \, dx$

Let: $u = e^x$ $\quad dv = \cos x dx$

$du = e^x dx$ $\quad v = \int \cos x dx = \sin x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Let: $u = e^x$ $\quad dv = \sin x dx$

$du = e^x dx$ $\quad v = \int \sin x dx = -\cos x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$\int e^x \cos x\, dx + \int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx + \int e^x \cos x\, dx$$

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x\, dx = \frac{1}{2}(\sin x + \cos x)\, e^x + C$$

Example 8

Evaluate the integral $\int e^{5x} \sin 3x\, dx$

Solution

$$\int \sin 3x\, dx$$

+	e^{5x}	$-\frac{1}{3}\cos 3x$
−	$5e^{5x}$	$-\frac{1}{9}\sin 3x$
+	$25e^{5x}$	$-\frac{1}{9}\int \sin 3x\, dx$

$$\int e^{5x} \sin 3x\, dx = \left(-\frac{1}{3}\sin x + \frac{5}{9}\cos x\right) e^{5x} - \frac{25}{9}\int e^{5x} \cos 3x\, dx$$

$$\left(1 + \frac{25}{9}\right)\int e^{5x} \sin 3x\, dx = \left(-\frac{1}{3}\sin x + \frac{5}{9}\cos x\right) e^{5x}$$

$$\frac{34}{9}\int e^{5x}\sin 3x\,dx = \frac{1}{9}(-3\sin x + 5\cos x)e^{5x}$$

$$\int e^{5x}\sin 3x\,dx = \frac{1}{34}(-3\sin x + 5\cos x)e^{5x} + C$$

Example 9

Obtain a formula that expresses the integral $\int \cos^n x\,dx$

Solution

Let:

$$u = \cos^{n-1} x \qquad dv = \cos x\,dx$$

$$du = (n-1)\cos^{n-2} x(-\sin x\,dx) = -(n-1)\cos^{n-2} x\sin x\,dx \qquad v = \int \cos x\,dx = \sin x$$

$$\int \cos^n x\,dx = \cos^{n-1} x\,\sin x - \int \sin x\left(-(n-1)\cos^{n-2} x\,\sin x\,dx\right) \qquad \int u\,dv = uv - \int v\,du$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right)dx$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$(1+n-1)\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Example: $\int \cos^3 x\, dx = \frac{1}{3}\cos^2 x\ \sin x + \frac{2}{3}\int \cos x\, dx$

$$= \frac{1}{3}\cos^2 x\ \sin x + \frac{2}{3}\sin x + C$$

6.1–3 Evaluating Definite Integrals by Parts

Example 10

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution

$$A = \int_0^4 x\, e^{-x}\, dx$$

$$\int e^{-x}\, dx$$

+	x	$-e^{-x}$
–	1	e^{-x}

$$Area = (-x-1)e^{-x}\Big|_0^4$$

$$= 1 - 5e^{-4} \quad unit^2$$

2nd *Method*

Let: $u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = \int e^{-x}dx = -e^{-x}$

$$\int_0^4 xe^{-x}dx = \left(-xe^{-x}\right|_0^4 - \int_0^4 \left(-e^{-x}\right) dx \qquad \int u\ dv = u\,v - \int v\ du$$

$$= -\left(4e^{-4} - 0\right) + \int_0^4 e^{-x}\ dx$$

$$= -4e^{-4} - \left(e^{-x}\right|_0^4$$

$$= -4e^{-4} - \left(e^{-4} - 1\right)$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4} \quad unit^2 \qquad \approx 0.91 \ unit^2$$

Example 11

Evaluate the definite integral $\displaystyle\int_0^1 e^{2x} \cos 4x \, dx$

Solution

$$\int \cos 4x \, dx$$

+	e^{2x}	$\frac{1}{4} \sin 4x$
−	$2e^{2x}$	$-\frac{1}{16} \cos 4x$
+	$4e^{2x}$	

$$\int e^{2x} \cos 4x \, dx = \left(\frac{1}{4}\sin 4x + \frac{1}{8}\cos 4x\right) e^{2x} - \frac{1}{4}\int e^{2x} \cos 4x \, dx$$

$$\left(1 + \frac{1}{4}\right)\int e^{2x} \cos 4x \, dx = \frac{1}{8}\left(2\sin 4x + \cos 4x\right) e^{2x}$$

$$\frac{5}{4}\int e^{2x} \cos 4x \, dx = \frac{1}{8}\left(2\sin 4x + \cos 4x\right) e^{2x}$$

$$\int_0^1 e^{2x} \cos 4x \, dx = \frac{1}{10}\left(2\sin 4x + \cos 4x\right) e^{2x} \Big|_0^1$$

$$= \frac{1}{10}\left(\left(2\sin 4 + \cos 4\right)e^2 - 1\right)$$

6.1–4 *Formula*

Evaluate the integral $\int x^n e^{ax}\, dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
–	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
–	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots \quad \vdots$	$\vdots \quad \vdots$

$$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \ldots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Note:

If $a < 0$, then all terms are negatives.

Otherwise, the signs alternate.

Example 12

Evaluate the definite integral $\displaystyle\int x^6\, e^{-2x}\, dx$

Solution

$$\begin{array}{ccccccc} \frac{1}{2}x^6 & \frac{3}{2}x^5 & \frac{15}{4}x^4 & \frac{15}{2}x^3 & \frac{45}{4}x^2 & \frac{45}{4}x & \frac{45}{8} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \nearrow \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}$$

$$\int x^6\, e^{-2x}\, dx = -\left(\frac{1}{2}x^6 + \frac{3}{2}x^5 + \frac{15}{4}x^4 + \frac{15}{2}x^3 + \frac{45}{4}x^2 + \frac{45}{4}x + \frac{45}{8}\right) e^{-2x} + C$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left(b \sin bx + a \cos bx\right) + C$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left(a \sin bx - b \cos bx\right) + C$$

Exercises *Section* 6.1 – Integration by Parts

$(\mathbf{1-80})$ Evaluate the integral

1. $\int x\ln x\,dx$

2. $\int \ln x^2\,dx$

3. $\int \ln(3x)\,dx$

4. $\int \frac{1}{x\ln x}\,dx$

5. $\int x(\ln x)^2\,dx$

6. $\int x^2(\ln x)^2\,dx$

7. $\int \frac{(\ln x)^3}{x}\,dx$

8. $\int x^2\ln x^3\,dx$

9. $\int \ln\left(x+x^2\right)dx$

10. $\int x\ln(x+1)\,dx$

11. $\int \frac{(\ln x)^2}{x}\,dx$

12. $\int x^5\ln 3x\,dx$

13. $\int x^5\ln x\,dx$

14. $\int \ln(x+1)\,dx$

15. $\int \frac{\ln x}{x^{10}}\,dx$

16. $\int x\,e^{2x}\,dx$

17. $\int x^3 e^x\,dx$

18. $\int \frac{2x}{e^x}\,dx$

19. $\int \frac{x^3e^{x^2}}{\left(x^2+1\right)^2}\,dx$

20. $\int x^2 e^{-3x}\,dx$

21. $\int \left(x^2-2x+1\right)e^{2x}dx$

22. $\int x^5 e^{x^3}\,dx$

23. $\int x\,e^{-4x}\,dx$

24. $\int \frac{xe^{2x}}{(2x+1)^2}\,dx$

25. $\int \frac{5x}{e^{2x}}\,dx$

26. $\int \frac{e^{1/x}}{x^2}\,dx$

27. $\int x^2e^{4x}\,dx$

28. $\int x^3 e^{-3x}\,dx$

29. $\int x^4 e^x\,dx$

30. $\int x^3 e^{4x}\,dx$

31. $\int (x+1)^2 e^x\,dx$

32. $\int 2x\,e^{3x}\,dx$

33. $\int x^2\sin x\,dx$

34. $\int \theta\cos\pi\theta\,d\theta$

35. $\int 4x\sec^2 2x\,dx$

36. $\int x^3\sin x\,dx$

37. $\int \left(x^3-2x\right)\sin 2x\,dx$

38. $\int x^2\sin 2x\,dx$

39. $\int x^2\sin(1-x)\,dx$

40. $\int x\sin x\cos x\,dx$

41. $\int x\cos x\,dx$

42. $\int x\csc x\cot x\,dx$

43. $\int x^2 \cos x \, dx$

44. $\int x^3 \cos 2x \, dx$

45. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

46. $\int x \sinh x \, dx$

47. $\int x^2 \cosh x \, dx$

48. $\int x^2 \sinh x \, dx$

49. $\int x^3 \cosh x \, dx$

50. $\int \cos x \cosh x \, dx$

51. $\int \sin x \cosh x \, dx$

52. $\int \sin x \sinh x \, dx$

53. $\int e^{2x} \cos 3x \, dx$

54. $\int e^{-3x} \sin 5x \, dx$

55. $\int e^{-x} \sin 4x \, dx$

56. $\int e^{-2\theta} \sin 6\theta \, d\theta$

57. $\int e^{-3x} \sin 4x \, dx$

58. $\int e^{4x} \cos 2x \, dx$

59. $\int e^{3x} \cos 3x \, dx$

60. $\int e^{3x} \cos 2x \, dx$

61. $\int e^{x} \sin x \, dx$

62. $\int e^{-2x} \sin 3x \, dx$

63. $\int \frac{x}{\sqrt{x-1}} dx$

64. $\int x\sqrt{x-5} \, dx$

65. $\int \frac{x}{\sqrt{6x+1}} dx$

66. $\int \frac{x}{2\sqrt{x+2}} dx$

67. $\int \frac{2x^2-3x}{(x-1)^3} dx$

68. $\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}} dx$

69. $\int \frac{x}{\sqrt{x+1}} dx$

70. $\int \frac{x^5}{\sqrt{1-2x^3}} dx$

71. $\int x\sqrt{1-3x} \, dx$

72. $\int \sin(\ln x) \, dx$

73. $\int \tan^{-1} y \, dy$

74. $\int \sin^{-1} y \, dy$

75. $\int x \tan^{-1} x \, dx$

76. $\int \sinh^{-1} x \, dx$

77. $\int \tan^{-1} 3x \, dx$

78. $\int \cos^{-1}\left(\frac{x}{2}\right) dx$

79. $\int x \sec^{-1} x \, dx$

80. $\int e^{-(s-2)t} \cos 3t \, dt$

$(\mathbf{81-104})$ Evaluate the definite integral.

81. $\displaystyle\int_{-1}^{0} 2x^2\sqrt{x+1}\,dx$

82. $\displaystyle\int_{0}^{1/\sqrt{2}} x\tan^{-1}x^2\,dx$

83. $\displaystyle\int_{1}^{e} x^2\ln x\,dx$

84. $\displaystyle\int_{-1}^{\ln 2} \frac{3t}{e^t}\,dt$

85. $\displaystyle\int_{\pi}^{2\pi} \cot\frac{x}{3}\,dx$

86. $\displaystyle\int_{0}^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx$

87. $\displaystyle\int_{1}^{e} x^3\ln x\,dx$

88. $\displaystyle\int_{0}^{1} x\sqrt{1-x}\,dx$

89. $\displaystyle\int_{0}^{\pi/3} x\tan^2 x\,dx$

90. $\displaystyle\int_{0}^{\pi} x\sin x\,dx$

91. $\displaystyle\int_{1}^{e} \ln 2x\,dx$

92. $\displaystyle\int_{0}^{\pi/2} x\cos 2x\,dx$

93. $\displaystyle\int_{0}^{\ln 2} x\,e^x\,dx$

94. $\displaystyle\int_{1}^{e^2} x^2\ln x\,dx$

95. $\displaystyle\int_{0}^{3} x\,e^{x/2}\,dx$

96. $\displaystyle\int_{0}^{2} x^2 e^{-2x}\,dx$

97. $\displaystyle\int_{0}^{\pi/4} x\cos 2x\,dx$

98. $\displaystyle\int_{0}^{\pi} x\sin 2x\,dx$

99. $\displaystyle\int_{1}^{4} e^{\sqrt{x}}\,dx$

100. $\displaystyle\int_{0}^{\frac{\pi}{2}} x^2\cos 2x\,dx$

101. $\displaystyle\int_{0}^{2} x^3 e^{3x}\,dx$

102. $\displaystyle\int_{0}^{1} x^4 e^{-2x}\,dx$

103. $\displaystyle\int_{0}^{2} \left(x^2-2x-1\right)e^{4x}\,dx$

104. $\displaystyle\int_{1}^{e} \frac{\ln x}{x}\,dx$

$(\mathbf{105-109})$ Use integration by parts to establish the reduction formula.

105. $\displaystyle\int x^n\sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$

106. $\displaystyle\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a\neq 0$

107. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

108. $\int_a^b \left(\int_x^b f(t)\, dt \right) dx = \int_a^b (x-a) f(x)\, dx$

109. $\int \sqrt{1-x^2}\, dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}}\, dx$

110. Find the indefinite integral: $\int 5x^n \ln ax\, dx \quad a \neq 0,\ n \neq -1$

111. Find the volume of the solid generated by the region bounded by $f(x) = x \ln x$, and the $x-axis$ on $\left[1,\ e^2\right]$ is revolved about the $y-axis$.

112. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

113. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^{-x}$, and the line $x = 1$, about:

a) the line $y-axis$

b) the line $x = 1$

114. Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the $y\text{-}axis$.

115. Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, and the $x-axis$ on $[1,\ \ln 2]$ is revolved about the line $x = \ln 2$.

116. Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the $x\text{-}axis$ on $[0,\ \pi]$ is revolved about the $y\text{-}axis$.

117. Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0,\ \frac{1}{2}\right]$.

118. Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and \quad x = e^2$

119. Determine the area of the shaded region bounded by

$$y = \ln x, \quad y = 2, \quad y = 0, \quad \textit{and } x = 0$$

120. The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ \textit{and}\ 3$.

a) Find the area of the region bounded by c $y = xe^{-x}$ and the x-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, b]. Because this area depends on a and b, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

121. Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s = 0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \ldots$

122. Given the region bounded by the graphs of $y = x\sin x, \quad y = 0, \quad x = 0, \quad x = \pi$, find:

a) The area of the region.
b) The volume of the solid generated by revolving the region about the x-*axis*.
c) The volume of the solid generated by revolving the region about the y-*axis*.
d) The centroid of the region.

123. The region R is bounded by the curve $y = \ln x$ and the x-*axis* on the interval $[1, e]$. Find the volume of the solid that is generated when R is revolved in the following ways.

a) About the x-*axis*
b) About the y-*axis*
c) About the line $x = 1$
d) About the line $y = 1$

124. A string stretched between the two points $(0, 0)$ and $(2, 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a ***Fourier Sine series*** whose coefficients are given by

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}\,dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}\,dx$$

Find b_n

Section 6.2 – Trigonometric Integrals

6.2–1 Products of Powers of *Sines* and *Cosines*

We begin with integrals of the form

$$\int \sin^m x \ \cos^n x \, dx$$

Example 1

Evaluate the integral $\displaystyle\int \sin^3 x \cos^2 x \, dx$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin x \sin^2 x \cos^2 x \, dx \\
&= \int \left(1-\cos^2 x\right)\cos^2 x \ \left(-d\left(\cos x\right)\right) \qquad d\left(\cos x\right) = -\sin x \, dx \\
&= -\int \left(\cos^2 x - \cos^4 x\right) \, d\left(\cos x\right) \\
&= -\left(\tfrac{1}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x\right) + C \\
&= \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C
\end{aligned}$$

Example 2

Evaluate the integral $\displaystyle\int \cos^5 x \, dx$

Solution

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \qquad \cos x \, dx = d\left(\sin x\right) \qquad \cos^2 x = 1 - \sin^2 x \\
&= \int \left(1-\sin^2 x\right)^2 \, d\left(\sin x\right) \\
&= \int \left(1 - 2\sin^2 x + \sin^4 x\right) \, d \sin x
\end{aligned}$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Example 3

Evaluate the integral $\int \sin^2 x \cos^4 x \, dx$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{8}\int (1-\cos 2x)\left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{8}\int \left(1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x\right)dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int\left(\cos^3 2x+\cos^2 2x\right)dx\right]$$

$$\int \cos^3 2x dx = \int \left(1-\sin^2 2x\right)\cos 2x dx$$

$$= \frac{1}{2}\int \left(1-\sin^2 2x\right)d\left(\sin 2x\right)$$

$$= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)$$

$$\int \cos^2 2x dx = \frac{1}{2}\int (1+\cos 4x)\,dx$$

$$= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)-\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+C$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 2x+\frac{1}{6}\sin^3 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x\right]+C$$

$$= \frac{1}{8}\left(\frac{1}{2}x + \frac{1}{6}\sin^3 2x - \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{16}\left(x + \frac{1}{3}\sin^3 2x - \frac{1}{4}\sin 4x\right) + C$$

Example 4

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx$

Solution

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$1+\cos 2\theta = 2\cos^2\theta$$

Let $\theta = 2x$

$$1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\ dx$$

$$\sqrt{2\cos^2 2x} = \sqrt{2}\ \sqrt{\cos^2 2x}$$
$$= \sqrt{2}\ |\cos 2x|$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\ dx \qquad \cos 2x \geq 0 \quad on \quad \left[0, \frac{\pi}{4}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\ dx$$

$$= \sqrt{2}\ \left(\frac{\sin 2x}{2}\right|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\ \left(\sin\frac{\pi}{2} - 0\right)$$

$$= \frac{\sqrt{2}}{2}$$

Example 5

Evaluate the integral $\int \sin^3 x \cos^{-2} x \, dx$

Solution

$$\int \sin^3 x \cos^{-2} x \, dx = \int \sin^2 x \cos^{-2} x \, \sin x \, dx$$

$$= -\int \left(1-\cos^2 x\right) \cos^{-2} x \, d(\cos x)$$

$$= -\int \left(\cos^{-2} x - 1\right) \, d(\cos x)$$

$$= -\left(-\cos^{-1} x - \cos x\right) + C$$

$$= \cos x + \sec x + C$$

Example 6

Evaluate the definite integral $\int_0^{\pi/2} \sin^5 5x \, dx$

Solution

$$\int_0^{\pi/2} \sin^5 5x \, dx = \int_0^{\pi/2} \sin^4 5x \sin 5x \, dx$$

$$= -\frac{1}{5} \int_0^{\pi/2} \left(\sin^2 5x\right)^2 \, d(\cos 5x)$$

$$= -\frac{1}{5} \int_0^{\pi/2} \left(1-\cos^2 5x\right)^2 \, d(\cos 5x)$$

$$= -\frac{1}{5} \int_0^{\pi/2} \left(1 - 2\cos^2 5x + \cos^4 5x\right) \, d(\cos 5x)$$

$$= -\frac{1}{5} \left(\cos 5x - \frac{2}{3}\cos^3 5x + \frac{1}{5}\cos^5 5x \right|_0^{\pi/2}$$

$$= -\frac{1}{5} \left(-1 + \frac{2}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{5}\left(\frac{15-10+3}{15}\right)$$

$$= \frac{8}{75}$$

6.2–2 Products of Powers of *tan x* and *sec x*

Example 7

Evaluate the integral $\int \tan^4 x\, dx$

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \int \tan^2 x \cdot \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

$$= \int \tan^2 x\, d(\tan x) - \int \sec^2 x\, dx + \int dx \qquad d(\tan x) = \sec^2 x\, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Example 8

Evaluate the integral $\int \sec^3 x\, dx$

Solution

Let: $u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \qquad v = \tan x$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \tan x\left(\sec x \tan x\, dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Example 9

Evaluate the integral $\int \tan^3 3x\ \sec 3x\, dx$

Solution

$$\int \tan^3 3x\ \sec 3x\, dx = \int \tan^2 3x\ \tan 3x\ \sec 3x\, dx \qquad d\left(\sec 3x\right) = 3\tan 3x\ \sec 3x\, dx$$

$$= \frac{1}{3}\int \left(\sec^2 3x - 1\right)\ d\left(\sec 3x\right) \qquad \tan^2 x = \sec^2 x\ -1$$

$$= \frac{1}{3}\left(\frac{1}{3}\sec^3 3x - \sec 3x\right) + C$$

$$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$$

Example 10

Evaluate the integral $\int \csc^3 x \cot^3 x \, dx$

Solution

$$\int \csc^3 x \cot^3 x \, dx = \int \csc^2 x \cot^2 x \left(\csc x \cot x\right) dx \qquad d\left(\csc x\right) = -\csc x \cot x \, dx$$

$$= -\int \csc^2 x \left(\csc^2 x - 1\right) d\left(\csc x\right) \qquad \cot^2 x = \csc^2 x - 1$$

$$= -\int \left(\csc^4 x - \csc^2 x\right) d\left(\csc x\right)$$

$$= -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$$

Example 11

Evaluate the integral $\int \sec^2 x \sqrt{5 + \tan x} \, dx$

Solution

$$\int \sec^2 x \sqrt{5 + \tan x} \, dx = \int \left(5 + \tan x\right)^{1/2} d\left(5 + \tan x\right) \qquad d\left(\tan x\right) = \sec^2 x \, dx$$

$$= \frac{2}{3}\left(5 + \tan x\right)^{3/2} + C$$

Example 12

Evaluate the integral $\int \sec x \sqrt{\sec x + \tan x} \, dx$

Solution

$$d\left(\sqrt{\sec x + \tan x}\right) = \frac{\sec x \tan x + \sec^2 x}{2\sqrt{\sec x + \tan x}} \, dx$$

$$= \frac{\sec x \left(\tan x + \sec x\right)}{2\sqrt{\sec x + \tan x}} \, dx$$

$$= \frac{1}{2}\sec x \sqrt{\sec x + \tan x} \, dx$$

$$\int \sec x \sqrt{\sec x + \tan x}\, dx = 2\int d\left(\sqrt{\sec x + \tan x}\right)$$

$$= 2\sqrt{\sec x + \tan x} + C$$

Example 13

Evaluate the integral $\int \tan^2 x \sec^3 x\, dx$

Solution

$$\int \tan^2 x \sec^3 x\, dx = \int \left(\sec^2 x - 1\right) \sec^3 x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \int \sec^5 x\, dx - \int \sec^3 x\, dx$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$\int \sec^5 x\, dx = \int \sec^2 x \sec^3 x\, dx$$

$$u = \sec^3 x \qquad v = \int \sec^2 x\, dx$$

$$du = 3\sec^3 x \tan x\, dx \qquad = \tan x$$

$$\int \sec^5 x\, dx = \tan x \sec^3 x - 3\int \sec^3 x \tan^2 x\, dx$$

$$\int \tan^2 x \sec^3 x\, dx = \tan x \sec^3 x - 3\int \sec^3 x \tan^2 x\, dx - \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$4\int \tan^2 x \sec^3 x\, dx = \tan x \sec^3 x - \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$\boxed{\int \tan^2 x \sec^3 x\, dx = \frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln\left|\sec x + \tan x\right| + C}$$

6.2–3 Products of Sines and Cosines

Recall the identities:

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example 14

Evaluate the integral $\int \sin 3x \cos 5x \, dx$

Solution

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int \left(\sin(-2x) + \sin 8x\right) dx \\ &= \frac{1}{2}\int \left(-\sin(2x) + \sin 8x\right) dx \\ &= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + C \\ &= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C\end{aligned}$$

Example 15

Evaluate the integral $\int \cos 6x \cos 4x \, dx$

Solution

$$\begin{aligned}\int \cos 6x \cos 4x \, dx &= \frac{1}{2}\int \left(\cos 2x + \cos 10x\right) dx \\ &= \frac{1}{2}\left(\frac{1}{2}\sin 2x + \frac{1}{10}\cos 8x\right) \\ &= \frac{1}{4}\sin 2x + \frac{1}{20}\cos 8x + C\end{aligned}$$

6.2–4 Guidelines for Cosine & Sine

Case* 1** If ***m is ***odd***, we write m as $2k+1$ and use the identity $\sin^2 x = 1-\cos^2 x$ to obtain:

$$\begin{aligned}\sin^m x &= \sin^{2k+1} x \\ &= \left(\sin^2 x\right)^k \sin x \\ &= \left(1-\cos^2 x\right)^k \sin x\end{aligned}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\ dx = -d\left(\cos x\right)$

Case* 2** If ***m is ***even*** **and** ***n*** is ***odd***, in $\int \sin^m x\ \cos^n x\ dx$ we write n as $2k+1$ and use the identity $\cos^2 x = 1-\sin^2 x$ to obtain

$$\begin{aligned}\cos^n x &= \cos^{2k+1} x \\ &= \left(\cos^2 x\right)^k \cos x \\ &= \left(1-\sin^2 x\right)^k \cos x\end{aligned}$$

Then we combine the single $\cos x$ with dx in the integral and set $\cos x\ dx = d\left(\sin x\right)$

Case* 3** If both ***m **and** ***n*** are ***even***, in $\int \sin^m x\ \cos^n x\ dx$, we substitute:

$$\int \left(\sin^2 x\right)^{m/2} \left(\cos^2 x\right)^{n/2}\ dx$$

$$\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right) \qquad \cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$$

To reduce the integrand to one in lower powers of $\cos ax$

$$\int \cos ax\ dx = \frac{1}{a}\sin ax + C$$

6.2–5 Guidelines for Tangent & Secant

Case* 1** When the power of the tangent is ***odd and positive.

$$\tan^2 x = \sec^2 x - 1$$

$$\int \sec^m x \ \tan^{2k+1} x \, dx = \int \sec^{m-1} x \left(\tan^2 x\right)^k \sec x \tan x \, dx$$

$$= \int \sec^{m-1} x \left(\sec^2 x - 1\right)^k d\left(\sec x\right)$$

Case* 2** When the power of the secant is ***even and positive.

$$\int \sec^{2k} x \ \tan^n x \, dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x \ \sec^2 x \, dx$$

$$= \int \left(1 + \tan^2 x\right)^{k-1} \tan^n x \, d\left(\tan x\right)$$

***Case* 3** When there are no secant factors

$$\int \tan^n x \, dx = \int \tan^{n-2} x \left(\tan^2 x\right) dx$$

$$= \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

Case* 4** When there is only ***secant, use integration by parts.

***Case* 5** Otherwise, convert to cosines and sines.

6.2−6 Inverse Trigonometric

Example 16

Evaluate the integral $\int \tan^{-1}(ax)\, dx$

Solution

$$Let: \quad y = \tan^{-1}(ax) \qquad ax = \tan y$$

$$dx = \frac{1}{a}\sec^2 y\, dy$$

$$\int \tan^{-1}(ax)\, dx = \frac{1}{a}\int y \sec^2 y\, dy$$

$$\int \sec^2 y\, dy$$

+	y	$\tan y$
−	1	$\ln\lvert \sec y \rvert$

$$\int \tan y\, dy = \int \frac{\sin y}{\cos y}\, dy$$

$$= -\int \frac{1}{\cos y}\, d(\cos y)$$

$$= -\ln\lvert \cos y \rvert$$

$$= \ln\lvert \sec y \rvert$$

$$\int \tan^{-1}(ax)\, dx = \frac{1}{a}\left(y \tan y - \ln\lvert \sec y \rvert\right)$$

$$= \frac{1}{a}\left(a x \tan^{-1}(a x) - \ln\left\lvert \sqrt{1+a^2x^2} \right\rvert\right)$$

$$= x\tan^{-1}(a x) - \frac{1}{a}\ln\sqrt{1+a^2x^2} + C$$

6.2–7 *Wallis*'s Formulas

1. If n is odd $(n \geq 3)$, then $$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right)$$

2. If n is even $(n \geq 2)$, then $$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Example 17

Evaluate the integral $$\int_0^{\frac{\pi}{2}} \cos^{15} x\, dx$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^{15} x\, dx = \frac{14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3}$$

$$= \frac{2^{11}}{6,435}$$

Example 18

Evaluate the integral $$\int_0^{\frac{\pi}{2}} \cos^{16} x\, dx$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^{16} x\, dx = \frac{\pi}{2} \frac{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}$$

$$= \frac{\pi}{2} \frac{15 \cdot 13 \cdot 11 \cdot 3}{16 \cdot 2 \cdot 4 \cdot 2 \cdot 8 \cdot 2 \cdot 4 \cdot 2}$$

$$= \frac{6,435}{2^{16}} \pi$$

6.2–8 *Formulas*

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$$

$$\int \sin^n x\, dx = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$$

$$\int \tan^n x\, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x\, dx$$

$$\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Exercises *Section* 6.2 – Trigonometric Integrals

(1 – 135) Evaluate the integral.

1. $\int \sin^5 \frac{x}{2}\, dx$

2. $\int \sin^4 6\theta\, d\theta$

3. $\int x^2 \sin^2 x\, dx$

4. $\int \sin^3 3x\, dx$

5. $\int \sin^5 x\, dx$

6. $\int 8\cos^4 2\pi x\, dx$

7. $\int x\cos^3 x\, dx$

8. $\int \cos^4 x\, dx$

9. $\int \cos^4 5x\, dx$

10. $\int \cos^2 3x\, dx$

11. $\int \cos^3 \frac{x}{3}\, dx$

12. $\int \cos^2 4x\, dx$

13. $\int \sqrt{1+\cos\frac{x}{2}}\, dx$

14. $\int \sec^4 2x\, dx$

15. $\int 6\sec^4 x\, dx$

16. $\int \sec^3 \pi x\, dx$

17. $\int \sec 4x\, dx$

18. $\int \sec^5 \theta\, d\theta$

19. $\int \sec^6 \theta\, d\theta$

20. $\int \sec^7 \theta\, d\theta$

21. $\int \csc^6 x\, dx$

22. $\int \tan^5 \frac{x}{2}\, dx$

23. $\int \tan^5 x\, dx$

24. $\int \tan^5 3x\, dx$

25. $\int \tan^6 3x\, dx$

26. $\int 20\tan^6 x\, dx$

27. $\int \tan^4 x\, dx$

28. $\int \tan^3 \theta\, d\theta$

29. $\int \tan^3 4x\, dx$

30. $\int \cot^3 2x\, dx$

31. $\int \cot^4 x\, dx$

32. $\int \cot^4 3x\, dx$

33. $\int \cot^5 3x\, dx$

34. $\int \sin^2 x\, \cos^2 x\, dx$

35. $\int \sin^2 x\, \cos^3 x\, dx$

36. $\int \sin^2 x\, \cos^4 x\, dx$

37. $\int \sin^2 x\, \cos^5 x\, dx$

38. $\int \sin^3 x\, \cos^5 x\, dx$

39. $\int \sin^3 x\, \cos^4 x\, dx$

40. $\int \sin^3 2x\, \cos^4 x\, dx$

41. $\int \sin^3 2x\, \cos^3 2x\, dx$

42. $\int \sin^4 x\cos^2 x\, dx$

43. $\int \sin^4 x \cos^3 x \, dx$

44. $\int \sin^4 x \, \cos^4 x \, dx$

45. $\int \sin^4 x \, \cos^5 x \, dx$

46. $\int \sin^5 x \, \cos^5 x \, dx$

47. $\int \sin^5 x \, \cos^{-2} x \, dx$

48. $\int \sin 3x \cos^6 3x \, dx$

49. $\int \sin^4 2x \cos 2x \, dx$

50. $\int \cos^3 2x \sin^5 2x \, dx$

51. $\int 16 \sin^2 x \cos^2 x \, dx$

52. $\int \sin 2x \cos 3x \, dx$

53. $\int \sin^2 \theta \cos 3\theta \, d\theta$

54. $\int \cos^3 \theta \sin 2\theta \, d\theta$

55. $\int \sin^{-3/2} x \, \cos^3 x \, dx$

56. $\int \sin^3 x \, \cos^{3/2} x \, dx$

57. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

58. $\int \sin 3x \, \cos 6x \, dx$

59. $\int \sin 3x \cos 7x \, dx$

60. $\int \sin 5x \cos 4x \, dx$

61. $\int \cos 2\theta \, \cos 6\theta \, d\theta$

62. $\int \cos 5\theta \, \cos 3\theta \, d\theta$

63. $\int \sin 2\theta \, \cos 4\theta \, d\theta$

64. $\int \sin(-7\theta) \, \cos 6\theta \, d\theta$

65. $\int \sin \theta \, \sin 3\theta \, d\theta$

66. $\int \sin 5\theta \, \sin 4\theta \, d\theta$

67. $\int \sin x \, \cos^5 x \, dx$

68. $\int \sin^7 2x \, \cos 2x \, dx$

69. $\int \sin^3 2x \, \sqrt{\cos 2x} \, dx$

70. $\int \sin^3 x \cos^2 x \, dx$

71. $\int \frac{\cos^3 \theta}{\sqrt{\sin \theta}} \, d\theta$

72. $\int \frac{\cos^5 \theta}{\sqrt{\sin \theta}} \, d\theta$

73. $\int \frac{\cos^2 x}{\sin^5 x} \, dx$

74. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

75. $\int \frac{\sin^4 x}{\cos^6 x} \, dx$

76. $\int \frac{2\cos x + 3\sin x}{\sin^3 x} \, dx$

77. $\int \frac{\sin 2x}{1 + \sin^2 x} \, dx$

78. $\int \frac{2 + \sin x + 2\cos x}{1 + \cos x} \, dx$

79. $\int \frac{dx}{1 - \cos x}$

80. $\int \frac{dx}{1 - \sin x}$

81. $\int \frac{\sin \theta \cos \theta}{2 - \cos \theta} \, d\theta$

82. $\int \tan^3 x \, \sec^3 x \, dx$

83. $\int \sec x \tan^2 x \, dx$

84. $\int \sec^2 x \tan^2 x \, dx$

85. $\int \sec^4 x \tan^2 x \, dx$

86. $\int \sec^6 4x \, \tan 4x \, dx$

87. $\int \sec^2 \frac{x}{2} \, \tan \frac{x}{2} \, dx$

88. $\int \tan^3 2x \, \sec^3 2x \, dx$

89. $\int \tan^5 2x \, \sec^4 2x \, dx$

90. $\int \tan^3 x \, \sec^5 x \, dx$

91. $\int \tan^3 x \sec^4 x \, dx$

92. $\int \tan^5 \theta \sec^4 \theta \, d\theta$

93. $\int \tan^5 \theta \sec^7 \theta \, d\theta$

94. $\int \tan^7 \theta \sec^5 \theta \, d\theta$

95. $\int \sec^4 3x \tan^3 3x \, dx$

96. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx$

97. $\int \sec^{-2} x \tan^3 x \, dx$

98. $\int \sqrt{\tan x} \sec^4 x \, dx$

99. $\int \tan^{3/2} x \sec^4 x \, dx$

100. $\int \tan^4 x \sec^4 x \, dx$

101. $\int \tan^5 \theta \csc^2 \theta \, d\theta$

102. $\int \csc^2 x \cot x \, dx$

103. $\int \csc^{10} x \cot x \, dx$

104. $\int (\cot 2x - \csc 2x)^2 \, dx$

105. $\int \csc^3 x \cot^3 x \, dx$

106. $\int \csc^4 2x \, dx$

107. $\int \operatorname{sech}^4 x \, dx$

108. $\int \sinh^3 x \cosh^2 x \, dx$

109. $\int \operatorname{sech}^2 x \sinh x \, dx$

110. $\int x \tanh x \, dx$

111. $\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx$

112. $\int \frac{\tan^2 x}{\sec x} \, dx$

113. $\int \frac{\sec x}{\tan^2 x} \, dx$

114. $\int \frac{\sec^2 x}{\tan^5 x} \, dx$

115. $\int \tan x \sqrt{\sec x} \, dx$

116. $\int \frac{\csc^4 x}{\cot^2 x} \, dx$

117. $\int \frac{\cot^3 x}{\csc x} \, dx$

118. $\int \frac{\sec^4 (\ln x)}{x} \, dx$

119. $\int e^x \sec\left(e^x + 1\right) \, dx$

120. $\int e^x \sec^3 e^x \, dx$

121. $\int e^x \sqrt{\tan^2 e^x + 1} \, dx$

122. $\int (1 + \cos 3x)^{3/2} \, dx$

123. $\int \sin^3 3x \cos^5 3x \, dx$

124. $\int \sin^4 3x \cos^2 3x \, dx$

125. $\int \sin^{-1}(ax) \, dx$

126. $\int x \sin^{-1}(ax) \, dx$

127. $\int x^2 \sin^{-1}(ax) \, dx$

128. $\int \cos^{-1}(ax) \, dx$

129. $\int x \cos^{-1}(ax) \, dx$

130. $\int x^2 \cos^{-1}(ax) \, dx$

131. $\int x \tan^{-1}(ax) \, dx$

132. $\int \cot^{-1}(ax) \, dx$

133. $\int \sec^{-1}(ax) \, dx$

134. $\int x \sec^{-1}(ax) \, dx$

135. $\int \csc^{-1}(ax) \, dx$

(136 – 176) Evaluate the definite integral.

136. $\int_0^{\sqrt{\frac{\pi}{2}}} x\ \sin^3\left(x^2\right)\ dx$

137. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\ dx$

138. $\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x}$

139. $\int_{\pi/6}^{\pi/3} \cot^3\theta\ d\theta$

140. $\int_0^{\pi/3} \tan^2 x\ dx$

141. $\int_0^{\pi/4} 6\tan^3 x\ dx$

142. $\int_0^{\pi/4} \tan^4 x\ dx$

143. $\int_0^{\pi} 8\sin^4 y\cos^2 y\ dy$

144. $\int_0^{\pi/6} 3\cos^5 3x\ dx$

145. $\int_0^{\pi/2} \sin^2 2\theta\ \cos^3 2\theta\ d\theta$

146. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\ dx$

147. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\ d\theta$

148. $\int_0^{\pi/6} \sqrt{1+\sin x}\ dx$

149. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta-1}\ d\theta$

150. $\int_0^{\pi} (1-\cos 2x)^{3/2}\ dx$

151. $\int_0^{\pi} \left(1-\cos^2 x\right)^{3/2}\ dx$

152. $\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2}\ dx$

153. $\int_{\pi/4}^{\pi/2} \csc^4\theta\ d\theta$

154. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\ dx$

155. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\ dx$

156. $\int_0^{\pi/4} \cos^5 2x\ \sin^2 2x\ dx$

157. $\int_0^{\pi/6} \sin^5 x\ dx$

158. $\int_{-\pi}^{\pi} \sin^2 x\ dx$

159. $\int_{-\pi/2}^{\pi/2} \left(\sin^2 x+1\right)\ dx$

160. $\int_0^{\pi/3} \sec^{3/2} x\ \tan x\ dx$

161. $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\ dx$

162. $\int_{\pi/6}^{\pi/3} \sin 6x\ \cos 4x\ dx$

163. $\int_{-\pi/2}^{\pi/2} 3\cos^3 x\ dx$

164. $\int_0^{\pi} \sec^2 x\ dx$

165. $\int_0^{\ln\left(\sqrt{3}+2\right)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\ dx$

166. $\int_0^{\pi/2} \cos^4 x\ dx$

167. $\int_0^{\pi/2} \cos^{10} x\ dx$

168. $\int_0^{\pi/2} \cos^7 x\ dx$

169. $\int_0^{\pi/2} \cos^9 x\ dx$

170. $\int_0^{\pi/2} \sin^5 x\ dx$

171. $\int_0^{\pi/2} \sin^6 x\ dx$

172. $\displaystyle\int_0^{\pi/2} \sin^8 x\, dx$

173. $\displaystyle\int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx$

174. $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1}(2x)\, dx$

175. $\displaystyle\int_0^{\sqrt{2}} x\, \sin^{-1}\left(\frac{1}{2}x\right) dx$

176. $\displaystyle\int_0^{\sqrt{3}} \cos^{-1}\left(\frac{1}{2}x\right) dx$

177. Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0, \frac{\pi}{4}\right]$

(178 – 181) Find the ***area*** of the region bounded by the graphs of the equations.

178. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

179. $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

180. $y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

181. $y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$

(182 – 183) Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*.

182. $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

183. $y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

(184 – 185) Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region.

184. $y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$

185. $y = \cos x, \quad y = \sin 0, \quad x = 0, \quad x = \frac{\pi}{2}$

186. For the given functions $y = \cos^2 x \quad y = \sin^2 x \quad$ *where* $\quad 0 \le x \le \pi$

a) Find the ***area*** bounded between the 2 curves.

b) Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*.

187. For the given functions $y = \cos^2 x \quad y = \sin^3 x \quad \textit{where} \quad 0 \le x \le \frac{\pi}{2}$

a) Find the ***area*** bounded between the 2 curves.

b) Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the $x\text{-}axis$.

Section 6.3 – Trigonometric Substitutions

6.3–1 Trigonometric

Trigonometric substitution is a technique that provide a method to evaluate an integral by reducing it to a simpler one.

This technique uses substitution to rewrite these integrals as trigonometric integrals.

$b\,x = a\,\tan\theta$	$b\,x = a\,\sin\theta$	$b\,x = a\,\sec\theta$
$\sqrt{a^2+(bx)^2} = a\,\sec\theta$	$\sqrt{a^2-(bx)^2} = a\,\cos\theta$	$\sqrt{(bx)^2-a^2} = a\tan\theta$

$$b\,x = a\,\tan\theta$$

$$\begin{aligned}\sqrt{a^2+(bx)^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2\left(1+\tan^2\theta\right)}\\ &= a\sqrt{\sec^2\theta}\\ &= a\,\sec\theta\end{aligned}$$

$x = a\tan\theta$ $\theta = \tan^{-1}\left(\frac{x}{a}\right) \quad with \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	

$x = a\sin\theta$ $\theta = \sin^{-1}\left(\frac{x}{a}\right) \quad with \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	
$x = a\sec\theta$ $\theta = \sec^{-1}\left(\frac{x}{a}\right) \quad with \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & if \ \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi & if \ \frac{x}{a} \le -1 \end{cases}$	

6.3–2 Procedure for a Trigonometric Substitution

1. Write down the substitution for x, Find the differential dx, and specify the selected values of θ for the substitution.

2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.

3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.

4. Reserve the substitution in the integration result and convert it back to the original variable x, try to avoid graphing the right triangle.

Example 1

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

Solution

Let $\quad x = 2\tan\theta \qquad \sqrt{4+x^2} = 2\sec\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$x = 2\tan\theta \quad \rightarrow \quad \tan\theta = \frac{x}{2}$$

$$\sqrt{4+x^2} = 2\sec\theta \quad \rightarrow \quad \sec\theta = \frac{\sqrt{4+x^2}}{2}$$

$$\boxed{= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

Example 2

Evaluate the integral $\displaystyle\int \frac{x^2dx}{\sqrt{9-x^2}}$

Solution

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta}{|3\cos\theta|}\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\theta - \frac{1}{2}2\sin\theta\cos\theta\right) + C$$

$$\sin\theta = \frac{x}{3} \qquad \cos\theta = \frac{\sqrt{9-x^2}}{3}$$

$$\theta = \sin^{-1}\frac{x}{3}$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C$$

Example 3

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}, \quad x > \frac{2}{5}$

Solution

$$5x = 2\sec\theta \qquad \sqrt{25x^2-4} = 2\tan\theta$$

$$dx = \frac{2}{5}\tan\theta\sec\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\frac{2}{5}\sec\theta\,\tan\theta\, d\theta}{2\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\sec\theta = \frac{5x}{2} \qquad \tan\theta = \frac{1}{2}\sqrt{25x^2-4}$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$

Example 4

Evaluate the integral $\displaystyle\int \frac{\sqrt{1+9x^2}}{x}\,dx$

Solution

$$3x = \tan\theta \qquad \sqrt{1+9x^2} = \sec\theta$$

$$dx = \frac{1}{3}\sec^2\theta\,d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{1+9x^2}}{x}\,dx &= \int \frac{\sec\theta}{\frac{1}{3}\tan\theta}\,\frac{1}{3}\sec^2\theta\,d\theta \\
&= \int \frac{\sec^3\theta}{\tan\theta}\,d\theta \\
&= \int \frac{1}{\cos^3\theta}\,\frac{\cos\theta}{\sin\theta}\,d\theta \\
&= \int \frac{1}{\cos^2\theta\,\sin\theta}\,d\theta \\
&= \int \left(\frac{\sin\theta}{\cos^2\theta} + \frac{1}{\sin\theta}\right)d\theta \\
&= -\int \frac{d(\cos\theta)}{\cos^2\theta} + \int \csc\theta\,d\theta \\
&= \frac{1}{\cos\theta} - \ln\left|\csc\theta + \cot\theta\right| \\
&= \sec\theta - \ln\left|\csc\theta + \frac{1}{\tan\theta}\right| \qquad \sec\theta = \sqrt{1+9x^2} \quad \tan\theta = 3x \\
&= \sqrt{1+9x^2} - \ln\left|\frac{\sqrt{1+9x^2}}{3x} + \frac{1}{3x}\right| + C
\end{aligned}$$

Example 5

Evaluate the integral $\displaystyle\int_4^5 \frac{x}{\sqrt{x^2-16}}\,dx$

Solution

$$x = 4\sec\theta \qquad \sqrt{x^2-16} = 4\tan\theta$$

$$dx = 4\tan\theta\sec\theta\,d\theta$$

$$\int_4^5 \frac{x}{\sqrt{x^2-16}}\,dx = \int_4^5 \frac{4\sec\theta}{4\tan\theta}\,4\tan\theta\sec\theta\,d\theta$$

$$= 4\int_4^5 \sec^2\theta\,d\theta$$

$$= 4\tan\theta\,\Big|_4^5 \qquad \sqrt{x^2-16} = 4\tan\theta$$

$$= \sqrt{x^2-16}\,\Big|_4^5$$

$$= 3$$

$$\int_4^5 \frac{x}{\sqrt{x^2-16}}\,dx = \frac{1}{2}\int_4^5 \left(x^2-16\right)^{-1/2}\,d\left(x^2-16\right)$$

$$= \sqrt{x^2-16}\,\Big|_4^5$$

$$= 3$$

Example 6

Evaluate the integral $\displaystyle\int_{-2}^{1} \frac{x}{\sqrt{5-x^2}}\,dx$

Solution

$$x = \sqrt{5}\,\sin\theta \qquad \sqrt{5-x^2} = \sqrt{5}\,\cos\theta$$

$$dx = \sqrt{5}\,\cos\theta\,d\theta$$

$$\int_{-2}^{1} \frac{x}{\sqrt{5-x^2}}\,dx = \int_{-2}^{1} \frac{\sqrt{5}\,\sin\theta}{\sqrt{5}\,\cos\theta}\,\sqrt{5}\,\cos\theta\,d\theta$$

$$= \sqrt{5}\int_{-2}^{1} \sin\theta\,d\theta$$

$$= -\sqrt{5}\,\cos\theta\,\Big|_{-2}^{1}$$

$$= -\sqrt{5}\ \frac{\sqrt{5-x^2}}{\sqrt{5}}\ \Bigg|_{-2}^{1}$$

$$= -\sqrt{5-x^2}\ \Bigg|_{-2}^{1}$$

$$= -(2-1)$$

$$= -1$$

$$\int_{-2}^{1} \frac{x}{\sqrt{5-x^2}}\ dx = -\frac{1}{2}\int_{-2}^{1} \left(5-x^2\right)^{-1/2}\ d\left(5-x^2\right)$$

$$= -\sqrt{5-x^2}\ \Bigg|_{-2}^{1}$$

$$= -(2-1)$$

$$= -1$$

Exercises *Section* 6.3 – Trigonometric Substitutions

$(\mathbf{1-119})$ Evaluate the integrals.

1. $\displaystyle\int \frac{3dx}{\sqrt{1+9x^2}}$

2. $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

3. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

5. $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

6. $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

7. $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

8. $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\,dx$

9. $\displaystyle\int \sqrt{16x^2+9}\,dx$

10. $\displaystyle\int x\sqrt{x^2+1}\,dx$

11. $\displaystyle\int \frac{\sqrt{25x^2+4}}{x^4}\,dx$

12. $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\,dx$

13. $\displaystyle\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx$

14. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4}}$

15. $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

16. $\displaystyle\int \frac{dx}{\left(1+4x^2\right)^{3/2}}$

17. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+9}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

19. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

20. $\displaystyle\int \frac{dx}{\left(x^2+4\right)^2}$

21. $\displaystyle\int \frac{dx}{\sqrt{4x^2+16}}$

22. $\displaystyle\int \frac{x^4}{x^2+1}\,dx$

23. $\displaystyle\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}}\,dx$

24. $\displaystyle\int \frac{dx}{1+\cos x}$

25. $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

26. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y>7$

27. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x>\frac{3}{5}$

28. $\displaystyle\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x>1$

29. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-100}}$

30. $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

31. $\displaystyle\int \frac{\left(1-x^2\right)^{1/2}}{x^4}\,dx$

32. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

33. $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\,dx$

34. $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

35. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}$

36. $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

37. $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

38. $\displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\,dx$

39. $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

40. $\displaystyle\int \sqrt{5x^2-1}\,dx$

41. $\displaystyle\int \frac{dx}{\sqrt{9x^2-25}}, \quad x>\frac{5}{3}$

42. $\displaystyle\int \frac{x\,dx}{\sqrt{4x^2-1}}$

43. $\displaystyle\int \frac{dx}{\sqrt{x^2-81}}$

44. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

45. $\displaystyle\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx$

46. $\displaystyle\int \frac{dx}{\left(x^2-36\right)^{3/2}}$

47. $\displaystyle\int \frac{dx}{\sqrt{36-x^2}}$

48. $\displaystyle\int \frac{dx}{2-\sqrt{3x}}$

49. $\displaystyle\int \frac{x\,dx}{1-\sqrt{x}}$

50. $\displaystyle\int \frac{dx}{\sqrt{1-2x^2}}$

51. $\displaystyle\int \frac{x^2}{\left(100-x^2\right)^{3/2}}\,dx$

52. $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

53. $\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx$

54. $\displaystyle\int x\sqrt{2x-x^2}\,dx$

55. $\displaystyle\int \frac{x}{\sqrt{4x-x^2}}\,dx$

56. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

57. $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

58. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

59. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

60. $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$

61. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

62. $\displaystyle\int \sqrt{25-4x^2}\,dx$

63. $\displaystyle\int e^x\sqrt{1-e^{2x}}\,dx$

64. $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$

65. $\displaystyle\int \frac{y\,dy}{\sqrt{16-y^2}}$

66. $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$

67. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

68. $\displaystyle\int \frac{dx}{\left(1-x^2\right)^{3/2}}$

69. $\displaystyle\int \frac{\left(1-x^2\right)^{5/2}}{x^8}\,dx$

70. $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

71. $\displaystyle\int \frac{1}{x^4+4x^2+4}\,dx$

72. $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx$

73. $\displaystyle\int \operatorname{arcsec} 2x\,dx \quad x>\frac{1}{2}$

74. $\displaystyle\int x\,\arcsin x\,dx$

75. $\displaystyle\int \frac{dx}{\left(x^2-16\right)^{3/2}}$

76. $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

77. $\displaystyle\int \ln\left(x^2+2\right)dx$

78. $\displaystyle\int \frac{\left(16-9x^2\right)^{3/2}}{x^2}\,dx$

79. $\displaystyle\int \frac{dx}{\sqrt{1-\sin 2x}}$

80. $\displaystyle\int \frac{\left(16-9x^2\right)^{3/2}}{x^6}$

81. $\displaystyle\int \frac{dx}{\left(9-x^2\right)^2}$

82. $\displaystyle\int \frac{\sqrt{16-x^2}}{x}\,dx$

83. $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

84. $\displaystyle\int \frac{x^2}{\left(4-x^2\right)^{5/2}}\,dx$

85. $\displaystyle\int \sqrt{x^2-8}\,dx$

86. $\displaystyle\int \sqrt{x^2+4}\,dx$

87. $\displaystyle\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

88. $\displaystyle\int x^3\sqrt{49-x^2}\,dx$

89. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+13}}$

90. $\displaystyle\int \frac{dx}{\left(4x-x^2\right)^{3/2}}$

91. $\displaystyle\int \frac{dx}{\left(4+x^2\right)^2}$

92. $\displaystyle\int \frac{x^2}{\left(a^2+x^2\right)^2}\,dx$

93. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}}$

94. $\displaystyle\int \frac{x^2}{\left(a^2-x^2\right)^2}\,dx$

95. $\displaystyle\int \frac{x^2}{\left(a^2-x^2\right)^{3/2}}\,dx$

96. $\displaystyle\int \frac{dx}{\left(a^2+x^2\right)^2}$

97. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$

98. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx$

99. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx$

100. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$

101. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx$

102. $\displaystyle\int \frac{x^3\,dx}{\left(x^2+a^2\right)^{3/2}}$

103. $\displaystyle\int \frac{dx}{x^2\left(a^2-x^2\right)}$

104. $\displaystyle\int \frac{x}{\sqrt{ax-x^2}}\,dx, \quad a>0$

105. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$

106. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx$

107. $\displaystyle\int \sqrt{a^2-x^2}\,dx$

108. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

109. $\displaystyle\int \frac{x\,dx}{\left(x^2-a^2\right)^2}$

110. $\displaystyle\int \frac{x^3}{\sqrt{x^2-a^2}}\,dx$

111. $\displaystyle\int \sqrt{x^2-a^2}\,dx$

112. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$

113. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx$

114. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx$

115. $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$

116. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$

117. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

118. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}$

119. $\displaystyle\int x\left(\sqrt{x^2-a^2}\right)^n dx$

(120 – 140) Evaluate the definite integrals.

120. $\displaystyle\int_0^2 \sqrt{1+4x^2}\,dx$

121. $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx$

122. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x\,dx}{\left(1+e^{2x}\right)^{3/2}}$

123. $\displaystyle\int_1^e \frac{e^x\,dx}{\left(1+e^{2x}\right)^{3/2}}$

124. $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

125. $\displaystyle\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

126. $\displaystyle\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2}\,dx$

127. $\displaystyle\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}\,(1+4x)}$

128. $\displaystyle\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$

129. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\,dx$

130. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

131. $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$

132. $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

133. $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}}\,dx$

134. $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$

135. $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

136. $\displaystyle\int_0^{3} \frac{dx}{\sqrt{9-x^2}}$

137. $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

138. $\displaystyle\int_{-1}^{1} \frac{x}{x^2+1}\,dx$

139. $\displaystyle\int_3^5 \sqrt{x^2-9}\,dx$

140. $\displaystyle\int_2^5 \frac{dx}{\sqrt{x^2-4}}$

141. Consider the region bounded by the graph $y=\sqrt{x\tan^{-1}x}$ and $y=0$ for $0\le x\le 1$. Find the volume of the solid formed by revolving this region about the x-axis.

142. Use two approaches to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$$

a) Find the area using geometry (no calculus).
b) Find the area using calculus

143. A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.

144. The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

145. The surface of a machine part is the region between the graphs of

$$y = |x| \quad and \quad x^2 + (y-k)^2 = 25$$

a) Find k when the circle is tangent to the graph of $y = |x|$

b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

146. Consider the function $f(x) = \left(9 + x^2\right)^{-1/2}$ and the region ***R*** on the interval [0, 4].

a) Find the area of R.

b) Find the volume of the solid generated when R is revolved about the x-*axis*.

c) Find the volume of the solid generated when R is revolved about the y-*axis*.

147. A total of ***Q*** is distributed uniformly on a line segment of length $2L$ along the y-*axis*. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$

b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

148. A long, straight wire of length $2L$ on the *y-axis* carries a current I. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}\,dy$$

Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure

a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is $B(a) = \dfrac{\mu_0 IL}{2\pi a\sqrt{a^2 + L^2}}$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L \to \infty)$?

149. The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.

A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$\textit{descent time} = \int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\,dt$$

Where g is the acceleration due to gravity, $t=0$ corresponds to the top of the wire, and $t=\pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b=\pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

150. Find the area of the region bounded by the curve $f(x)=\left(16+x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$.

151. Find the length of the curve $y=ax^2$ from $x=0$ to $x=10$, where $a>0$ is a real number.

152. Find the arc length of the graph of $f(x)=\frac{1}{2}x^2$ from $x=0$ to $x=1$.

153. A solid obtained by the region within the parabola $x=9-y^2$ and between $y=x-7$ and the $y-axis$, by revolving about $y-axis$.

a) Find the volume of the solid.

b) Find the surface.

154. A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x-*axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y=-\frac{1}{2}kx^2+y_{max} \quad where\ \ k=\frac{g}{(V\cos\theta)^2}$$

$$and \qquad y_{max}=\frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $\left(0,\ y_{max}\right)$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\int_0^a \sqrt{1+k^2x^2}\ dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta+\tan\theta)=1$.

155. Let $F(x)=\int_0^x \sqrt{a^2-t^2}\,dt$. The figure shows that

$F(x) = \textit{area of sector OAB} + \textit{area of triangle OBC}$

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

156. A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.

157. The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.

b) Graph the function in part (*a*).

c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, *and* $\frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

158. The field strength *H* of a magnet of length 2*L* on a particle *r* unit(s) from the center of the magnet is

$$H = \frac{2mL}{\left(r^2 + L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to *R* ***units*** from the center by evaluating the integral:

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr$$

Section 6.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called ***partial fractions***.

6.4–1 Decompose $\frac{P}{Q}$, where Q has Only Non-repeated Linear Factor

Under the assumption that Q has only non-repeated linear factors, the polynomial Q has the form

$$Q(x) = (x - a_1)(x - a_2) \quad \cdots\cdots \quad (x - a_n)$$

Where no 2 of the number $a_1, a_2, \ldots, a_n$ are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P}{Q} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \quad \cdots\cdots + \frac{A_n}{x - a_n}$$

Where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example 1

Evaluate the integral $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x - 3 = (A+B)x - 3A + B$$

$$\begin{array}{ll} x & A + B = 5 \\ x^0 & \underline{-3A + B = -3} \\ & 4A = 8 \\ & \underline{A = 2, \quad B = 3} \end{array}$$

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx$$

$$\underline{= 2\ln|x+1| + 3\ln|x-3| + C}$$

Example 2

Evaluate the integral $\displaystyle\int \frac{2x}{(x+4)(x-5)}\,dx$

Solution

$$\frac{2x}{(x+4)(x-5)} = \frac{A}{x+4} + \frac{B}{x-5}$$

$$2x = A(x-5) + B(x+4)$$

$x \qquad A + B = 2$

$x^0 \qquad -5A + 4B = 0$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -5 & 4 \end{vmatrix} = 9 \quad \Delta_A = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8 \quad \Delta_B = \begin{vmatrix} 1 & 2 \\ -5 & 3 \end{vmatrix} = 10$$

$$A = \frac{8}{9}, \quad B = \frac{10}{9}$$

$$\int \frac{2x}{(x+4)(x-5)}\,dx = \frac{8}{9}\int \frac{dx}{x+4} + \frac{10}{9}\int \frac{dx}{x-5}$$

$$= \frac{8}{9}\ln|x+4| + \frac{10}{9}\ln|x-5| + C$$

Example 3

Use partial fractions to evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$x^2 \qquad A + B + C = 1$

$x \qquad 4A + 2B = 4$

$x^0 \qquad 3A - 3B - C = 1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -16 \qquad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = -12$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -8 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 4$$

$$A = \frac{-12}{-16} = \frac{3}{4}$$

$$B = \frac{-8}{-16} = \frac{1}{2}$$

$$C = \frac{4}{-16} = -\frac{1}{4}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \left(\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right) dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K$$

Example 4

Evaluate the integral $\displaystyle\int \frac{3x+2}{x^3-x^2-2x}\,dx$

Solution

$$x^3 - x^2 - 2x = x\left(x^2 - x - 2\right)$$

$$\frac{3x+2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\begin{array}{lll} x^2 & A+B+C=0 & (1) \\ x & -A-2B+C=3 & (2) \\ x^0 & -2A=2 & A=-1 \end{array}$$

$$\begin{cases} (1) & B+C=1 \\ (2) & -2B+C=2 \end{cases}$$

$$B = -\frac{1}{3}, \qquad C = \frac{4}{3}$$

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int \left(-\frac{1}{x} - \frac{1}{3}\frac{1}{x+1} + \frac{4}{3}\frac{1}{x-2}\right) dx$$

$$= -\ln|x| - \frac{1}{3}\ln|x+1| + \frac{4}{3}\ln|x-2| + K$$

6.4–2 Method of Partial Fractions has Repeated Linear Factors

Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divide $g(x)$. Then,

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

Example 5

Use partial fractions to evaluate $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

Solution

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$6x+7=A(x+2)+B$$

$$\begin{cases} \underline{A=6} \\ 2A+B=7 \end{cases}$$

$$B=7-12\ \underline{=-5}$$

$$\int \frac{6x+7}{(x+2)^2}\,dx=\int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx \qquad d(x+2)=dx$$

$$=\int \frac{6}{x+2}\,dx-5\int (x+2)^{-2}\,d(x+2)$$

$$=6\ln|x+2|+5(x+2)^{-1}+C$$

$$\underline{=6\ln|x+2|+\frac{5}{x+2}+C}$$

Example 6

Use partial fractions to evaluate $\displaystyle\int \frac{x^2-2x+3}{(x+3)(x-2)^2}\,dx$

Solution

$$\frac{x^2-2x+3}{(x+3)(x-2)^2}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$$x^2-2x+3=A\left(x^2-4x+4\right)+B(x+3)(x-2)+C(x+3)$$

$$\begin{array}{ll} x^2 & A+B=1 \\ x & -4A+B+C=-2 \\ x^0 & 4A-6B+3C=3 \end{array}$$

$$\Delta=\begin{vmatrix}1&1&0\\-4&1&1\\4&-6&3\end{vmatrix}=25 \qquad \Delta_A=\begin{vmatrix}1&1&0\\-2&1&1\\3&-6&3\end{vmatrix}=18 \qquad \Delta_B=\begin{vmatrix}1&1&0\\-4&-2&1\\4&3&3\end{vmatrix}=7$$

$$A=\frac{18}{25} \qquad B=\frac{7}{25} \qquad C=\frac{3}{5}$$

$$\int\frac{x^2-2x+3}{(x+3)(x-2)^2}\,dx=\frac{18}{25}\int\frac{dx}{x+3}+\frac{7}{25}\int\frac{dx}{x-2}+\frac{3}{5}\int\frac{dx}{(x-2)^2}$$

$$=\frac{18}{25}\ln|x+3|+\frac{7}{25}\ln|x-2|-\frac{3}{5}\frac{1}{x-2}+K$$

6.4–3 Non-repeated Irreducible Quadratic Factor

Let x^2+px+q be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $\left(x^2+px+q\right)^n$ is the highest power. Then

$$\frac{B_1x+C_1}{\left(x^2+px+q\right)}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

Example 7

Use partial fractions to evaluate $\displaystyle\int\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}\,dx$

Solution

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$$

$$-2x+4=(Ax+B)(x-1)^2+C(x-1)\left(x^2+1\right)+D\left(x^2+1\right)$$

$$=(Ax+B)\left(x^2-2x+1\right)+C\left(x^3-x^2+x-1\right)+Dx^2+D$$

$$=(A+C)x^3+(-2A+B-C+D)x^2+(A-2B+C)x+B-C+D$$

x^3 $\quad A+C=0 \quad \rightarrow \underline{C=-A}$

x^2 $\quad -2A+B-C+D=0 \quad (1)$

x $\quad A-2B+C=-2 \quad \rightarrow -2B=-2 \quad \underline{B=1}$

x^0 $\quad B-C+D=4 \quad (2)$

$$\begin{array}{l} -A+D=-1 \\ \underline{A+D=3} \\ 2D=2 \end{array}$$

$\underline{D=1, \quad A=2}$

$\underline{C=-2}$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \left(\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$= \int \frac{1}{x^2+1}\,d\left(x^2+1\right)\,dx + \int \left(\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$\underline{= \ln\left(x^2+1\right)+\tan^{-1}x-2\ln|x-1|-\frac{1}{x-1}+K}$$

6.4–3 Repeated Irreducible Quadratic Factor

If ***denominator*** contains a repeated irreducible quadratic factor of the form $\left(ax^2+bx+c\right)^n$, $n \geq 2$, n an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

Where the numbers $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are to be determined.

Example 8

Use partial fractions to evaluate $\displaystyle\int \frac{dx}{x\left(x^2+1\right)^2}$

Solution

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2}$$

$$1 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} x^4 & A+B=0 & \rightarrow \underline{B=-1} \\ x^3 & \underline{C=0} \\ x^2 & 2A+B+D=0 & \rightarrow \underline{D=-1} \\ x & C+E=0 & \rightarrow \underline{E=0} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int \frac{dx}{x\left(x^2+1\right)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{\left(x^2+1\right)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right) - \frac{1}{2}\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$\boxed{= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)} + K}$$

6.4–4 Dividing before Applying Partial Fractions

Example 9

Use partial fractions to evaluate $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

Solution

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & \underline{2x^3-4x^2-6x} \\ & \qquad\qquad 5x-3 \end{array}$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$\begin{array}{ll} x & A+B=5 \\ x^0 & \underline{-3A+B=-3} \\ & 4A=8 \\ & \underline{A=2,\ \ B=3} \end{array}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$\underline{= x^2 + 2\ln|x+1| + 3\ln|x-3| + C}$$

Example 10

Evaluate the integral $\displaystyle\int \frac{\cos x}{\sin^2 x - \sin x}\,dx$

Solution

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx = \int \frac{1}{\sin^2 x - \sin x}\,d(\sin x)$$

$$\frac{1}{\sin^2 x - \sin x} = \frac{A}{\sin x} + \frac{B}{\sin x - 1}$$

$$A\sin x - A + B\sin x = 1$$

$$\begin{cases} \sin x & A + B = 0 \quad \rightarrow \quad \underline{B = 1} \\ & -A = 1 \qquad \underline{A = -1} \end{cases}$$

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = \int \left(\frac{-1}{\sin x} + \frac{1}{\sin x - 1} \right) d(\sin x)$$

$$= -\ln|\sin x| + \ln|\sin x - 1| + C$$

$$\underline{= \ln\left| \frac{\sin x - 1}{\sin x} \right| + C}$$

Example 11

Evaluate the definite integral $\displaystyle\int_2^4 \frac{4x-1}{x^2+x-2}\, dx$

Solution

$$\frac{4x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\begin{array}{ll} x & A + B = 4 \\ x^0 & \underline{2A - B = -1} \\ & 3A = 3 \\ & \underline{A = 1,\ B = 3} \end{array}$$

$$\int_2^4 \frac{4x-1}{x^2+x-2}\, dx = \int_2^4 \frac{dx}{x-1} + \int_2^4 \frac{3}{x+2}\, dx$$

$$= \ln(x-1) + 3\ln(x+2) \Big|_2^4$$

$$= \ln 3 + 3\ln 6 - 3\ln 4$$

$$= \ln 3 + \ln\left(\frac{6}{4}\right)^3$$

$$\underline{= \ln\frac{81}{8}}$$

Example 12

Evaluate the definite integral $\displaystyle\int_0^1 \frac{x^3}{\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$(Ax+B)\left(x^2+1\right) + Cx + D = x^3$$

$$\begin{array}{lll} x^3 & & \underline{A=1} \\ x^2 & & \underline{B=0} \\ x^1 & A+C=0 & \underline{C=-1} \\ x^0 & B+D=0 & \underline{D=0} \end{array}$$

$$\int_0^1 \frac{x^3}{\left(x^2+1\right)^2}\,dx = \int_0^1 \frac{x}{x^2+1}\,dx - \int_0^1 \frac{x}{\left(x^2+1\right)^2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,d\left(x^2+1\right) - \frac{1}{2}\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$= \frac{1}{2}\left(\ln\left(x^2+1\right) + \frac{1}{x^2+1}\right|_0^1$$

$$= \frac{1}{2}\left(\ln 2 + \frac{1}{2} - 0 - 1\right)$$

$$\underline{= \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right)}$$

Example 13

Evaluate the definite integral $\displaystyle\int_0^1 \frac{e^x}{\left(e^{2x}-4\right)^2}\,dx$

Solution

Let $u = e^x$

$du = e^x\,dx$

$$\int_0^1 \frac{e^x}{\left(e^{2x}-4\right)^2}\,dx = \int_0^1 \frac{1}{\left(u^2-4\right)^2}\,du$$

$$\frac{1}{\left(u^2-4\right)^2} = \frac{1}{(u+2)^2(u-2)^2}$$

$$= \frac{A}{u+2} + \frac{B}{(u+2)^2} + \frac{C}{u-2} + \frac{D}{(u-2)^2}$$

$$1 = A(u+2)\left(u^2-4u+4\right) + B\left(u^2-4u+4\right)$$
$$+ C(u-2)\left(u^2+4u+4\right) + D\left(u^2+4u+4\right)$$

u^3 $\quad A+C=0 \quad \rightarrow \quad A=-C$

u^2 $\quad -2A+B+2C+D=0$

u^1 $\quad -4A-4B-4C+4D=1$

u^0 $\quad 8A+4B-8C+D=0$

$$-4A+B+D=0$$
$$-4B+4D=1$$
$$16A+4B+D=0$$

$$\Delta = \begin{vmatrix} -4 & 1 & 1 \\ 0 & -4 & 4 \\ 16 & 4 & 1 \end{vmatrix} = 208 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -4 & 4 \\ 0 & 4 & 1 \end{vmatrix} = 3 \quad \Delta_B = \begin{vmatrix} -4 & 0 & 1 \\ 0 & 1 & 4 \\ 16 & 0 & 1 \end{vmatrix} = -20 \quad \Delta_D = \begin{vmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 16 & 4 & 0 \end{vmatrix} = -32$$

$$A = \frac{3}{208}, \quad B = -\frac{5}{52}, \quad C = -\frac{3}{208}, \quad D = -\frac{2}{13}$$

$$\int_0^1 \frac{e^x}{\left(e^{2x}-4\right)^2}\,dx = \frac{3}{208}\int \frac{du}{u+2} - \frac{5}{52}\int \frac{du}{(u+2)^2} - \frac{3}{208}\int \frac{du}{u-2} - \frac{2}{13}\int \frac{du}{(u-2)^2}$$

$$= \frac{3}{208}\ln(u+2) + \frac{5}{52}\frac{1}{u+2} - \frac{3}{208}\ln|u-2| + \frac{2}{13}\frac{1}{u-2}\Bigg|_0^1$$

$$= \frac{3}{208}\ln\left(e^x+2\right) + \frac{5}{52}\frac{1}{e^x+2} - \frac{3}{208}\ln\left|e^x-2\right| + \frac{2}{13}\frac{1}{e^x-2}\Bigg|_0^1$$

$$= \frac{3}{208}\ln(e+2) + \frac{5}{52}\frac{1}{e+2} - \frac{3}{208}\ln|e-2| + \frac{2}{13}\frac{1}{e-2} - \frac{3\ln 3}{208} + \frac{5}{156} - \frac{2}{13}$$

$$= \frac{3}{208}\ln\left|\frac{e+2}{e-2}\right| + \frac{5}{52(e+2)} + \frac{2}{13(e-2)} - \frac{3\ln 3}{208} - \frac{19}{156}$$

Exercises *Section* 6.4 – Partial Fractions

(1 – 124) Express the integrand as a sum of partial fractions and evaluate the integrals.

1. $\displaystyle\int \frac{dx}{x^2+2x}$

2. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

3. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

4. $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}\,dx$

5. $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}\,dx$

6. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$

7. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}\,d\theta$

8. $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

9. $\displaystyle\int \frac{16x^3}{4x^2-4x+1}\,dx$

10. $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}\,dx$

11. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

12. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx$

13. $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

14. $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx$

15. $\displaystyle\int \frac{4x^2+2x+4}{x+1}\,dx$

16. $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$

17. $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}\,dx$

18. $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx$

19. $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}\,dx$

20. $\displaystyle\int \frac{dx}{1+\sin x}$

21. $\displaystyle\int \frac{dx}{2+\cos x}$

22. $\displaystyle\int \frac{dx}{1-\cos x}$

23. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

24. $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

25. $\displaystyle\int \frac{1}{x^2-5x+5}\,dx$

26. $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

27. $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

28. $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

29. $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}\,dx$

30. $\displaystyle\int \frac{5\cos x}{\sin^2 x+3\sin x-4}\,dx$

31. $\displaystyle\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\,dx$

32. $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\,dx$

33. $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

34. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

35. $\displaystyle\int \frac{1}{x^2-9}\,dx$

36. $\displaystyle\int \frac{2}{9x^2-1}\,dx$

37. $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

38. $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

39. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

40. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

41. $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

42. $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

43. $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

44. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{\tan x\left(\tan x+1\right)}\,dx$

46. $\displaystyle\int \frac{x\ dx}{x^2+4x+3}$

47. $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

48. $\displaystyle\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

49. $\displaystyle\int \frac{8x+5}{2x^2+3x+1}\,dx$

50. $\displaystyle\int \frac{3x^3+4x^2+6x}{(x+1)^2\left(x^2+4\right)}\,dx$

51. $\displaystyle\int \frac{x^2-4}{x^2+4}\,dx$

52. $\displaystyle\int \frac{dx}{x^2-2x-15}$

53. $\displaystyle\int \frac{3x^2+x-3}{x^2-1}\,dx$

54. $\displaystyle\int \frac{2x^2-4x}{x^2-4}\,dx$

55. $\displaystyle\int \frac{dx}{x^3-2x^2}$

56. $\displaystyle\int \frac{dx}{x^2-x-2}$

57. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

58. $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

59. $\displaystyle\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx$

60. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx$

61. $\displaystyle\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx$

62. $\displaystyle\int \frac{81}{x^3-9x^2}\,dx$

63. $\displaystyle\int \frac{10x}{x^2-2x-24}\,dx$

64. $\displaystyle\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx$

65. $\displaystyle\int \frac{1+x^2}{(x+1)^3}\,dx$

66. $\displaystyle\int \frac{6}{x^2-1}\,dx$

67. $\displaystyle\int \frac{21x^2}{x^3-x^2-12x}\,dx$

68. $\displaystyle\int \frac{x+1}{x^3+3x^2-18x}\,dx$

69. $\displaystyle\int \frac{x^2+12x-4}{x^3-4x}\,dx$

70. $\displaystyle\int \frac{6x^2}{x^4-5x^2+4}\,dx$

71. $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

72. $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

73. $\displaystyle\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx$

74. $\displaystyle\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx$

75. $\displaystyle\int \frac{2}{x\left(x^2+1\right)^2}\,dx$

76. $\displaystyle\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx$

77. $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

78. $\displaystyle\int \frac{3x+11}{(x+2)(x+3)}\,dx$

79. $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

80. $\displaystyle\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

81. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

82. $\displaystyle\int \frac{3x+6}{x^3+2x^2-3x}\,dx$

83. $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,dx$

84. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx$

85. $\displaystyle\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\,dx$

86. $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

87. $\displaystyle\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx$

88. $\displaystyle\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx$

89. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx$

90. $\displaystyle\int \frac{x^5-x^4+x^3-4x^2+8x-4}{\left(x^2+2\right)^3}\,dx$

91. $\displaystyle\int \frac{2x^2+7x+4}{x^3+2x^2+2x}\,dx$

92. $\displaystyle\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$

93. $\displaystyle\int \frac{2x^3+3}{\left(x^2+1\right)^2}\,dx$

94. $\displaystyle\int \frac{x-1}{x\left(x^2+1\right)\left(x^2+2\right)}\,dx$

95. $\displaystyle\int \frac{x^3+1}{(x-1)^3\,(x+2)}\,dx$

96. $\displaystyle\int \frac{\left(2+\tan^2\theta\right)\,\sec^2\theta}{1+\tan^3\theta}\,d\theta$

97. $\displaystyle\int \frac{\sin\theta}{\cos\theta\left(1+\cos^2\theta\right)}\,d\theta$

98. $\displaystyle\int \frac{dx}{e^{2x}-3e^x}$

99. $\displaystyle\int \frac{x^6+7x^5+14x^4+32x^3+23x^2+25x-3}{\left(x^2+1\right)^2\left(x^2+x+2\right)^2}\,dx$

100. $\displaystyle\int \frac{x^3+x^2-5x+15}{\left(x^2+5\right)\left(x^2+2x+3\right)}\,dx$

101. $\displaystyle\int \frac{x^4+8x^3-x^2+2x+1}{x(x+1)\left(x^3+1\right)}\,dx$

102. $\displaystyle\int \frac{2x^3+x^2+4}{\left(x^2+4\right)^2}\,dx$

103. $\displaystyle\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx$

104. $\displaystyle\int \frac{x^3+x^2+x+3}{\left(x^2+1\right)\left(x^2+3\right)}\,dx$

105. $\displaystyle\int \frac{dx}{\sqrt{\sqrt{x}+1}}$

106. $\displaystyle\int \frac{x^2+3}{(x-1)^3(x+1)}\,dx$

107. $\displaystyle\int \frac{x^2+3}{(x+1)^3(x-1)}\,dx$

108. $\displaystyle\int \frac{x-1}{x^3+2x^2-x-2}\,dx$

109. $\displaystyle\int \frac{dx}{b+c\sin ax}$

110. $\displaystyle\int \frac{dx}{b+c\cos ax}$

111. $\displaystyle\int \frac{dx}{1+\sin ax}$

112. $\displaystyle\int \frac{dx}{1-\sin ax}$

113. $\displaystyle\int \frac{dx}{1+\cos ax}$

114. $\displaystyle\int \frac{dx}{1-\cos ax}$

115. $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

116. $\displaystyle\int \frac{dx}{ax^2+bx}$

117. $\displaystyle\int \frac{x}{(ax+b)^2}\,dx$

118. $\displaystyle\int \frac{x^2}{a^2-x^2}\,dx$

119. $\displaystyle\int \frac{dx}{x\left(a^2-x^2\right)}$

120. $\displaystyle\int \frac{dx}{\left(a^2-x^2\right)^2}$

121. $\displaystyle\int \frac{dx}{x^2\left(a^2-x^2\right)}$

122. $\displaystyle\int \frac{x^2\,dx}{x^2-a^2}$

123. $\displaystyle\int \frac{dx}{\left(x^2-a^2\right)^2}$

124. $\displaystyle\int \frac{x\,dx}{\left(x^2-a^2\right)^2}$

$(\mathbf{125-147})$ Evaluate the definite integral.

125. $\displaystyle\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$

126. $\displaystyle\int_{0}^{5} \frac{2}{x^2-4x-32}\,dx$

127. $\displaystyle\int_{0}^{1} \frac{dx}{(x+1)\left(x^2+1\right)}$

128. $\displaystyle\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx$

129. $\displaystyle\int_{0}^{2} \frac{3}{4x^2+5x+1}\,dx$

130. $\displaystyle\int_{1}^{5} \frac{x-1}{x^2(x+1)}\,dx$

131. $\displaystyle\int_{1}^{2} \frac{x+1}{x\left(x^2+1\right)}\,dx$

132. $\displaystyle\int_{0}^{1} \frac{x^2-x}{x^2+x+1}\,dx$

133. $\displaystyle\int_{4}^{8} \frac{y\,dy}{y^2-2y-3}$

134. $\displaystyle\int_{1}^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

135. $\displaystyle\int_{0}^{\pi/2} \frac{dx}{\sin x+\cos x}$

136. $\displaystyle\int_{0}^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

137. $\displaystyle\int_{0}^{\pi/2} \frac{\cos\theta}{2-\cos\theta}\,d\theta$

138. $\displaystyle\int_{0}^{\pi/4} \frac{4}{1+\tan\theta}\,d\theta$

139. $\displaystyle\int_{2}^{3} \frac{d\theta}{1-\theta^2}$

140. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\csc x}{3+4\tan x}\,dx$

141. $\displaystyle\int_{0}^{4} \frac{1}{x^2-5x+6}\,dx$

142. $\displaystyle\int_{1}^{2} \frac{dx}{e^{2x}-1}$

143. $\displaystyle\int_{1}^{2} \frac{dx}{e^{2x}+1}$

144. $\displaystyle\int_{0}^{1} \sqrt{1+e^x}\,dx$

145. $\displaystyle\int_{0}^{3} \frac{dx}{x^3+27}$

146. $\displaystyle\int_{0}^{3} \frac{dx}{x^3-27}$

147. $\displaystyle\int_{0}^{2} \frac{dx}{(x-1)\left(x^2+4\right)}$

148. Find the volume of the solid generated by the revolving the shaded region about x-axis.

$(\mathbf{149-150})$ Find the area of the region bounded by the graphs of:

149. $y=\dfrac{12}{x^2+5x+6}$, $\quad y=0,\quad x=0,\quad and\ \ x=1$

150. $y=\dfrac{7}{16-x^2}\quad and\ \ y=1$

151. The region in the first quadrant that is enclosed by the x-axis, the curve $y=\dfrac{5}{x\sqrt{5-x}}$, and the lines $x=1$ and $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

152. Find the length of the graph of the function $y=\ln\left(1-x^2\right),\quad 0\le x\le \frac{1}{2}$.

153. Consider the region bounded by the graphs $y=\dfrac{2x}{x^2+1}$, $\quad y=0,\quad x=0,\quad and\ \ x=3$.

a) Find the volume of the solid generated by revolving the region about the *x-axis*.
b) Find the centroid of the region.

154. Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the *x-axis*.

155. A single infected individual enters a community of *n* susceptible individuals. Let *x* be the number of newly infected individuals at time *t*. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain:

$$\int \frac{1}{(x+1)(n-x)}\, dx = \int k\ dt$$

Solve for *x* as a function of *t*.

156. Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\, dx$ in ***two*** different ways.

Section 6.5 – Numerical Integration

6.5–1 Absolute and Relative Error

Definition

Suppose c is a computed numerical solution to a problem having an exact solution x.
There are two common measures of the error in c as an approximation to x:

$$\textbf{\textit{absolute error}} = |c - x| \qquad \& \qquad \textbf{\textit{relative error}} = \frac{|c - x|}{|x|} \quad (if\ \ x \neq 0)$$

Example 1

The ancient Greeks used $\frac{22}{7}$ to approximate the value of π. Determine the absolute and relative error in this approximation to π.

Solution

$$absolute\ error = \left|\frac{22}{7} - \pi\right|$$
$$\approx 0.00126$$

$$relative\ error = \frac{\left|\frac{22}{7} - \pi\right|}{\pi}$$
$$\approx .000402$$
$$\approx .04\%$$

6.5–2 Midpoint Rule

Definition

Suppose f is defined and integrable on $[a,\ b]$. The ***midpoint Rule Approximation*** to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a,\ b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$
$$= \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x$$

Where $\Delta x = \dfrac{b-a}{n}$,

$$x_0 = a, \qquad x_k = a + k\Delta x$$

$$m_k = \frac{x_{k-1} + x_k}{2} \text{ is the midpoint of } \left[x_{k-1},\ x_k\right], \text{ for } k = 1,\ 2,\ \ldots,\ n.$$

Example 2

Approximate $\displaystyle\int_2^4 x^2\,dx$ using the Midpoint Rule with $n = 4$ subinterval.

Solution

With $a = 2,\ b = 4$

$$\Delta x = \frac{4-2}{4} = 0.5$$

The grid points are:

$$x_0 = 2$$

$$x_1 = 2 + 0.5 = 2.5$$

$$x_2 = 2 + 2(.5) = 3$$

$$x_3 = 2 + 3(0.5) = 3.5$$

$$x_4 = 2 + 4(0.5) = 4$$

$$m_1 = \frac{2.5+2}{2} = 2.25$$

$$m_2 = \frac{3+2.5}{2} = 2.75$$

$$m_3 = \frac{2+3.5}{2} = 3.25$$

$$m_4 = \frac{3.5+4}{2} = 3.75$$

$$M(4) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x$$

$$= \left(2.25^2 + 2.75^2 + 3.25^2 + 3.75^2\right)(0.5)$$

$$= 18.625$$

Midpoint Rule $\left(n = 4 \quad \& \quad \Delta x = \frac{1}{2}\right)$

$$Exact = \int_2^4 x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_2^4$$

$$= \frac{56}{3}$$

$$absolute\ error = \left|18.625 - \frac{56}{3}\right|$$

$$\approx 0.0417$$

$$relative\ error = \frac{\left|18.625 - \frac{56}{3}\right|}{\frac{56}{3}}$$

$$\approx .00223 \ = .223\%$$

6.5–3 Trapezoid Approximations

The ***Trapezoid Rule*** for the value of a definite integral is based on approximating the region between a curve and the *x*-axis with trapezoids instead of rectangles.

The length of each subinterval is $\Delta x = \dfrac{b-a}{n}$ is called the ***step size*** or ***mesh size***.

The area of a trapezoid: $\Delta x \cdot \left(\dfrac{y_{i-1} + y_i}{2} \right)$

The area is the approximation by adding the areas of all trapezoids:

$$T = \tfrac{1}{2}\left(y_0 + y_1\right)\Delta x + \tfrac{1}{2}\left(y_1 + y_2\right)\Delta x + \cdots + \tfrac{1}{2}\left(y_{n-2} + y_{n-1}\right)\Delta x + \tfrac{1}{2}\left(y_{n-1} + y_n\right)\Delta x$$

$$= \tfrac{1}{2}\Delta x\left(y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n\right)$$

$$= \tfrac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n\right)$$

6.5–4 The Trapezoid Rule

If f is continuous on [*a*, *b*] and if a regular partition of [*a*, *b*] is determined by the numbers $a = x_0,\ x_1,\ \ldots,\ x_n = b$, then

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + \ldots + 2f\left(x_{n-1}\right) + f\left(x_n\right)\right]$$

$$T(n) = \left(\frac{1}{2} f\left(x_0\right) + \sum_{k=1}^{n-1} f\left(x_k\right) + \frac{1}{2} f\left(x_n\right) \right) \Delta x$$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ \ldots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

Where $\Delta x = \dfrac{b-a}{n}$ and $x_0 = a, \quad x_k = a + k\Delta x$

6.5–5 Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $|f''(x)| \le M$ for all x in $[a, b]$, then the error involved in using the Trapezoidal Rule is not greater than $\dfrac{M(b-a)^3}{12n^2}$

Example 3

Use the Trapezoid Rule with $n = 4$ to estimate $\displaystyle\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{2-1}{4}$$

$$= \frac{1}{4}$$

$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = 1 + 2\left(\frac{1}{4}\right) = \frac{6}{4}$$

$$x_3 = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4}$$

$$x_4 = 2$$

$$T = \frac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

$$= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32}$$

$$\approx 2.34375$$

$$\int_1^2 x^2\, dx = \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}\left(2^3 - 1^3\right)$$

$$= \frac{7}{3}$$

$$\approx 2.3333$$

The difference: $2.34375 - 2.3333 \approx 0.01042$

The percentage error: $\dfrac{2.34375 - 2.33333}{2.33333} \approx 0.004466$

$$0.446\%$$

6.5–6 *Simpson's Rule*: Approximations Using Parabolas

We partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = \dfrac{b-a}{n}$ $\quad n : \textit{even number}$

The parabola has an equation of the form: $y = Ax^2 + Bx + C$

So, the area under it from $x = -h$ to $x = h$ is

$$A_p = \int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx$$

$$= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big|_{-h}^{h}$$

$$= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h)\right)$$

$$= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

$$= \frac{h}{3}\left(2Ah^2 + 6C\right)$$

Since the curve passes through the three points $\left(-h,\ y_0\right)$, $\left(0,\ y_1\right)$, *and* $\left(h,\ y_2\right)$

$$\begin{cases} y_0 = Ah^2 - Bh + C \\ y_1 = C \\ y_2 = Ah^2 + Bh + C \end{cases}$$

$$C = y_1, \qquad Ah^2 - Bh = y_0 - y_1$$

$$\underline{Ah^2 + Bh = y_2 - y_1}$$

$$2Ah^2 = y_0 - 2y_1 + y_2$$

$$A_p = \frac{h}{3}\left(2Ah^2 + 6C\right)$$

$$= \frac{h}{3}\left(y_0 - 2y_1 + y_2 + 6y_1\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Computing the areas under all the parabolas and adding the results gives the approximation:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

6.5–7 Simpson's Rule Formula

To approximate $\int_a^b f(x)dx$,

Use $S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$

$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$

Where $\Delta x = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

6.5–8 Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $\left|f^{(4)}(x)\right| \le M$ for all x in $[a, b]$, then the error involved in using the Simpson's Rule is not greater than $\frac{M(b-a)^5}{180n^4}$.

Example 4

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution

$\Delta x = \frac{2-0}{4}$ $\qquad$ $\Delta x = \frac{b-a}{n}$

$= \frac{1}{2}$

$x_0 = 0$

$x_2 = 0 + 2\frac{1}{2} = 1$

$x_3 = 0 + 3\frac{1}{2} = \frac{3}{2}$

$x_4 = 2$

$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$

$$= \frac{1}{3}\frac{1}{2}\left(5(0)^4 + 4(5)\left(\frac{1}{2}\right)^4 + 2(5)(1)^4 + 4(5)\left(\frac{3}{2}\right)^4 + 5(2)^4\right)$$

$$= \frac{1}{6}\left(0 + \frac{5}{4} + 10 + \frac{405}{4} + 80\right)$$

$$= \frac{1}{6}\left(\frac{385}{2}\right)$$

$$= \frac{385}{12}$$

≈ 32.08333 *The exact value is* 32.

Example 5

The table lists rates of change $s'(t)$ in global sea level $s(t)$ in various years from 1995 $(t=0)$ to 2011 $(t=16)$, with rates of change reported in *mm/yr*.

Years	1995	1998	2000	2002	2003	2007	2009	2011
t	0	3	5	7	8	12	14	16
$s'(t)$ (*mm/yr*)	0.51	5.19	4.39	2.21	5.24	0.63	4.19	2.38

a) Assuming $s'(t)$ is continuous on $[0,\ 16]$, explain how a definite integral can be used to find the net change in sea level from 1995 to 2011; then write the definite integral.

b) Use the data in the table and generalize the trapezoid Rule to estimate the value of the integral from part (*a*).

Solution

a) The net charge in any quantity Q over the interval $[a,\ b]$ is $Q(b) - Q(a)$

Net change in

$$s(t) = S(b) - S(a)$$

$$= \int_0^{16} s'(t)\ dt$$

b) From the figure the values accompanied by 7 trapezoids whose area approximates

$$\int_0^{16} s'(t)\ dt$$

Area of the ***first*** trapezoid:

$$T_1 = \frac{1}{2}\left(s'(0) + s'(3)\right)\cdot 3$$

$$= \frac{1}{2}(0.51 + 5.19) \cdot 3$$

$$= 8.55$$

$$T_2 = \frac{1}{2}\left(s'(3) + s'(5)\right) \cdot 2$$

$$= \frac{1}{2}(5.19 + 4.39) \cdot 2$$

$$= 9.58$$

$$T_3 = \frac{1}{2}\left(s'(5) + s'(7)\right) \cdot 2$$

$$= \frac{1}{2}(4.39 + 2.21) \cdot 2$$

$$= 6.6$$

$$T_4 = \frac{1}{2}\left(s'(7) + s'(8)\right) \cdot 1$$

$$= \frac{1}{2}(2.21 + 5.24)$$

$$= 3.725$$

$$T_5 = \frac{1}{2}\left(s'(8) + s'(12)\right) \cdot 4$$

$$= \frac{1}{2}(5.24 + 0.63) \cdot 4$$

$$= 11.74$$

$$T_6 = \frac{1}{2}\left(s'(12) + s'(14)\right) \cdot 2$$

$$= \frac{1}{2}(0.63 + 4.19) \cdot 2$$

$$= 4.82$$

$$T_7 = \frac{1}{2}\left(s'(14) + s'(16)\right) \cdot 2$$

$$= \frac{1}{2}(4.19 + 2.38) \cdot 2$$

$$= 6.57$$

$$T(7) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$$

$$\approx 51.585 \quad mm$$

Exercises *Section* 6.5 – Numerical Integration

$(\mathbf{1-4})$ Find the *Midpoint* Rule approximations to:

1. $\int_0^1 \sin \pi x \, dx \quad n=6$ *subintervals*

2. $\int_0^{\pi} x^2 \sin x \, dx \quad n=8$ *subintervals*

3. $\int_0^1 e^{-\sqrt{x}} \, dx \quad n=6$ *subintervals*

4. $\int_0^1 e^{-x} \, dx \quad n=8$ *subintervals*

$(\mathbf{5-7})$ Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the *Trapezoid* Rule and (***b***) *Simpson's* Rule.

5. $\int_1^3 (2x-1) \, dx$

6. $\int_{-1}^1 \left(x^2+1\right) dx$

7. $\int_2^4 \frac{1}{(s-1)^2} \, ds$

$(\mathbf{8-27})$ Find the *Trapezoid* & *Simpson's* Rule approximations and error to:

8. $\int_0^1 \sin \pi x \, dx \quad n=6$ *subintervals*

9. $\int_0^1 e^{-x} \, dx \quad n=8$ *subintervals*

10. $\int_1^5 \left(3x^2-2x\right) dx \quad n=8$ *subintervals*

11. $\int_0^{\pi/4} 3\sin 2x \, dx \quad n=8$ *subintervals*

12. $\int_0^8 e^{-2x} \, dx \quad n=8$ *subintervals*

13. $\int_{-1}^1 \sqrt{x^2+1} \, dx \quad n=8$ *subintervals*

14. $\int_0^{1/2} \sin\left(x^2\right) dx \quad n=4$ *subintervals*

15. $\int_{\pi/2}^{\pi} \frac{\sin x}{x} \, dx \quad n=6$ *subintervals*

16. $\int_0^{\pi/4} x \tan x \, dx \quad n=6$ *subintervals*

17. $\int_0^1 e^{-x^2} \, dx \quad n=10$ *subintervals*

18. $\int_0^2 \frac{1}{\sqrt{1+x^2}} \, dx \quad n=10$ *subintervals*

19. $\int_0^{1/2} \sin\left(e^{x/2}\right) dx \quad n=8$ *subintervals*

20. $\int_2^3 \frac{1}{\ln x}\,dx \quad n=10 \quad subintervals$

21. $\int_1^2 e^{1/x}\,dx \quad n=4 \quad subintervals$

22. $\int_0^1 \ln\left(1+e^x\right)\,dx \quad n=8 \quad subintervals$

23. $\int_0^1 x^5 e^x\,dx \quad n=10 \quad subintervals$

24. $\int_0^4 \sqrt{x}\,\sin x\,dx \quad n=8 \quad subintervals$

25. $\int_0^3 \frac{1}{1+x^5}\,dx \quad n=6 \quad subintervals$

26. $\int_1^4 \frac{e^x}{x}\,dx \quad n=10 \quad subintervals$

27. $\int_1^2 \frac{dx}{x} \quad n=10 \quad subintervals$

28. $\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\cos x}{x}\,dx \quad n=10 \quad subintervals$

29. $\int_0^1 e^x \cos x\,dx \quad n=10 \quad subintervals$

30. A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).

a) Estimate the surface area of the paneling using the Trapezoid Rule
b) Estimate the surface area of the paneling using a left Riemann sum.
c) Could two identical pieces be cut from a 9-*in* by 9-*in* piece of wood?

31. The region bounded by the curves $y=\frac{1}{1+e^{-x}}$, $x=0$ and $x=10$ is rotated about $x-axis$. Use Trapezoid and Simpson's Rule with $n=10$ to estimate the surface area for $0 \le x \le \frac{\pi}{4}$.

$(\mathbf{32-34})$ Find the area under the curve $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Use Simpson's Rule with $n = 20$ to estimate area under the probability density given function.

32. $0 \le x \le .4$

33. $-1 \le x \le 1 \quad (1 - sigma)$

34. $-3 \le x \le 3 \quad (3 - sigma)$

35. A pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

Where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1\ m$ and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.

36. The length of an ellipse is given by $L = \frac{8}{3} \int_0^{\frac{\pi}{2}} \sqrt{9 - \cos^2 t}\ dt$. Use Simpson's rule to estimate the length.

Section 6.6 – Improper Integrals

6.6–1 *Definition*

Integrals with infinite limits of integration are ***improper integrals***.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

In each case, if the limit is finite, we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit fails to exist, the improper integral ***diverges***.

Notes:

$$\frac{1}{\infty} = 0 \qquad e^{-\infty} = 0 \qquad \ln(0) = -\infty$$

$$\frac{1}{0} = \infty \qquad e^{\infty} = \infty \qquad \ln(\infty) = \infty$$

Example 1

Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution

$$u = \ln x \qquad v = \int \frac{dx}{x^2}$$

$$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$$

$$Area = \int_1^{\infty} \frac{\ln x}{x^2}\, dx$$

$$= -\frac{1}{x}\ln x \Big|_1^{\infty} + \int_1^{\infty} \frac{1}{x^2}\, dx$$

$$= -\left(\frac{1}{\infty} - \ln 1\right) - \frac{1}{x}\Big|_1^{\infty}$$

$$= -(0) - \left(\frac{1}{\infty} - 1\right)$$

$$= 1 \quad unit^2$$

$$\int_1^b \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x \Big|_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$$

$$= -\left(\frac{1}{b}\ln b - \ln 1\right) + \int_1^b \frac{1}{x^2}\, dx$$

$$= -\frac{1}{b}\ln b + \left(-\frac{1}{x}\Big|_1^b\right.$$

$$= -\frac{1}{b}\ln b - \left(\frac{1}{b} - 1\right)$$

$$= -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\, dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}\ln b - \frac{1}{b} + 1\right)$$

$$= -\lim_{b\to\infty}\left(\frac{1/b}{1}\right) - 0 + 1 \qquad \textit{L'Hôpital Rule}$$

$$= -0 + 1$$

$$= 1 \quad unit^2$$

Example 2

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan x \Big|_{-\infty}^{\infty}$$

$$= \arctan\infty - \arctan(-\infty)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

Example 3

Evaluate the integral $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

Solution

$$\int_{-\infty}^{0} e^{3x}\,dx = \frac{1}{3}e^{3x}\Big|_{-\infty}^{0}$$

$$= \frac{1}{3}\left(e^{0} - e^{-\infty}\right)$$

$$= \frac{1}{3}(1-0)$$

$$= \frac{1}{3}$$

6.6–2 Integrands with Vertical Asymptotes

Definition

Integrals of functions that become infinite at a point within the interval of integration are ***improper integrals***.

If the limit is finite, we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit does not exist, the integral ***diverges***.

1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

3. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{c \to p^-} \int_a^c f(x)dx + \lim_{d \to p^+} \int_d^b f(x)dx$$

Example 4

Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

Solution

$$\int_0^1 \frac{1}{1-x}\,dx = -\int_0^1 \frac{1}{1-x}\,d(1-x)$$

$$= -\ln|1-x| \Big|_0^1$$

$$= -\ln 0 + \ln 1$$

$$= \infty$$

The limit is infinite, so the integral diverges.

Example 5

Evaluate the integral $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^3 (x-1)^{-2/3}\,d(x-1)$$

$$= 3(x-1)^{1/3} \Big|_0^3$$

$$= 3\left(\sqrt[3]{2}+1\right)$$

$$= 3+3\sqrt[3]{2}$$

Example 6

Does the integral $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converge?

Solution

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$$

$$= -e^{-b} + e^{-1}$$

$$< e^{-1}$$

$$= \frac{1}{e}$$

$$\approx 0.36788$$

The integral *converges*.

6.6–3 *Theorem* – Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges*

2. $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges*

6.6–4 *Theorem* – Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^\infty f(x)\,dx \quad \textit{and} \quad \int_a^\infty g(x)\,dx$$

Both converge or both diverge.

Example 7

Show that $\int_1^{\infty} \frac{dx}{1+x^2}$ converges by comparison with $\int_1^{\infty} \frac{dx}{x^2}$.

Find and compare the two integral values.

Solution

The functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{1+x^2}$ are positive and continuous on [1, ∞). Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= 1$$

Therefore, $\int_1^{\infty} \frac{dx}{1+x^2}$ converges because $\int_1^{\infty} \frac{dx}{x^2}$ converges.

$$\int_1^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_1^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Example 8

Let R be the region bounded by the graph of $y=\frac{1}{x}$ and the x-*axis*, for $x \geq 1$.

a) What is the volume of the solid generated when R is revolved about the x-*axis*?

b) What is the surface area of the solid generated when R is revolved about the x-*axis*?

c) What is the volume of the solid generated when R is revolved about the y-*axis*?

Solution

a) $V=\pi\int_1^{\infty}\frac{1}{x^2}\,dx$ $\qquad V=\pi\int_a^b (f(x))^2\,dx$

$$=-\pi\frac{1}{x}\Big|_1^{\infty}$$

$$=-\pi(0-1)$$

$$=\pi \quad unit^3$$

b) $S=2\pi\int_1^{\infty}\frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$ $\qquad S=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$

$$=2\pi\int_1^{\infty}\frac{1}{x^3}\sqrt{x^4+1}\,dx$$

$$>2\pi\int_1^{\infty}\frac{x^2}{x^3}\,dx \qquad \sqrt{x^4+1}>\sqrt{x^4}=x^2$$

$$=2\pi\int_1^{\infty}\frac{1}{x}\,dx$$

$$=2\pi\left(\ln x\right|_1^{\infty}$$

$$= \infty \quad unit^2$$

c) $V = 2\pi \int_1^{\infty} x \frac{1}{x}\, dx$ $\qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (Shell\ method)$

$$= 2\pi x \Big|_1^{\infty}$$

$$= \infty \quad unit^3$$

Example 9

Find the area of the region bounded by the function $f(x) = \frac{1}{x^2 - 1}$ and the $x - axis$, where $x \geq 3$.

Solution

$$Area = \int_3^{\infty} \frac{1}{x^2 - 1}\, dx$$

$$\frac{1}{x^2 - 1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x \quad A + B = 0$$

$$x^0 \quad A - B = 1$$

$$= \int_3^{\infty} \left(\frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1} \right) dx$$

$$= \frac{1}{2} \left(\ln(x-1) - \ln(x+1) \right) \Big|_3^{\infty}$$

$$= \frac{1}{2} \left(\ln \frac{x-1}{x+1} \right. \Big|_3^{\infty}$$

$$\lim_{x \to \infty} \ln\left(\frac{x-1}{x+1} \right) = \lim_{x \to \infty} \ln\left(\frac{x}{x} \right)$$

$$= \ln 1$$

$$= 0$$

$$= \frac{1}{2} \left(-\ln \frac{2}{4} \right)$$

$$= \frac{\ln 2}{2} \quad unit^2$$

Example 10

Evaluate the integral $\displaystyle\int_0^{\infty} e^{-x}\cos 2x\, dx$.

Solution

$$\int \cos 2x\, dx$$

+	e^{-x}	$\frac{1}{2}\sin 2x$
−	$-e^{-x}$	$-\frac{1}{4}\cos 2x$
+	e^{-x}	

$$\int e^{-x}\cos 2x\, dx = \left(\frac{1}{2}\sin 2x - \frac{1}{4}\cos 2x\right)e^{-x} - \frac{1}{4}\int e^{-x}\cos 2x\, dx$$

$$\left(1+\frac{1}{4}\right)\int e^{-x}\cos 2x\, dx = \frac{1}{4}\left(2\sin 2x - \cos 2x\right)e^{-x}$$

$$\frac{5}{4}\int e^{-x}\cos 2x\, dx = \frac{1}{4}\left(2\sin 2x - \cos 2x\right)e^{-x}$$

$$\int_0^{\infty} e^{-x}\cos 2x\, dx = \frac{1}{5}\left(2\sin 2x - \cos 2x\right)e^{-x}\Big|_0^{\infty}$$

$$= \frac{1}{5}\left((*)e^{-\infty} - \cos 0\right)$$

$$= -\frac{1}{5}$$

Example 11

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{dx}{x^2+3x+2}$

Solution

$$\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\begin{cases} A+B=0 \\ 2A+B=1 \end{cases}$$

$A = 1 \quad B = -1$

$$\int_0^\infty \frac{dx}{x^2+3x+2} = \int_0^\infty \frac{d(x+1)}{x+1} - \int_0^\infty \frac{d(x+2)}{x+2}$$

$$= \ln(x+1) - \ln(x+2) \Big|_0^\infty$$

$$= \ln\left(\frac{x+1}{x+2}\right) \Big|_0^\infty$$

$$= \lim_{x\to\infty} \ln\left(\frac{x+1}{x+2}\right) - \ln\frac{1}{2}$$

$$= \lim_{x\to\infty} \ln\left(\frac{x}{x}\right) + \ln 2$$

$$= \ln 1 + \ln 2$$

$$= \ln 2$$

Example 12

Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{x^2+2x+1}$

Solution

$$\int_0^\infty \frac{dx}{x^2+2x+1} = \int_0^\infty \frac{d(x+1)}{(x+1)^2}$$

$$= -\frac{1}{x+1} \Big|_0^\infty$$

$$= -(0-1)$$

$$= 1$$

Exercises *Section* 6.6 – Improper Integrals

(1 – 120) Evaluate the integrals.

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

3. $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{\left(x^2+4\right)^{3/2}}$

5. $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

6. $\displaystyle\int_{-\infty}^{\infty} 2x\,e^{-x^2}\,dx$

7. $\displaystyle\int_0^{1} (-\ln x)\,dx$

8. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

9. $\displaystyle\int_0^{\infty} e^{-3x}\,dx$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

11. $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

12. $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

13. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)^3}$

14. $\displaystyle\int_{-\infty}^{0} e^x\,dx$

15. $\displaystyle\int_1^{\infty} 2^{-x}\,dx$

16. $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

17. $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

18. $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p>1$

19. $\displaystyle\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\,dp$

20. $\displaystyle\int_{-1}^{1} \ln y^2\,dy$

21. $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

22. $\displaystyle\int_0^{\infty} x\,e^{-x}\,dx$

23. $\displaystyle\int_0^{1} x\ln x\,dx$

24. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

25. $\displaystyle\int_1^{\infty} (1-x)e^x\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

27. $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x}}$

28. $\displaystyle\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\,dx$

29. $\displaystyle\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\,dx$

30. $\displaystyle\int_0^{2} \frac{dx}{x^3}$

31. $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

32. $\displaystyle\int_1^{\infty} \frac{6}{x^4}\,dx$

33. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

34. $\displaystyle\int_{-\infty}^{0} x\,e^{-4x}\,dx$

35. $\displaystyle\int_{0}^{\infty} x\,e^{-x/3}\,dx$

36. $\displaystyle\int_{0}^{\infty} x^2\,e^{-x}\,dx$

37. $\displaystyle\int_{0}^{\infty} e^{-x}\cos x\,dx$

38. $\displaystyle\int_{4}^{\infty} \frac{1}{x\,(\ln x)^3}\,dx$

39. $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x}\,dx$

40. $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx$

41. $\displaystyle\int_{0}^{\infty} \frac{x^3}{\left(x^2+1\right)^2}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$

43. $\displaystyle\int_{0}^{\infty} \frac{e^x}{1+e^x}\,dx$

44. $\displaystyle\int_{0}^{\infty} \cos \pi x\,dx$

45. $\displaystyle\int_{0}^{\infty} \sin\frac{x}{2}\,dx$

46. $\displaystyle\int_{1}^{\infty} \frac{dx}{(x+1)^9}$

47. $\displaystyle\int_{1}^{\infty} \frac{3x-1}{4x^3-x^2}\,dx$

48. $\displaystyle\int_{-\infty}^{\infty} \frac{4}{x^2+16}\,dx$

49. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$

50. $\displaystyle\int_{0}^{\infty} x\,e^{-x}\,dx$

51. $\displaystyle\int_{0}^{\infty} \frac{6x}{1+x^6}\,dx$

52. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt[3]{|x-1|}}$

53. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

54. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

55. $\displaystyle\int_{0}^{\infty} \cos x\,dx$

56. $\displaystyle\int_{2}^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$

57. $\displaystyle\int_{-\infty}^{a} \sqrt{e^x}\,dx$

58. $\displaystyle\int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$

59. $\displaystyle\int_{1}^{\infty} \frac{dx}{x(x+1)}$

60. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^2(x+1)}$

61. $\displaystyle\int_{1}^{\infty} \frac{3x^2+1}{x^3+x}\,dx$

62. $\displaystyle\int_{1}^{\infty} \frac{1}{x^2}\sin\frac{\pi}{x}\,dx$

63. $\displaystyle\int_{2}^{\infty} \frac{dx}{(x+2)^2}$

64. $\displaystyle\int_{1}^{\infty} \frac{\tan^{-1}x}{x^2+1}\,dx$

65. $\displaystyle\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}}$

66. $\displaystyle\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

67. $\displaystyle\int_{0}^{\ln 3} \frac{e^x}{\left(e^x-1\right)^{2/3}}\,dx$

68. $\displaystyle\int_{1}^{2} \frac{dx}{\sqrt{x-1}}$

69. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$

70. $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

71. $\displaystyle\int_{-1}^2 \frac{dx}{(x-1)^2}$

72. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$

73. $\displaystyle\int_0^\infty x\,e^{-x^2}\,dx$

74. $\displaystyle\int_{-\infty}^\infty x\,e^{-x^2}\,dx$

75. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

76. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$

77. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+6x+12}$

78. $\displaystyle\int_0^\infty \frac{1}{x^2+4}\,dx$

79. $\displaystyle\int_0^\infty e^{-x}\sin x\,dx$

80. $\displaystyle\int_{-\infty}^0 e^{5x}\,dx$

81. $\displaystyle\int_{-1}^0 \frac{dx}{x^{1/3}}$

82. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

83. $\displaystyle\int_5^\infty \frac{dx}{\sqrt{x^2-25}}$

84. $\displaystyle\int_0^3 \frac{dx}{3-x}$

85. $\displaystyle\int_1^\infty \frac{\ln x}{x^p}\,dx$

86. $\displaystyle\int_1^\infty x^p\,e^{-x}\,dx$

87. $\displaystyle\int_e^\infty \frac{dx}{(\ln x)^p}$

88. $\displaystyle\int_2^\infty \frac{1}{x\ln^2 x}\,dx$

89. $\displaystyle\int_2^\infty \frac{1}{x\ln^k x}\,dx$

90. $\displaystyle\int_e^\infty \frac{dx}{x\ln x\sqrt{\ln x}}$

91. $\displaystyle\int_0^\infty x^3\,e^{-x}\,dx$

92. $\displaystyle\int_0^{\frac{\pi}{2}} \sec x\,dx$

93. $\displaystyle\int_0^\infty \cos x\,dx$

94. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}}\,dx$

95. $\displaystyle\int_{\frac{1}{\pi}}^\infty \frac{1}{x^2}\sin\frac{1}{x}\,dx$

96. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1-\sin x}$

97. $\displaystyle\int_0^{\frac{\pi}{2}} \cot x\,dx$

98. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{\sin x^2}\,dx$

99. $\displaystyle\int_1^\infty \left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+3}}\right)dx$

100. $\displaystyle\int_0^9 \frac{dx}{(x-1)^{2/3}}$

101. $\displaystyle\int_0^\infty \frac{x}{x^4+1}\,dx$

102. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

103. $\displaystyle\int_1^\infty e^{-2x}dx$

104. $\displaystyle\int_1^\infty \frac{x^2}{e^{x/2}}\,dx$

105. $\displaystyle\int_1^\infty x^2\,e^{-3x^3}\,dx$

106. $\displaystyle\int_2^{\infty} \frac{x-4}{x^2-2x+1}\,dx$

107. $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2-3x+2}\,dx$

108. $\displaystyle\int_4^{\infty} \frac{x+18}{x^2+x-12}\,dx$

109. $\displaystyle\int_2^{\infty} \frac{dx}{x^2-x}$

110. $\displaystyle\int_0^{\infty} \operatorname{sech} x\,dx$

111. $\displaystyle\int_0^{\infty} \operatorname{csch} x\,dx$

112. $\displaystyle\int_1^{\infty} \frac{x}{\sqrt{x^2+4}}\,dx$

113. $\displaystyle\int_1^{\infty} \frac{2^x+3^x}{4^x}\,dx$

114. $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x\,\left(\ln\left(\ln x\right)\right)^2}$

115. $\displaystyle\int_1^{\infty} \frac{x}{\left(x^2+4\right)^3}\,dx$

116. $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt[3]{x+9}}$

117. $\displaystyle\int_0^{\infty} \tan^{-1} x\,dx$

118. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

119. $\displaystyle\int_0^{\infty} \frac{dx}{\left(16+x^2\right)^{3/2}}$

120. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{9+x^2}}$

121. $\displaystyle\int_0^{\infty} x\tan^{-1} x\,dx$

122. $\displaystyle\int_0^{\infty} \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}\,dx$

123. $\displaystyle\int_1^{\infty} \log x\,dx$

124. $\displaystyle\int_0^{\infty} e^{-ax}\,dx$

125. $\displaystyle\int_0^{1} \frac{1}{x^a}\,dx$

126. $\displaystyle\int_1^{\infty} \frac{1}{x^a}\,dx$

127. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a^2}$

128. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$

129. $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2+a^2}}$

130. $\displaystyle\int_0^{a} \frac{dx}{a^2-x^2}$

131. $\displaystyle\int_0^{a} \frac{dx}{\sqrt{a^2-x^2}}$

132. $\displaystyle\int_{a+1}^{\infty} \frac{dx}{x^2-a^2}$

133. $\displaystyle\int_0^{\infty} \frac{x^2}{x^2-a^2}\,dx$

134. $\displaystyle\int_0^{\infty} \frac{x}{x^2-a^2}\,dx$

135. $\displaystyle\int_0^{\infty} \frac{x}{\left(x^2-a^2\right)^2}\,dx$

136. $\displaystyle\int_a^{3a} \frac{x}{\left(x^2-a^2\right)^{3/2}}\,dx$

137. $\displaystyle\int_0^{3a} \frac{x}{\left(x^2-a^2\right)^{2/3}}\,dx$

138. $\displaystyle\int_a^{\infty} \frac{dx}{\sqrt{x^2-a^2}}$

139. $\displaystyle\int_a^{\infty} \frac{x^2\,dx}{\sqrt{x^2-a^2}}$

140. $\displaystyle\int_a^{\infty} \frac{dx}{x^2\sqrt{x^2-a^2}}$

(141 – 142) Evaluate $\int_0^2 f(x)\,dx$ where:

141. $f(x)=\begin{cases} x^2 & for \quad 0\le x<1 \\ 2x^2 & for \quad 1\le x\le 2 \end{cases}$

142. $f(x)=\begin{cases} 2x & for \quad 0\le x<1 \\ \frac{1}{2}x & for \quad 1\le x\le 2 \end{cases}$

143. $f(x)=\begin{cases} x & for \quad 0\le x<1 \\ x^2 & for \quad 1\le x\le 2 \end{cases}$

144. $f(x)=\begin{cases} \cos\frac{\pi}{4}x & for \quad 0\le x<1 \\ \sin\frac{\pi}{4}x & for \quad 1\le x\le 2 \end{cases}$

(145 – 148) Find the area of the unbounded shaded region.

145. $y=e^x, \quad -\infty<x\le 1$

146. $y=-\ln x$

147. $y=\dfrac{1}{x^2+1}$

148. $y=\dfrac{8}{x^2+4}$

149. Find the area of the region R between the graph of $f(x)=\dfrac{x}{\sqrt{1-x^2}}$ and the x-*axis* on the interval $0\le x\le 1$.

150. Find the area of the region R between the graph of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $-3\le x\le 3$.

151. Find the area of the region R of the graph of $f(x)=e^{-2x}$ in the first quadrant.

152. Find the area of the graph of $f(x)=\dfrac{1}{x^2+6x+10}$ in the first quadrant.

153. Find the area between the curves $f(x)=\dfrac{1}{x}$ and $f(x)=\dfrac{1}{x+1}$ to the right of the line $x=1$.

154. Find the volume of the region R of the graph of $xy=9$ and to the right of $x=1$ revolving about the $x-axis$.

155. Find the volume of the region R of the graph of $f(x)=e^{-2x}$ in the first quadrant revolving about the $x-axis$.

156. Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the x-$axis$ on the interval $[2,\ \infty)$ is revolved about the $x-axis$.

157. Find the volume of the region bounded by $f(x)=\sqrt{\dfrac{x+1}{x^3}}$ and the x-$axis$ on the interval $[1,\ \infty)$ is revolved about the x-$axis$.

158. Find the volume of the region bounded by $f(x)=(x+1)^{-3}$ and the x-$axis$ on the interval $[0,\ \infty)$ is revolved about the y-$axis$.

159. Find the volume of the region bounded by $f(x)=\dfrac{1}{\sqrt{x}\ln x}$ and the x-$axis$ on the interval $[2,\ \infty)$ is revolved about the x-$axis$.

160. Find the volume of the region bounded by $f(x)=\dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the x-$axis$ on the interval $[0,\ \infty)$ is revolved about the x-$axis$.

161. Find the volume of the region bounded by $f(x)=\left(x^2-1\right)^{-1/4}$ and the x-$axis$ on the interval $(1,\ 2]$ is revolved about the y-$axis$.

162. Find the volume of the region bounded by $f(x)=\tan x$ and the x-$axis$ on the interval $\left[0,\ \dfrac{\pi}{2}\right)$ is revolved about the x-$axis$.

163. Find the volume of the region bounded by $f(x)=-\ln x$ and the x-$axis$ on the interval $(0,\ 1]$ is revolved about the x-$axis$.

164. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the *x-axis*.

165. The region between the *x-axis* and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x\ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the *x-axis* to generate the solid. Find the volume of the solid.

(166 – 167) Consider the region satisfying the inequalities.

a) Find the area of the region.
b) Find the volume of the solid generated by revolving the region about the *x-axis*.
c) Find the volume of the solid generated by revolving the region about the *y-axis*.

166. $y \le e^{-x}, \quad y \ge 0, \quad x \ge 0$

167. $y \le \dfrac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

168. Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$.

169. Find the arc length of the graph $y = \sqrt{16 - x^2}$ over the interval $[0,\ 4]$.

170. The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

171. Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0,\ \infty)$ about the *x-axis*.

172. The magnetic potential *P* at a point on the axis of a circular coil is given by:

$$P = \frac{2\pi N I r}{k} \int_c^{\infty} \frac{1}{\left(r^2 + x^2\right)^{3/2}}\, dx$$

Where *N*, *I*, *r*, *k*, and *c* are constants. Find *P*.

173. A "semi-infinite" uniform rod occupies the nonnegative $x\text{-}axis$. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a,\ 0)$. The gravitational force F that the rod exerts on the mass is given by:

$$F = \int_0^{\infty} \frac{GM\delta}{(a+x)^2}\, dx$$

Where G is the gravitational constant. Find F.

174. Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the $x\text{-}axis$.

a) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

175. The solid formed by revolving (about the $x\text{-}axis$) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the $x\text{-}axis$ $(x \ge 1)$ is called ***Gabriel's Horn***.

Show that this solid has a finite volume and an infinite surface area.

176. Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

177. Let $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$, where a is a real number.

a) Evaluate $I(a)$ and show that its value is independent of a.

(***Hint***: split the integral into two integrals over $[0,\ 1]$ and $[1,\ \infty)$; then use a change of variables to convert the second integral into an integral over $[0,\ 1]$.)

b) Let f be any positive continuous function on $\left[0,\ \frac{\pi}{2}\right]$

Evaluate $\displaystyle\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x)+f(\sin x)}\,dx$

(***Hint***: Use the identity $\cos\left(\frac{\pi}{2}-x\right)=\sin x$)

178. Let R be the region bounded by $y=\ln x$, the *x-axis*, and the line $x=a$, where $a>1$.

a) Find the volume $V_1(a)$ of the solid generated when R is revolved about the *x-axis* (as a function of a).

b) Find the volume $V_2(a)$ of the solid generated when R is revolved about the *y-axis* (as a function of a).

c) Graph V_1 and V_2. For what values of $a>1$ is $V_1(a)>V_2(a)$?

179. Let R be the region bounded by the graph of $f(x)=x^{-p}$ and the *x-axis*, for $x\geq 1$. Let V_1 and V_2 be the volumes of the solids generated when R is revolved about the *x-axis* and the *y-axis*, respectively, if they exist.

a) For what values of p (if any) is $V_1=V_2$?

b) Repeat part (a) on the interval $(0,\ 1]$.

180. Let R_1 be the region bounded by the graph of $y=e^{-ax}$ and the *x-axis* on the interval $[0,\ b]$ where $a>0$ and $b>0$. Let R_2 be the region bounded by the graph of $y=e^{-ax}$ and the *x-axis* on the interval $[b,\ \infty)$. Let V_1 and V_2 be the volumes of the solids generated when R_1 and R_2 are revolved about the *x-axis*. Find and graph the relationship between a and b for which $V_1=V_2$.

181. Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 *feet.* long. As the boat travels along the positive *y*-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown.

a) Show that $f'(x) = -\dfrac{\sqrt{900 - x^2}}{x}$

(*Hint*: Assume that the skier is always pointed directly at the boat and the rope is online is on a line tangent to the path $y = f(x)$.)

b) Solve the equation in part (a) for $f(x)$, using $f(30) = 0$

182. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation.

$$\frac{dx}{dt} = k(a - x)(b - x) \quad \textit{or} \quad \frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k$$

Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$.

First Order Differential Equations

Section 6.7 – First-Order Linear Equations

In the mathematical, engineering and sciences models are developed to assist in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a *differential equation*.

6.7−1 General First-Order Differential Equations and Solutions

A ***first-order differential equation*** is an equation.

$$\frac{dy}{dx} = f(x, y)$$

In which $f(x, y)$ is a function of two variables defined on a region in the xy−plane.

Example 1

Show that every member of the family of functions $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$ on the interval (0, ∞), where C is any constant.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{C}{x} + 2\right)\\ &= -\frac{C}{x^2}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}(2 - y)\\ -\frac{C}{x^2} &= \frac{1}{x}\left(2 - \left(\frac{C}{x} + 2\right)\right)\\ -\frac{C}{x^2} &= \frac{1}{x}\left(2 - \frac{C}{x} - 2\right)\\ -\frac{C}{x^2} &= \frac{1}{x}\left(-\frac{C}{x}\right)\end{aligned}$$

$$-\frac{C}{x^2} = -\frac{C}{x^2} \quad \checkmark$$

Therefore, for every value of C, the function $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$.

Example 2

Show that the function $y = x + 1 - \frac{1}{3}e^x$ is a solution of the first-order initial value problem $\frac{dy}{dx} = y - x \quad y(0) = \frac{2}{3}$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right)$$
$$= 1 - \frac{1}{3}e^x$$

$$y - x = 1 - \frac{1}{3}e^x$$

$$y = x + 1 - \frac{1}{3}e^x$$

$$y(0) = (0 + 1) - \frac{1}{3}e^0$$
$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3}$$

6.7–2 *Slope Fields*: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the solution curve is required to pass through the point (x_0, y_0) and to have a slope $f(x_0, y_0)$ there.

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

$$y' = y^2 - xy + 2x$$

$$y' = 2y, \quad with\ y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$$

6.7–3 Euler's Method

Euler's method named after *Leonhard Euler* is an example of a ***fixed-step*** solver.
Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.

$$y' = f(x,y) \quad y(x_0) = y_0$$

The setting size: $h = \dfrac{b-a}{k} > 0$; $k = 1,\ 2,\ 3,\ \ldots$

Then, $x_0 = a$

$$x_1 = x_0 + h = a + h$$

$$x_k = x_{k-1} + h = a + kh$$

Last point $x_k = a + kh = b$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : slope$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step}.f\left(x_k, y_k\right)$$

$$\boxed{y_{k+1} = y_k + f\left(x_k, y_k\right)dx}$$

This method is known as *Euler's Method* with step size h.

Example 3

Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \quad y(0) = 1$$

Starting at $x_0 = 0$ with $dx = 0.1$.

Solution

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)dx \\ &= y_0 + (1 + y_0)dx \\ &= 1 + (1+1)(0.1) \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)dx \\ &= y_1 + (1 + y_1)dx \\ &= 1.2 + (1+1.2)(0.1) \\ &= 1.42 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + f(x_2, y_2)dx \\ &= y_2 + (1 + y_2)dx \\ &= 1.42 + (1+1.42)(0.1) \\ &= 1.662 \end{aligned}$$

Example 4

Use Euler's method to solve $y' = 1 + y$, $y(0) = 1$

On the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking

a) $dx = 0.1$.
b) $dx = 0.05$.

Compare the approximations with the values of the exact solution $y = 2e^x - 1$

Solution

a) Euler Method ***dx* = 0.1**

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.21034184	0.01034184
0.20	1.42000000	1.44280552	0.02280552
0.30	1.66200000	1.69971762	0.03771762
0.40	1.92820000	1.98364940	0.05544940
0.50	2.22102000	2.29744254	0.07642254
0.60	2.54312200	2.64423760	0.10111560
0.70	2.89743420	3.02750541	0.13007121
0.80	3.28717762	3.45108186	0.16390424
0.90	3.71589538	3.91920622	0.20331084
1.00	4.18748492	4.43656366	0.24907874

b) Euler Method ***dx* = 0.05**

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10254219	0.00254219
0.10	1.20500000	1.21034184	0.00534184
0.15	1.31525000	1.32366849	0.00841849
0.20	1.43101250	1.44280552	0.01179302
0.25	1.55256313	1.56805083	0.01548771
0.30	1.68019128	1.69971762	0.01952633
0.35	1.81420085	1.83813510	0.02393425
0.40	1.95491089	1.98364940	0.02873851
0.45	2.10265643	2.13662437	0.03396794

0.50	2.25778925	2.29744254	0.03965329
0.55	2.42067872	2.46650604	0.04582732
0.60	2.59171265	2.64423760	0.05252495
0.65	2.77129828	2.83108166	0.05978337
0.70	2.95986320	3.02750541	0.06764222
0.75	3.15785636	3.23400003	0.07614367
0.80	3.36574918	3.45108186	0.08533268
0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825

6.7–4 *Definition*

A ***first-order linear*** differential equation is one that can be written in the ***standard form***

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Where P and Q are continuous functions of x

6.7–5 Separable Equation

Solution of the homogenous equation

$$\frac{dy}{dx} + P(x)\,y = 0$$

$$\frac{dy}{dx} = -P(x)\,y$$

$$\int \frac{dy}{y} = -\int P(x)\,dx$$ ***Integrate both sides***

$$\ln|y| = -\int P(x)dx + C$$ ***Convert to exponential form***

$$y(x) = e^{-\int P(x)dx + C}$$

$$= e^{-\int P(x)dx} e^{C}$$

$$\boxed{y(x) = A.e^{-\int P(x)\,dx}}$$

Example 5

Solve the differential equation $y' = t\,y^2$

Solution

$$\frac{dy}{dt} = t\,y^2$$

$$\frac{dy}{y^2} = t\,dt$$

$$\int y^{-2}dy = \int t\,dt$$

$$-y^{-1} = \tfrac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2+2C}{2}$$ *Cross multiplication*

$$y(t) = -\frac{2}{t^2 + C_1}$$ $C_1 = 2C$

6.7–6 Separable Strategy

To apply a separable differential equation:

1. Separate the variables
2. Integrate both sides
3. Solve for the solution $y(t)$, if possible

Example 6

Find the general solution of the differential equation. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2-1}dx$$ $d\left(x^2-1\right) = 2x\,dx$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2-1\right)}{x^2-1}$$

$$\ln|y+1| = \ln\left|x^2-1\right| + \ln C$$

$$= \ln\left(C\left|x^2-1\right|\right)$$

$$y+1 = C\left|x^2-1\right|$$

$$y(x) = C\left|x^2-1\right| - 1$$

Example 7

Solve the differential equation $x' = (2t+1)\left(x^2 - 5x + 4\right)$ $\quad x(0) = 0$

Solution

$$x' = (2t+1)\left(x^2 - 5x + 4\right)$$

$$\frac{dx}{dt} = (2t+1)\left(x^2 - 5x + 4\right)$$

$$\frac{1}{x^2 - 5x + 4}\,dx = (2t+1)\,dt$$

$$\int \frac{1}{(x-1)(x-4)}\,dx = \int (2t+1)\,dt$$

$$\frac{1}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4}$$

$$\begin{cases} A + B = 0 \\ -4A - B = 1 \end{cases}$$

$$A = -\frac{1}{3} \quad B = \frac{1}{3}$$

$$-\frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{x-4}\,dx = t^2 + t$$

$$-\frac{1}{3}\ln|x-1| + \frac{1}{3}\ln|x-4| = t^2 + t$$

$$\frac{1}{3}\left(\ln|x-4| - \ln|x-1|\right) = t^2 + t$$

$$\frac{1}{3}\ln\left|\frac{x-4}{x-1}\right| = t^2 + t$$

$$\ln\left|\frac{x-4}{x-1}\right| = 3t^2 + 3t$$

$$\frac{x-4}{x-1} = e^{3t^2+3t+C}$$

$$x(0) = 0$$

$$4 = C\,e^0$$

$$C = 4$$

$$\frac{x-4}{x-1} = 4\,e^{3t^2+3t}$$

6.7–7 *Solution* of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \textit{Homogeneous Solution} \\ y_p & \textit{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h$$

$$\boxed{y_h = e^{-\int p\,dx}}$$

$$y_p = u(x)y_h$$

$$= u\,e^{\int p\,dx}$$

$$y'_p + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u\left(y'_h + py_h\right) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f\,e^{\int p\,dx}dx$$

$$y_p = f\,e^{\int p\,dx}$$

$$u = \left(\int f \, e^{\int p\,dx} dx \right) e^{-\int p\,dx}$$

$$y_p = e^{-\int p\,dx} \int f \, e^{\int p\,dx} dx \qquad y = y_h + y_p$$

$$y = Ce^{-\int p\,dx} + e^{-\int p\,dx} \int f \, e^{\int p\,dx} dx$$

$$\boxed{y(x) = e^{-\int p\,dx} \left(\int f . e^{\int p\,dx} dx + C \right)}$$

Example 8

Solve the equation $x\dfrac{dy}{dx} = x^2 + 3y, \quad x > 0$

Solution

$$y' - \frac{3}{x} y = x$$

$$e^{\int p dx} = e^{-3\int \frac{dx}{x}}$$

$$= e^{-3\ln|x|}$$

$$= e^{\ln x^{-3}}$$

$$= x^{-3}$$

$$\int x.x^{-3} dx = \int x^{-2} dx$$

$$= -\frac{1}{x}$$

$$y(x) = \frac{1}{x^{-3}} \left(-\frac{1}{x} + C \right)$$

$$= x^3 \left(-\frac{1}{x} + C \right)$$

$$= -x^2 + Cx^3 \quad x > 0$$

Example 9

Solve the equation $3xy' - y = \ln x + 1, \quad x > 0$, satisfying $y(1) = -2$

Solution

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}$$

$$e^{\int p dx} = e^{\int\left(-\frac{1}{3x}\right)dx} = e^{-\frac{1}{3}\ln x}$$

$$= e^{\ln x^{-1/3}}$$

$$= x^{-1/3}$$

$$\int\left(x^{-1/3}\right)\frac{\ln x + 1}{3x}dx = \frac{1}{3}\int(\ln x + 1)x^{-4/3}\,dx$$

$$u = \ln x + 1 \qquad dv = \int x^{-4/3}dx$$

$$du = \frac{1}{x}dx \qquad v = -3x^{-1/3}$$

$$= \frac{1}{3}\left(-3x^{-1/3}(\ln x + 1) + 3\int\left(x^{-4/3}\right)dx\right)$$

$$= \frac{1}{3}\left(-3x^{-1/3}(\ln x + 1) - 9x^{-1/3}\right)$$

$$= -x^{-1/3}(\ln x + 1) - 3x^{-1/3}$$

$$y(x) = x^{1/3}\left(-x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C\right)$$

$$= -\ln x - 4 + Cx^{1/3}$$

$$y(1) = -\ln(1) - 4 + C(1)^{1/3}$$

$$-2 = -0 - 4 + C$$

$$C = 2$$

$$y = 2x^{1/3} - \ln x - 4$$

Example 10

Find the general solution of $x' = x\sin t + 2t\,e^{-\cos t}$ and the solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2t\, e^{-\cos t}$$

$$x_h = e^{-\int \sin t\, dt}$$

$$= e^{\cos t}$$

$$\int 2t\, e^{-\cos t}\, e^{\cos t}\, dt = \int 2t\, dt$$

$$= t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$x(t) = \left(t^2 + e\right)e^{-\cos t}$$

Example 11

Find the general solution of $x' = x\tan t + \sin t$ and the solution that satisfies $x(0) = 2$.

Solution

$$x' = x\tan t + \sin t$$

$$x' - (\tan t)x = \sin t$$

$$e^{-\int \tan t\, dt} = e^{\ln(\cos t)}$$

$$= \cos t$$

$$\int (\sin t)(\cos t)\, dt = -\int \cos t\, d(\cos t)$$

$$= -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right)$$

$$= -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$= -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2$$

$$C = \frac{5}{2}$$

$$x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}$$

Example 12

Find the general solution of $t\,y' + 3y = 5t^2 - 3t$ and the solution that satisfies $y(1) = 0$.

Solution

$$t\,y' + 3y = 5t^2 - 3t$$

$$y' + \frac{3}{t}y = 5t - 3$$

$$e^{\int \frac{3}{t}\,dt} = e^{3\ln t}$$

$$= e^{\ln t^3}$$

$$= t^3$$

$$\int (4t-3)t^3\,dt = \int \left(5t^4 - 3t^3\right)dt$$

$$= t^5 - \frac{3}{4}t^4$$

$$y(t) = \frac{1}{t^3}\left(t^5 - \frac{3}{4}t^4 + C\right)$$

$$= t^2 - \frac{3}{4}t + \frac{C}{t^3}$$

$$y(0) = 1 - \frac{3}{4} + C = 0$$

$$C = -\frac{1}{4}$$

$$y(t) = t^2 - \frac{3}{4}t - \frac{1}{4t^3}$$

Example 13

Find the general solution of $y' = -\frac{2}{x}\, y + \ln(x)$.

Solution

$$y' + \frac{2}{x}\, y = \ln x$$

$$e^{\int \frac{2}{x}\, dx} = e^{2\ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

$$\int x^2 \ln x \, dx$$

$$u = \ln x \qquad v = \int x^2\, dx$$

$$du = \frac{1}{x}\, dx \qquad = \frac{1}{3} x^3$$

$$\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2\, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3$$

$$y(x) = \frac{1}{x^2} \left(\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C \right)$$

$$= \frac{1}{3} x \ln x - \frac{1}{9} x + \frac{C}{x^2}$$

Exercises *Section* 6.7 – First-Order Linear Equations

$(\mathbf{1-2})$ Write an equivalent first-order differential equation and initial condition for y.

1. $y = \int_1^x \frac{1}{t}\,dt$

2. $y = 2 - \int_0^x \left(1 + y(t)\right) \sin t\,dt$

$(\mathbf{3-6})$ Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals.

3. $y' = 1 - \frac{y}{x}$, $y(2) = -1$, $dx = 0.5$

4. $y' = x(1-y)$, $y(1) = 0$, $dx = 0.2$

5. $y' = y^2(1+2x)$, $y(-1) = 1$, $dx = 0.5$

6. $y' = ye^x$, $y(0) = 2$, $dx = 0.5$

7. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

8. Use Euler's Method to solve $y' = 1 + y$, $y(0) = 1$ on the interval $0 \le x \le 1$ and taking $dx = 0.05$. Compare the approximation to the values of the exact solution.

9. Use Euler's Method to solve $y' = 2xy + 2y$, $y(0) = 3$ on the interval $0 \le x \le 1$ and taking $dx = 0.1$. Compare the approximation to the values of the exact solution.

$(\mathbf{10-16})$ Verify that the given function y is a solution of the differential equation that follows it. Assume that C, C_1, *and* C_2 are arbitrary constants.

10. $y = Ce^{-5t}$; $y'(t) + 5y = 0$

11. $y = Ct^{-3}$; $ty'(t) + 3y = 0$

12. $y = C_1 \sin 4t + C_2 \cos 4t$; $y''(t) + 16y = 0$

13. $y = C_1 e^{-x} + C_2 e^x$; $y''(x) - y = 0$

14. $y' + 4y = \cos t$, $y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}$, $y(0) = -1$

15. $ty' + (t+1)y = 2te^{-t}$, $y(t) = e^{-t}\left(t + \frac{C}{t}\right)$, $y(1) = \frac{1}{e}$

16. $y' = y(2+y)$, $y(t) = \frac{2}{-1 + Ce^{-2t}}$, $y(0) = -3$

$(\mathbf{17-20})$ Verify that the given function y is a solution of the initial value problem that follows it.

17. $y = 16e^{2t} - 10$; $y' - 2y = 20$, $y(0) = 6$

18. $y = 8t^6 - 3$; $ty' - 6y = 18$, $y(1) = 5$

19. $y = -3\cos 3t$; $y'' + 9y = 0$, $y(0) = -3$, $y'(0) = 0$

20. $y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right)$; $y'' - 4y = 0$, $y(0) = 0$, $y'(0) = 1$

$(\mathbf{21-115})$ Solve the differential equations.

21. $y' = xy$

22. $xy' = 2y$

23. $y' = e^{x-y}$

24. $y' = \left(1 + y^2\right)e^x$

25. $y' = xy + y$

26. $y' = ye^x - 2e^x + y - 2$

27. $y' = \dfrac{x}{y+2}$

28. $y' = \dfrac{xy}{x-1}$

29. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

30. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

31. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

32. $(1+x)y' + y = \sqrt{x}$

33. $e^{2x}y' + 2e^{2x}y = 2x$

34. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

35. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

36. $y' = \cos x - y\sec x$

37. $\left(1 + x^3\right)y' = 3x^2y + x^2 + x^5$

38. $\dfrac{dy}{dt} - 2y = 4 - t$

39. $y' + y = \dfrac{1}{1+e^t}$

40. $y' = 3y - 4$

41. $y' = -2y - 4$

42. $y' = -y + 2$

43. $y' = 2y + 6$

44. $x(x-1)dy - ydx = 0$

45. $xy' + 2y = 1 - x^{-1}$

46. $xy' - y = 2x\ln x$

47. $\left(1 + e^x\right)dy + \left(ye^x + e^{-x}\right)dx = 0$

48. $\left(x + 3y^2\right)dy + ydx = 0$

49. $y' = \dfrac{y^2 + ty + t^2}{t^2}$

50. $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$

51. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

52. $\dfrac{dy}{dx} = \sin 5x$

53. $\dfrac{dy}{dx} = (x+1)^2$

54. $dx + e^{3x}dy = 0$

55. $dy-(y-1)^2\,dx=0$

56. $x\dfrac{dy}{dx}=4y$

57. $\dfrac{dx}{dy}=y^2-1$

58. $\dfrac{dy}{dx}=e^{2y}$

59. $\dfrac{dy}{dx}+2xy^2=0$

60. $\dfrac{dy}{dx}=e^{3x+2y}$

61. $e^x y\dfrac{dy}{dx}=e^{-y}+e^{-2x-y}$

62. $y\ln x\dfrac{dx}{dy}=\left(\dfrac{y+1}{x}\right)^2$

63. $\dfrac{dy}{dx}=\left(\dfrac{2y+3}{4x+5}\right)^2$

64. $\csc y\,dx+\sec^2 x\,dy=0$

65. $\sin 3x\,dx+2y\cos^3 3x\,dy=0$

66. $\dfrac{dy}{dx}=(64xy)^{1/3}$

67. $\dfrac{dy}{dx}=2x\sec y$

68. $\dfrac{dy}{dx}=\dfrac{x}{ye^{x+2y}}$

69. $y'-y=3e^t$

70. $y'-y=e^{2t}-1$

71. $y'+y=te^{-t}+1$

72. $y'+y=1+e^{-x}\cos 2x$

73. $y'+y\cot x=\cos x$

74. $y'+y\sin t=\sin t$

75. $y'+(\cot t)\,y=2t\,\csc t$

76. $y'+(1+\sin t)\,y=0$

77. $y'+\left(\dfrac{1}{2}\cos x\right)y=-\dfrac{3}{2}\cos x$

78. $\dfrac{dy}{dx}+y=e^{3x}$

79. $y'-ty=t$

80. $y'=2y+x^2+5$

81. $xy'+2y=3$

82. $y'+2y=1$

83. $y'+2y=e^{-t}$

84. $y'+2y=e^{-2t}$

85. $y'-2y=e^{3t}$

86. $y'+2y=e^{-x}+x+1$

87. $y'+2xy=x$

88. $y'-2ty=t$

89. $y'+2ty=5t$

90. $y'-2xy=e^{x^2}$

91. $y'+2xy=x^3$

92. $y'-2y=t^2e^{2t}$

93. $x'-2\dfrac{x}{t+1}=(t+1)^2$

94. $y'+\dfrac{2}{t}y=\dfrac{\cos t}{t^2}$

95. $y'-2(\cos 2t)\,y=0$

96. $y'+2y=\cos 3t$

97. $y'-3y=5$

98. $y'+3y=2xe^{-3x}$

99. $y'+3x^2y=x^2$

100. $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

101. $y' + \frac{3}{x}y = 1 + \frac{1}{x}$

102. $y' + \frac{3}{2}y = \frac{1}{2}e^x$

103. $y' + 5y = t + 1$

104. $xy' - y = x^2 \sin x$

105. $\frac{dr}{d\theta} = \left(r + e^{-\theta}\right)\tan\theta$

106. $\tan\theta \frac{dr}{d\theta} - r = \tan^2\theta$

107. $\frac{dy}{dx} - \frac{2x}{x^2+4}y = 1$

108. $y' - \lambda y = e^{-t}$

109. $\frac{dy}{dx} + \frac{1}{\left(x^2+1\right)\arctan x}y = 1$

110. $(\cos x)y' + y = 1 - \sin x$

111. $y' + 2y = xe^{-x} + 1$

112. $y' + 3y = x + e^{-2x}$

113. $y' + 2y = x^2 + e^{3x} + \sin 2x$

114. $y' + 2y = xe^{-x} + 1$

115. $\left(x^2+9\right)y' + 2xy = x^2\left(x^2+9\right)$

(116 – 220) Solve the initial value problem.

116. $y' = \frac{1-2t}{y}, \quad y(1) = -2$

117. $y' = y^2 - 4, \quad y(0) = 0$

118. $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$

119. $y' = \frac{x}{1+2y}, \quad y(-1) = 0$

120. $\left(e^{2y} - y\right)\cos x \frac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0$

121. $\frac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$

122. $\frac{dy}{dx} + 2y = 1, \quad y(0) = \frac{5}{2}$

123. $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

124. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1) = 0$

125. $e^{-2t}\frac{dy}{dt} = \frac{1+e^{-2t}}{y}, \quad y(0) = 0$

126. $\frac{dy}{dt} = \frac{t+2}{y}, \quad y(0) = 2$

127. $\frac{1}{t^2}\frac{dy}{dt} = y, \quad y(0) = 1$

128. $\frac{dy}{dt} = -y^2e^{2t}; \quad y(0) = 1$

129. $\frac{dy}{dt} - (2t+1)y = 0; \quad y(0) = 2$

130. $\frac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

131. $\frac{dy}{dx} = ye^x; \quad y(0) = 2e$

132. $\frac{dy}{dx} = 3x^2\left(y^2+1\right); \quad y(0) = 1$

133. $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}; \quad y(5) = 2$

134. $\frac{dy}{dx} = 4x^3y - y; \quad y(1) = -3$

135. $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

136. $(\tan x)\frac{dy}{dx} = y; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

137. $x\frac{dy}{dx} - y = 2x^2y; \quad y(1) = 1$

138. $\frac{dy}{dx} = 2xy^2 + 3x^2y^2; \quad y(1) = -1$

139. $\frac{dy}{dx} = 6e^{2x-y}; \quad y(0) = 0$

140. $2\sqrt{x}\dfrac{dy}{dx}=\cos^2 y;\quad y(4)=\dfrac{\pi}{4}$

141. $y'+3y=0;\quad y(0)=-3$

142. $2y'-y=0;\quad y(-1)=2$

143. $\sqrt{y}dx+(1+x)dy=0;\quad y(0)=1$

144. $\dfrac{dy}{dx}=6y^2x,\quad y(1)=\dfrac{1}{25}$

145. $\dfrac{dy}{dx}=\dfrac{3x^2+4x-4}{2y-4},\quad y(1)=3$

146. $y'-3y=4\,;\quad y(0)=2$

147. $y'=y+2xe^{2x};\quad y(0)=3$

148. $\left(1+t^2\right)y'+4ty=\left(1+t^2\right)^{-2},\quad y(1)=0$

149. $y'+y=e^t,\quad y(0)=1$

150. $y'+\dfrac{1}{2}y=t,\quad y(0)=1$

151. $y'=x+5y,\quad y(0)=3$

152. $y'=2x-3y,\quad y(0)=\dfrac{1}{3}$

153. $xy'+y=e^x,\quad y(1)=2$

154. $y\dfrac{dx}{dy}-x=2y^2,\quad y(1)=5$

155. $xy'+y=4x+1,\quad y(1)=8$

156. $y'+4xy=x^3e^{x^2},\quad y(0)=-1$

157. $(x+1)y'+y=\ln x,\quad y(1)=10$

158. $y'-(\sin x)y=2\sin x,\quad y\left(\dfrac{\pi}{2}\right)=1$

159. $y'+y=2,\quad y(0)=0$

160. $y'-2y=3e^{2x},\quad y(0)=0$

161. $xy'+2y=3x,\quad y(1)=5$

162. $xy'+5y=7x^2,\quad y(2)=5$

163. $xy'-y=x,\quad y(1)=7$

164. $xy'+y=3xy,\quad y(1)=0$

165. $xy'+3y=2x^5,\quad y(2)=1$

166. $y'+y=e^x,\quad y(0)=1$

167. $xy'-3y=x^3,\quad y(1)=10$

168. $y'+2xy=x,\quad y(0)=-2$

169. $y'=(1-y)\cos x,\quad y(\pi)=2$

170. $(1+x)y'+y=\cos x,\quad y(0)=1$

171. $y'=1+x+y+xy,\quad y(0)=0$

172. $xy'=3y+x^4\cos x,\quad y(2\pi)=0$

173. $y'=2xy+3x^2e^{x^2},\quad y(0)=5$

174. $\left(x^2+4\right)y'+3xy=x,\quad y(0)=1$

175. $\left(x^2+1\right)y'+3x^3y=6xe^{-3x^2/2},\quad y(0)=1$

176. $y'-2y=e^{3x};\quad y(0)=3$

177. $y'-3y=6;\quad y(0)=1$

178. $2y'+3y=e^x;\quad y(0)=0$

179. $y'+y=1+e^{-x}\cos 2x;\quad y\left(\dfrac{\pi}{2}\right)=0$

180. $2y'+(\cos x)y=-3\cos x;\quad y(0)=-4$

181. $y'+2y=e^{-x}+x+1;\quad y(-1)=e$

182. $y'+\dfrac{y}{x}=xe^{-x};\quad y(1)=e-1$

183. $y'+4y=e^{-x};\quad y(1)=\dfrac{4}{3}$

184. $x^2y'+3xy=x^4\ln x+1;\quad y(1)=0$

185. $y'+\dfrac{3}{x}y=3x-2\quad y(1)=1$

186. $y'-(\sin x)y=2\sin x,\quad y\left(\dfrac{\pi}{2}\right)=1$

187. $y'+(\tan x)y=\cos^2 x,\quad y(0)=-1$

188. $t\,y'+2y=t^2-t+1\quad y(1)=\dfrac{1}{2}$

189. $y'+(\cos t)y=\cos t\,;\quad y(\pi)=2$

190. $y'+2ty=2t\,;\quad y(0)=1$

191. $y'+y=\dfrac{e^{-t}}{t^2}\,;\quad y(1)=0$

192. $ty'+2y=\sin t\,;\quad y(\pi)=\dfrac{1}{\pi}$

193. $t\dfrac{dy}{dt}+2y=t^3,\quad t>0,\quad y(2)=1$

194. $\theta\dfrac{dy}{d\theta}+y=\sin\theta,\quad \theta>0,\quad y\left(\frac{\pi}{2}\right)=1$

195. $\dfrac{dy}{dx}+xy=x,\quad y(0)=-6$

196. $y'=\dfrac{y}{x},\quad y(1)=-2$

197. $y'=\dfrac{\sin x}{y},\quad y\left(\frac{\pi}{2}\right)=1$

198. $y'=y+2xe^{2x};\quad y(0)=3$

199. $\left(x^2+1\right)y'+3xy=6x;\quad y(0)=-1$

200. $y'=\left(4t^3+1\right)y,\quad y(0)=4$

201. $y'=\dfrac{e^t}{2y},\quad y(\ln 2)=1$

202. $(\sec x)y'=y^3,\quad y(0)=3$

203. $\dfrac{dy}{dx}=e^{x-y},\quad y(0)=\ln 3$

204. $y'=2e^{3y-t},\quad y(0)=0$

205. $y'=3y-6,\quad y(0)=9$

206. $y'=-y+2,\quad y(0)=-2$

207. $y'=-2y-4,\quad y(0)=0$

208. $\dfrac{dy}{dx}+3x^2y=x^2,\quad y(0)=-1$

209. $xdy+(y-\cos x)dx=0,\quad y\left(\frac{\pi}{2}\right)=0$

210. $\dfrac{dy}{dt}=\dfrac{t+1}{2ty},\quad y(1)=4$

211. $\dfrac{dy}{dt}=\sqrt{y}\ \sin t,\quad y(0)=4$

212. $y'(t)+3y=0,\quad y(0)=6$

213. $y'(t)=2y+4,\quad y(0)=8$

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214. $y'+(\tan x)y=\sec x+\cos x\quad y(0)=1$

215. $\sin\theta\dfrac{dr}{d\theta}+(\cos\theta)r=\theta\sin\theta\quad r\left(\frac{\pi}{2}\right)=2$

216. $\left(1+t^2\right)y'+4ty=\left(1+t^2\right)^{-2},\quad y(1)=0$

217. $\left(x^2+1\right)y'+3x^3y=6xe^{-3x^2/2},\quad y(0)=1$

218. $(\cos x)y'+y\sin x=2x\cos^2 x\quad y\left(\frac{\pi}{4}\right)=\dfrac{-15\sqrt{2}\pi^2}{32}$

219. $(\cos x)y'+(\sin x)y=2\cos^3 x\sin x-1\quad y\left(\frac{\pi}{4}\right)=3\sqrt{2}$

220. $t\,y'-2y=t^5\sin 2t-t^3+4t^4\quad y(\pi)=\frac{3}{2}\pi^4$

Section 6.8 – Applications

6.8–1 Motion with Resistance Proportional to Velocity

An object with a mass ***m*** is moving along a coordinate line with position function ***s*** and velocity ***v*** at time ***t***.
From ***Newton's second law*** of motion, the resistance force opposing the motion is

$$\begin{aligned} Force &= mass \times acceleration \\ &= m\frac{dv}{dt} \\ &= m\,a \end{aligned}$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \quad or \quad \frac{dv}{dt} = -\frac{k}{m}v \quad (k > 0)$$

This is a separable differential equation, the solution with initial condition $v = v_0 \quad at \quad t = 0$ is

$$v = v_0 e^{-(k/m)t}$$

In the English system, where the weight is measured in pounds, mass is measured in slugs. Thus,

$$Pounds = slugs \times 32$$

Example 1

For a 192-*lb* ice skater, the ***k*** is about $\frac{1}{3}$ slug/sec and $m = \frac{192}{32} = 6 \;\; slugs$.

a) How long will it take the skater to coast from 11 *ft/sec* (7.5 *mph*) to 1 *ft/sec*?
b) What is the distance the skater traveled?
c) How far will the skater coast before coming to a complete stop?

Solution

a) $\frac{k}{m} = \frac{\frac{1}{3}}{6}$

$= \frac{1}{18}$

$v = v_0 e^{-(k/m)t}$

$1 = 11e^{-(1/18)t}$

$$e^{-(1/18)t} = \frac{1}{11}$$

$$-\frac{1}{18}t = \ln\left(\frac{1}{11}\right)$$

$$= -\ln 11$$

$$t = 18\ln 11$$

$$\approx 43 \quad \text{sec}$$

b) $v = 11e^{-t/18}$

$$s = 11\int_0^{18\ \ln 11} e^{-t/18}\,dt$$

$$= -198\ e^{-t/18}\Big|_0^{18\ \ln 11}$$

$$= -198\left(e^{-\ln 11} - 1\right)$$

$$= -198\left(\frac{1}{11} - 1\right)$$

$$= -198\left(-\frac{10}{11}\right)$$

$$= 180 \quad ft$$

c) $Distance = \dfrac{v_0 m}{k}$

$$= \frac{11(6)}{1/3}$$

$$= 198 \quad ft$$

6.8–2 Mixtures Problems

A typical mixing problem investigates the behavior of a mixed solution of some substance. Typically, the solution is being mixed in a tank. A solution of a given concentration enters the mixture at some fixed rate and is thoroughly mixed in the tank. The tank is also being drained at some fixed rate.

The physical representation of the rate of change:

$$\frac{dx}{dt} = rate\ of\ change$$
$$= rate\ in - rate\ out$$

This is referred to as a ***balance law***.

The rate in and out is given by:

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb./gal*)

If $y(t)$ is the amount of chemical in the container at time *t* and $V(t)$ is the total volume of liquid in the container at time *t*, then the departure rate of the chemical at time *t* is

$$Departure\ Rate = \frac{y(t)}{V(t)} \bullet (outflow\ rate)$$
$$= \begin{pmatrix} concentration\ in \\ container\ at\ \ t \end{pmatrix} \bullet (outflow\ rate)$$

Example 2

In an oil refinery, a storage tank contains 2,000 *gal* of gasoline that initially has 100 *lb*. of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 *lb*. of additive per gallon is pumped into the tank at a rate of 40 *gal/min*. The well-mixed solution is pumped out at a rate of 45 *gal/min*. How much of the additive is in the tank 20 *min* after the pumping process begins?

Solution

Let *y* be the amount (in *lb*.) of additive in the tank at time *t* and $y(0) = 100$

$$V(t) = 2{,}000 + \left(40\frac{gal}{\min} - 45\frac{gal}{\min}\right)(t\ \min)$$
$$= 2{,}000 - 5t$$

$$Rate\ out = \frac{y(t)}{V(t)} \bullet outflow\ rate$$

$$= \left(\frac{y}{2{,}000-5t}\right)(45)$$

$$= \frac{45y}{2{,}000-5t}\ \frac{lb}{\min}$$

$$= \frac{9y}{400-t}\ \frac{lb}{\min}$$

$$Rate\ in = \left(2\ \frac{lb}{gal}\right)\left(40\ \frac{gal}{\min}\right)$$

$$= 80\ \frac{lb}{\min}$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2{,}000-5t}$$

$$\frac{dy}{dt} + \frac{45}{2{,}000-5t}y = 80$$

$$e^{\int \frac{45}{2{,}000-5t}\,dt} = e^{\int \frac{45}{2{,}000-5t}\left(-\frac{1}{5}d(2{,}000-5t)\right)} \qquad d(2000-5t) = -5dt$$

$$= e^{\int \frac{45}{2{,}000-5t}\left(-\frac{1}{5}d(2{,}000-5t)\right)}$$

$$= e^{-\int \frac{9}{2{,}000-5t}d(2{,}000-5t)}$$

$$= e^{-9\ln(2{,}000-5t)}$$

$$= (2{,}000-5t)^{-9}$$

$$y(t) = \frac{1}{(2{,}000-5t)^{-9}}\left(\int 80.(2{,}000-5t)^{-9}\,dt + C\right)$$

$$= (2{,}000-5t)^{9}\left(80\int (2{,}000-5t)^{-9}\left(-\frac{1}{5}d(2{,}000-5t)\right) + C\right)$$

$$= (2{,}000-5t)^{9}\left(-16\ \frac{(2{,}000-5t)^{-8}}{-8} + C\right)$$

$$= (2{,}000-5t)^{9}\left(2(2{,}000-5t)^{-8} + C\right)$$

$$= 2(2{,}000 - 5t) + C(2{,}000 - 5t)^9$$

$$y(0) = 2(2{,}000 - 5(0)) + C(2{,}000 - 5(0))^9$$

$$100 = 4{,}000 + C(2{,}000)^9$$

$$C(2{,}000)^9 = -3{,}900$$

$$C = -\frac{3{,}900}{(2{,}000)^9}$$

$$y = 2(2{,}000 - 5t) - \frac{3{,}900}{(2{,}000)^9}(2{,}000 - 5t)^9$$

$$y(20) = 2(2{,}000 - 5(20)) - \frac{3{,}900}{(2{,}000)^9}(2{,}000 - 5(20))^9$$

$$\approx 1{,}342 \quad lbs.$$

6.8–3 *RL* Circuits

The diagram represents an electrical circuit whose total resistance is a constant ***R*** ohm and whose self-inductance is coil, is ***L*** henries.

Ohm's Law: $V = RI$

$$L\frac{di}{dt} + Ri = V$$

i: current in amperes
t: time in seconds

Example 3

The switch in the *RL* circuit is closed at time $t = 0$. How will the current flow as a function of time?

Solution

$$\frac{di}{dt}+\frac{R}{L}i=\frac{V}{L}$$

$$e^{\int p\,dt}=e^{\int\left(\frac{R}{L}\right)dt}$$

$$=e^{\frac{R}{L}t}$$

$$i(t)=\frac{1}{e^{\frac{R}{L}t}}\left(\int\frac{V}{L}.e^{\frac{R}{L}t}dt+C\right)$$

$$=e^{-\frac{R}{L}t}\left(\frac{V}{L}\frac{L}{R}e^{\frac{R}{L}t}+C\right)$$

$$=e^{-\frac{R}{L}t}\left(\frac{V}{R}e^{\frac{R}{L}t}+C\right)$$

$$i(0)=0$$

$$i=e^{-\frac{R}{L}(0)}\left(\frac{V}{R}e^{\frac{R}{L}(0)}+C\right)$$

$$0=\frac{V}{R}+C$$

$$C=-\frac{V}{R}$$

$$i(t)=e^{-\frac{R}{L}t}\left(\frac{V}{R}e^{\frac{R}{L}t}-\frac{V}{R}\right)$$

$$=\frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}$$

$$\lim_{t\to\infty} i=\lim_{t\to\infty}\left(\frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}\right)$$

$$=\frac{V}{R}-\frac{V}{R}\cdot 0$$

$$=\frac{V}{R}$$

The current approaches the ***steady-state value*** $\frac{V}{R}$.

Exercises *Section* 6.8 – Applications

1. A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg\ /\ \sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

2. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59{,}000\ kg\ /\ \sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 *m/sec*?

3. A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

4. A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

5. An Executive conference room of a corporation contains 4,500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3\ /\ \min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3\ /\ \min$. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

6. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \textit{or} \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

c) If $a = b$

d) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$.

7. The tank initially holds 100 *gal* of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

8. The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

9. A tank with a 2,000-*gal* capacity initially contains 500 *gal* of brine containing 100 *lbs*. of salt starting at time $t = 0$, brine containing 0.1 *lb/gal* of salt is added at a rate of 60 *gal/min* and the mixed solution is drained off at a rate of 40 *gal/min*. How much salt is in the tank when it reaches the point of overflowing?

10. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem.

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \ \ for\ t \geq 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.

b) What is the steady-state level of the drug?

c) When does the drug level reach 90% of the steady-state value?

11. A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for \ \ t \geq 0$$

Where t is measured in years.

a) Find the fish population for $t \geq 0$ in terms of the harvesting rate b.

b) Graph the solution in the case that $b = 40\ fish\ /\ yr$. Describe the solution.

c) Graph the solution in the case that $b = 60\ fish\ /\ yr$. Describe the solution.

12. A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \geq 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

a) Find the solution of the initial value problem.

b) What is the steady-state population?

13. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \;\; P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

14. An object of mass ***m*** is released from a balloon. Find the distance it falls in ***t*** seconds, if the force of resistance due to the air is directly proportional to the speed of the object.

15. A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to *g*, the earth's acceleration?

16. A body falls from an initial velocity of $1{,}000\ ft/s$. What distance has it traveled after 3 *sec*. if subject to $g = 32\ ft/s^2$, the earth's acceleration?

17. A projectile is fired straight upwards with an initial velocity of $1{,}600\ ft/s$. What is its velocity at $40{,}000\ ft$. $\left(g = 32\ ft/s^2\right)$

18. A projectile is fired straight upwards with an initial velocity of $1{,}000\ ft/s$. What is its velocity at $8{,}000\ ft$. $\left(g = 32\ ft/s^2\right)$

19. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

20. A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

21. A ball is projected vertically upward with initial velocity from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

c) What is the speed of the ball when it impacts with the ground on its return?

22. An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 *seconds*.

b) How long does it take the object to reach 80% of its terminal velocity?

23. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the object falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance, and the positive direction is downward. One common assumption (often used for motion v_0 in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0\ and\ \ 0 < v(t)^2 < \frac{g}{a}$ for $t \geq 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2,\ \ m = 1\ kg,\ \ and\ \ k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

24. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$

a) Find the solution of the initial value problem.
b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.
c) In general, how long does it take the tank to drain in terms of k and H?

25. The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \geq 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a) Show that for a first-order reaction $k = 0.1\text{n}\ (n = 1)$, the concentration obeys an exponential decay law.
b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$
c) Graph and compare the concentration for a first-order and second-order reaction with and $y_0 = 1$

26. The consumption of a substrate in a reaction involving an enzyme is often modeled using Michaelis-Menton Kinetics, which involves the initial value problem $\frac{ds}{dt} = \frac{Qs}{K+s}$, $s(0) = s_0$, where $s(t)$ is the amount of substrate present at time $t \geq 0$, and Q and K are positive constants. Solve the initial value problem with $Q = 10$, $K = 5$, and $s_0 = 50$. Notice that the solution can be expressed explicitly only with t as a function of s. Describe how s behaves as $t \to \infty$.

27. An investment account, which earns interest and has regular deposits, can be modeled by the initial value problem $B'(t) = aB + m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the monthly interest rate, m is the rate of monthly deposits, and B_0 is the initial balance in the account. Solve the initial value problem with $a = 0.005$, $m = \$100 / month$, and $B_0 = \$100$. After how many months does the account have a balance of $7,500?

28. The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln \frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM \ln \frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

29. An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a) Solve the initial value problem with $a = 0.05$, $m = \$1000 / yr.$ and $B_0 = \$15,000$. Does the balance in the account increase or decrease?

b) If $a = 0.05$ and $B_0 = \$50,000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

30. The halibut fishery has been modeled by the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

31. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$ where t is measured in *years*.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

32. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

33. An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time t for the given $E(t) = 10 - 2t$

34. An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\cos 3t$

35. An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\sin 2\pi t$

36. An RL circuit with a $1-\Omega$ resistor and a $0.01\text{-}H$ inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

37. An RL circuit with a $5-\Omega$ resistor and a $0.05\text{-}H$ inductor is driven by a voltage $E(t) = 5\cos 120t\ V$. If the initial inductor current is 1 A, determine the subsequence resistor and inductor current and the voltages.

38. For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

39. For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

40. A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = \frac{dQ}{dt}$, so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 F, a battery gives voltage of 60 V and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 F, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

41. A 30–volt electromotive force is applied to an LR-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0) = 0$

b) Determine the current as $t \to \infty$

c) Solve the equation when $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$

42. A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of $4\ gal/\min$. As the second solution is being added, the tank is being drained at a rate of $5\ gal/\min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?

43. A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing $0.5\ lb/gal$ enters the tank at the rate of $5\ gal/\min$, and well-stirred mixture is withdrawn at the rate of $3\ gal/\min$.

a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

44. A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing $1\ lb/gal$ enters the tank at the rate of $5\ gal/\min$, and well-stirred mixture is withdrawn at the rate of $3\ gal/\min$.

a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

45. A 200-*gallon* tank is full of a concentrate solution containing $25\ lb$. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of $10\ gal/\min$, and well-stirred mixture is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of *t*.
b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.
c) Find the quantity of the concentrate in the solution as $t \to \infty$.

46. A tank contains 300 *litres* of fluid in which 20 *grams* of salt is dissolved. Brine containing 1 *g* of salt per *litre* is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate. Find the number $x(t)$ of grams of salt in the tank at time *t*.

47. A 1,500-*gallon* tank initially contains 600 *gallons* of water with 5 *lbs*. of salt dissolved in it. Water enters the tank at a rate of 9 *gal/hr.* and the water entering the tank at a rate has a salt concentration of $\frac{1}{5}(1+\cos t)$ *lbs./gal*. If a well-mixed solution leaves the tank at a rate of 6 *gal/hr.*, how much salt is in the tank when it overflows?

48. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly always mixed.

a) What will be the sugar content in the tank after 20 *minutes*?
b) How long will it take the sugar content in the tank to reach 15 *lbs.*?
c) What will be the eventual sugar content in the tank?

49. A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lbs*. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal* per *minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.

a) How much sugar is in the tank after 10 *minutes*?
b) How long will it take the sugar content in the tank to dip below 20 *lb*.?
c) What will be the eventual sugar content in the tank?

50. Suppose that in the cascade tank 1 initially 100 *gal* of pure ethanol and tank 2 initially contains 100 *gal* of pure water. Pure water flows into tank 1 at 10 *gal/min*, and the other two flow rates are also 10 *gal/min*.

a) Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.

b) Find the maximum amount of ethanol ever in tank 2.

Summary

$\sin^2 x + \cos^2 x = 1$ $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\quad \tan^2 x = \sec^2 x - 1$

$\sin ax = \dfrac{\tan(ax)}{\sqrt{1 + \tan^2(ax)}}$ $\quad \cos ax = \dfrac{1}{\sqrt{1 + \tan^2(ax)}}$

$\displaystyle\int \frac{du}{u^2} = -\frac{1}{u}$ $\quad \displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ $\quad \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$ $\quad \displaystyle\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

$\displaystyle\int u\, dv = u\, v - \int v\, du$ $\quad \displaystyle\int \tan^n x\, dx = \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$

$$\int x^n e^{ax}\, dx = e^{ax} \sum_{k=0}^{n} (-1)^k \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{a^{k+1}} \cdot x^{n-k}$$

If n is odd $(n \geq 3)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right)$

If n is even $(n \geq 2)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

$\sqrt{a^2 + (bx)^2}$	$\sqrt{a^2 - (bx)^2}$	$\sqrt{(bx)^2 - a^2}$
$b\,x = a\,\tan\theta$	$b\,x = a\,\sin\theta$	$b\,x = a\,\sec\theta$
$\sqrt{a^2 + (bx)^2} = a\,\sec\theta$	$\sqrt{a^2 - (bx)^2} = a\,\cos\theta$	$\sqrt{(bx)^2 - a^2} = a\tan\theta$

Exercises *Lecture* 6 – Review

(1 – 71) Evaluate the integral.

1. $\int x^5 e^{4x}\,dx$

2. $\int \cos 2x\, e^{3x} dx$

3. $\int x^6 \cos 4x\,dx$

4. $\int \cos(\ln x)\,dx$

5. $\int \sin^4 x\,dx$

6. $\int \cos^7 x\,dx$

7. $\int \frac{dx}{\sqrt{x^2-25}}$

8. $\int \frac{dx}{x^2\sqrt{x^2+36}}$

9. $\int \frac{dx}{2-\sqrt{3x}}$

10. $\int \theta\cos(2\theta+1)\,d\theta$

11. $\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\,dx$

12. $\int \cot x\,\ln(\sin x)\,dx$

13. $\int \frac{x^4}{x^{10}+16}\,dx$

14. $\int \frac{dx}{4-5\sin x}$

15. $\int \frac{1}{x+4+4\sqrt{x+1}}\,dx$

16. $\int \frac{dx}{x\left(x^4+1\right)}$

17. $\int \cos x\,\cos 2x\,\cos 3x\,dx$

18. $\int \left(x^3+4x^2-3\right)\sin 2x\,dx$

19. $\int \frac{x}{x^4-16}\,dx$

20. $\int x^2\sqrt[3]{x+5}\,dx$

21. $\int \frac{x^2+1}{x^3-x^2}\,dx$

22. $\int \frac{x}{\sqrt{16-x^4}}\,dx$

23. $\int (x+\sin x)^3\,dx$

24. $\int t^5 e^{-2t}\,dt$

25. $\int x^5\cos x\,dx$

26. $\int \frac{x+8}{x^2+64}\,dx$

27. $\int \frac{x+2}{x^2+x+3}\,dx$

28. $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}}$

29. $\int \frac{x}{\left(x^2+1\right)\left(x^2+4\right)}\,dx$

30. $\int e^{\sqrt[3]{x}}\,dx$

31. $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

32. $\int \frac{x^3}{(x+1)^{10}}\,dx$

33. $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

34. $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

35. $\int \sqrt{\frac{1-x}{1+x}}\,dx$

36. $\int \sqrt{1+x-x^2}\,dx$

37. $\int x\sec x\,\tan x\,dx$

38. $\int \frac{3x+8}{x-5}\,dx$

39. $\int e^{x+e^x}\,dx$

40. $\displaystyle\int \frac{\sqrt{x-3}}{x+3}\,dx$

41. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$

42. $\displaystyle\int \frac{x}{x^2-2x+4}\,dx$

43. $\displaystyle\int \frac{x}{x^2+5x-6}\,dx$

44. $\displaystyle\int \sin\sqrt{x}\,dx$

45. $\displaystyle\int 2e^{3x}\cos 5x\,dx$

46. $\displaystyle\int x^2\ln(1+x)\,dx$

47. $\displaystyle\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$

48. $\displaystyle\int \frac{e^x}{e^{2x}-1}\,dx$

49. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$

50. $\displaystyle\int \frac{1}{\sqrt{9x^2+12x-5}}\,dx$

51. $\displaystyle\int \frac{1}{e^x-e^{-x}}\,dx$

52. $\displaystyle\int \sin^6 x\cos^4 x\,dx$

53. $\displaystyle\int \frac{1+\cos x}{\sin x}\,dx$

54. $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+4x}\,dx$

55. $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

56. $\displaystyle\int \ln\left(1+2x^2\right)dx$

57. $\displaystyle\int \frac{\sqrt{49-x^2}}{x}\,dx$

58. $\displaystyle\int \frac{dx}{x\sqrt{2x-36}}$

59. $\displaystyle\int \sec^3 ax\,dx$

60. $\displaystyle\int \cos 4x\cos 7x\,dx$

61. $\displaystyle\int x^5 e^{-x^2}\,dx$

62. $\displaystyle\int x\tan^{-1}x\,dx$

63. $\displaystyle\int \frac{x}{x^4+2x^2+10}\,dx$

64. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

65. $\displaystyle\int \frac{dx}{x^3+x^2+x+1}$

66. $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$

67. $\displaystyle\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$

68. $\displaystyle\int \frac{1}{x^3-27}\,dx$

69. $\displaystyle\int \frac{x^4-x^3-x-1}{x^3-x^2}\,dx$

70. $\displaystyle\int \frac{x^3-3}{x^4+x^2+2}\,dx$

71. $\displaystyle\int \frac{e^{3x}}{e^{2x}+4e^x-5}\,dx$

$(\mathbf{72-89})$ Evaluate the definite integral.

72. $\displaystyle\int_0^{\pi} e^{\cos t}\sin 2t\,dt$

73. $\displaystyle\int_0^{\pi/2} \cos^{15} x\,dx$

74. $\displaystyle\int_0^{\pi/2} \cos^{12} x\,dx$

75. $\displaystyle\int_1^{2} x^4\ln x\,dx$

76. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

77. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x\sec^4 x\,dx$

78. $\displaystyle\int_0^4 \frac{dx}{(x-1)^2}$

79. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

80. $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{x-1}}$

81. $\displaystyle\int_0^{\pi/2} \sec t\ dt$

82. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\ d\theta$

83. $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\ dx$

84. $\displaystyle\int_2^{\infty} \frac{dx}{x\ln^2 x}$

85. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(5-x)^2}$

86. $\displaystyle\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\ dx$

87. $\displaystyle\int_0^{\infty} x^3 e^{-x}\ dx$

88. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$

89. $\displaystyle\int_0^{\infty} \frac{x}{\left(1+4x^2\right)^2}\ dx$

(90 – 94) Solve the differential equation.

90. $x(x-1)\,dy - y\,dx = 0$

91. $xy' + 2y = 1 - x^{-1}$

92. $xy' - y = 2x\ln x$

93. $\left(1+e^x\right)dy + \left(ye^x + e^{-x}\right)dx = 0$

94. $\left(x+3y^2\right)dy + y\,dx = 0$

(95 – 98) Solve the differential equation with initial value.

95. $\dfrac{dy}{dx} + 3x^2 y = x^2, \quad y(0) = -1$

96. $xdy + (y - \cos x)dx = 0, \quad y\left(\frac{\pi}{2}\right) = 0$

97. $\dfrac{dy}{dt} = \dfrac{t+1}{2ty}, \quad y(1) = 4$

98. $\dfrac{dy}{dt} = \sqrt{y}\sin t, \quad y(0) = 4$

99. Find the length of the graph of the function $y = \ln\left(1-x^2\right), \quad 0 \le x \le \frac{1}{2}$

100. The region in the first quadrant that is enclosed by the x-axis, the curve $y = \dfrac{5}{x\sqrt{5-x}}$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid.
Find the volume of the solid.

101. The region between the x-axis and the curve

$$f(x)=\begin{cases} 0, & x=0 \\ x\ln x, & 0<x\le 2 \end{cases}$$

is revolved about the x-axis to generate the solid.

Find the volume of the solid.

102. A tank with a 2,000-*gal* capacity initially contains 500 *gal* of brine containing 100 *lbs*. of salt starting at time $t = 0$, brine containing 0.1 *lb/gal* of salt is added at a rate of 60 *gal/min* and the mixed solution is drained off at a rate of 40 *gal/min*. How much salt is in the tank when it reaches the point of overflowing?

103. An object of mass ***m*** is released from a balloon. Find the distance it falls in ***t*** seconds, if the force of resistance due to the air is directly proportional to the speed of the object.

Practice *Exam A*

Evaluate the integrals.

1. $\int x^4 e^{3x}\, dx$

2. $\int e^{2x} \cos 3x\, dx$

3. $\int x^2 \sin 3x\, dx$

4. $\int \sin^3 x\, dx$

5. $\int \sin^2 x \cos^2 x\, dx$

6. $\int_0^{\pi/2} \cos^9 x\, dx$

7. $\int \frac{2\, dx}{\sqrt{1-4x^2}}$

8. $\int \frac{dx}{\sqrt{x^2-81}}$

9. $\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2}\, dx$

10. $\int \frac{dx}{x^2-2x-3}$

11. $\int \frac{8x+5}{2x^2+3x+1}\, dx$

12. $\int_{-\infty}^{\infty} \frac{x\, dx}{\left(x^2+4\right)^{3/2}}$

13. $\int_0^{\infty} x e^{-x}\, dx$

14. Solve the differential equation: $y' - 2xy = e^{x^2}$.

15. Solve the differential equation: $y'(t) + 3y = 0, \quad y(0) = 6$.

Practice *Exam B*

Evaluate the integrals.

1. $\int x^3 e^{-4x}\, dx$

2. $\int e^{3x} \sin 2x\, dx$

3. $\int x^3 \cos 4x\, dx$

4. $\int \cos^3 x\, dx$

5. $\int \sin^2 x \cos^2 x\, dx$

6. $\int_0^{\pi/2} \cos^{10} x\, dx$

7. $\int \frac{dx}{\sqrt{4-x^2}}$

8. $\int \frac{dx}{\sqrt{4x^2-49}}$

9. $\int_0^{\sqrt{3}/2} \frac{1}{25+4x^2}\, dx$

10. $\int \frac{x\, dx}{x^2+3x+2}$

11. $\int \frac{8x+5}{2x^2+3x+1}\, dx$

12. $\int_{-\infty}^{\infty} \frac{x\, dx}{\left(x^2+4\right)^{3/2}}$

13. $\int_0^{\infty} x^2 e^{-x}\, dx$

14. Solve the differential equation: $y' + \frac{2}{x} y = 8x$.

15. Solve the differential equation: $2t\, y' + y = 0$, $y(4) = 1$.

Lecture Seven

Series

Section 7.1 – Infinite Sequences and Series

7.1−1 Sequences *Definition*

A sequence is a list of numbers

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

An ***infinite sequence*** of numbers is a function whose domain is the set of positive integers. These are the ***terms*** of the sequence.

The integer ***n*** is called the ***index*** of a_n .

***Example* 1**

$$a_n = \sqrt{n}$$

$$\left\{a_n\right\} = \left\{\sqrt{1},\ \sqrt{2},\ \sqrt{3},\ \ldots,\ \sqrt{n},\ \ldots\right\}$$

Also, we can write: $\left\{a_n\right\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}$

Example 2

$$a_n = (-1)^{n+1}\frac{1}{n}$$

$$\{a_n\} = \left\{1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots,\ (-1)^{n+1}\frac{1}{n},\ \ldots\right\}$$

$$a_n = (-1)^{n+1}\frac{1}{n}$$

Example 3

$$a_n = \frac{1}{n}$$

$$\{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$$

$$a_n = \frac{1}{n}$$

7.1–2 Series *Definition*

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is an ***infinite series***. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ is defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

$$\vdots$$

Is the ***sequence of partial sums*** of the series, the number $\{s_n\}$ being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L.

In this case, we also write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

7.1–3 Convergence and Divergence

$\left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ 1-\frac{1}{n},\ \ldots\right\}$ Terms approach 1.

$\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$ Terms approach 0.

7.1–4 *Definition*

The sequence $\{a_n\}$ ***converges*** to the number L if for every positive number ε there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \varepsilon$$

If no such number L exists, we say $\{a_n\}$ ***diverges***.

The $\{a_n\}$ ***converges*** to L, we write $\lim\limits_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the ***limit*** of the sequence.

Example 4

Show that $\lim_{n\to\infty} \frac{1}{n} = 0$

Solution

Let ε > 0 be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \varepsilon$$

This implication will hold if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. If N is any integer greater than $\frac{1}{\varepsilon}$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$

Example 5

Show that $\lim_{n\to\infty} k = k \quad (\textit{any constant } k)$

Solution

Let ε > 0 be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |k - k| < \varepsilon$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty} k = k$

7.1–5 *Definition*

The sequence $\{a_n\}$ ***diverges*** to infinity if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds, we write

$$\lim_{n\to\infty} a_n = \infty \quad or \quad a_n \to \infty$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ ***diverges to negative infinity*** and write

$$\lim_{n\to\infty} a_n = -\infty \quad or \quad a_n \to -\infty$$

7.1–6 The n*th*-Term Test for a Divergent Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example 6

a) The series $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$ diverges because each term is greater than 1,

so the sum of the n terms is greater than n.

$\frac{n+1}{n} \to 1$

b) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ doesn't exist.

d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

7.1–7 *Theorem*

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = A + B$

Difference Rule: $\lim_{n\to\infty} (a_n - b_n) = A - B$

Constant Multiple Rule: $\lim_{n\to\infty} (ka_n) = kA$

Product Rule: $\lim_{n\to\infty} (a_n \bullet b_n) = A \bullet B$

Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} \quad if \quad B \neq 0$

7.1–8 *Theorem*

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

Sum Rule:
$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$
$$= A + B$$

Difference Rule:
$$\sum (a_n - b_n) = \sum a_n - \sum b_n$$
$$= A - B$$

Constant Multiple Rule:
$$\sum ka_n = k \sum a_n$$
$$= kA$$

- Every nonzero constant multiple of a divergent series diverges.
- If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Example 7

Evaluate $\lim\limits_{n\to\infty} \left(-\frac{1}{n}\right)$

Solution

$$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right)$$

$$= -1(0)$$

$$= 0$$

Example 8

Evaluate $\lim\limits_{n\to\infty} \left(\frac{n-1}{n}\right)$

Solution

$$\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}$$

$$= 1 - 0$$

$$= 1$$

Example 9

Evaluate $\lim\limits_{n\to\infty} \left(\frac{5}{n^2}\right)$

Solution

$$\lim_{n\to\infty} \left(\frac{5}{n^2}\right) = 5 \cdot \lim_{n\to\infty} \left(\frac{1}{n^2}\right)$$

$$= 5 \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right) \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right)$$

$$= -1 \bullet 0 \bullet 0$$

$$= 0$$

Example 10

Evaluate $\lim\limits_{n\to\infty}\left(\dfrac{4-7n^6}{n^6+3}\right)$

Solution

$$\lim_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = \lim_{n\to\infty}\left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) \qquad \lim_{n\to\infty}\left(\frac{-7n^6}{n^6}\right)$$

$$= \frac{0-7}{1+0}$$

$$= -7$$

7.1–9 *Theorem* – The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ also.

Example 11

Since $\frac{1}{n} \to 0$, we know that

a) $\dfrac{\cos n}{n} \to 0$ *because* $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$

b) $\dfrac{1}{2^n} \to 0$ *because* $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$

c) $(-1)^n \frac{1}{n} \to 0$ *because* $-\frac{1}{n} \le (-1)^n \le \frac{1}{n}$

7.1–10 Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} a\, r^{n-1}$$

In which a and r are fixed real numbers, and $a \neq 0$.

The series can also be written as $\sum_{n=0}^{\infty} ar^n$

7.1–11 *Definition* of Geometric Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \frac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the nth Term of a Geometric Sequence:

$$a_n = a_1 r^{n-1}$$

7.1–12 *Theorem*: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$S_n = a \frac{1 - r^n}{1 - r}$$

Proof

By definition, the nth partial sum S_n of a geometric sequence is:

$$\begin{aligned} S_n &= a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \\ -\ rS_n &= ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ \hline S_n - rS_n &= a - ar^n \end{aligned}$$

$$(1-r)S_n = a\left(1-r^n\right)$$

$$S_n = a\frac{1-r^n}{1-r}$$

7.1–13 *Definition*

If $|r| < 1$, the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ converges to $\frac{a}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, the series diverges.

- The sequence may converge to a single value, which is the limit of the sequence.
- The sequence terms may increase in magnitude without bound (either with one sign or with mixed signs), in which case the sequence diverges.
- The sequence terms may remain bounded but settle into an oscillating pattern in which the terms approach two or more values, then the sequence diverges.
- The terms of a sequence may remain bounded, but wander chaotic forever without pattern, then the sequence diverges in this case.

Example 12

Find the geometric series with $a = \frac{1}{9}$ *and* $r = \frac{1}{3}$

Solution

$$\frac{1}{9} + \frac{1}{9}\frac{1}{3} + \frac{1}{9}\left(\frac{1}{3}\right)^2 + \cdots = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1}$$

$$= \frac{1/9}{1-1/3}$$

$$= \frac{1}{9}\,\frac{3}{2}$$

$$= \frac{1}{6}$$

Example 13

The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

Determine if the geometric series converges or diverges

Solution

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n$$

$\therefore$ By the *Geometric Series* $\left(|r| = \frac{1}{4} < 1\right)$, the given series ***converges*** with a sum

$$S = \frac{a}{1-r}$$

$$= \frac{5}{1-\left(-\frac{1}{4}\right)}$$

$$= \frac{5}{\frac{5}{4}}$$

$$= 5 \cdot \frac{4}{5}$$

$$= 4$$

Example 14

Find the sum of the "telescoping" series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$s_k = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1} \qquad \lim_{k\to\infty} \frac{1}{k+1} = 0$$

$$s_k = 1 - 0$$

$$= 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Example 15

Find the sums of the series $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

***Example* 16**

Find the sums of the series $\sum_{n=0}^{\infty} \frac{4}{2^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= 4\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 4\frac{1}{1-\frac{1}{2}}$$

$$= 4(2)$$

$$= 8$$

7.1–14 *Theorem* – The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example 17

Show that $\sqrt{\frac{n+1}{n}} \to 1$

Solution

We know that $\frac{n+1}{n} \to 1$.

Taking $f(x) = \sqrt{x}$ and $L = 1$ that gives

$$\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$$

Example 18

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

By taking $a_n = \frac{1}{n}$, $f(x) = 2^x$, and $L = 0$.

We see that

$$2^{1/n} = f\left(\frac{1}{n}\right)$$

$$f(L) = 2^0 = 1$$

The sequence $\left\{2^{1/n}\right\}$ converges to 1.

7.1–15 Using L'Hôpital's Rule

Theorem

Suppose that $f(x)$ is a function for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Example 19

Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

Solution

The function $\frac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers.

Therefore,

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

Example 20

Does the sequence whose nth term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n\to\infty} a_n$

Solution

The limit leads to the indeterminate form 1^∞.

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n\ln\left(\frac{n+1}{n-1}\right) \qquad \infty.0 \textit{ form}$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad 0.0 \textit{ form}$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \left(\ln\frac{n+1}{n-1}\right)' = \frac{\frac{n-1-(n+1)}{(n-1)^2}}{\frac{n+1}{n-1}} = \frac{-2}{(n+1)(n-1)}$$

$$= \lim_{n\to\infty} \frac{\frac{-2}{n^2-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1}$$

$$= 2$$

$$\lim_{n\to\infty} a_n = e^2$$

7.1–16 *Theorem*

The following six sequences converge to the limits listed below:

1. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim\limits_{n\to\infty} x^{1/n} = 1 \quad x > 0$

4. $\lim\limits_{n\to\infty} x^{n} = 1 \quad |x| < 1$

5. $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^{n} = e^{x} \quad (any\ x)$

6. $\lim\limits_{n\to\infty} \dfrac{x^{n}}{n!} = 0 \quad (any\ x)$

7.1–17 Bounded Monotonic Sequences

Definitions

A sequence $\{a_n\}$ is ***bounded from above*** if there exists a number M such that $a_n \le M$ for all n. The number M is an ***upper bound*** for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the ***least upper bound*** for $\{a_n\}$.

A sequence $\{a_n\}$ is ***bounded from below*** if there exists a number m such that $a_n \ge m$ for all n. The number m is an ***lower bound*** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the ***greatest lower bound*** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, the $\{a_n\}$ is ***bounded***.

If $\{a_n\}$ is not bounded, then $\{a_n\}$ is an ***unbounded*** sequence.

7.1–18 *Definition*

A sequence $\{a_n\}$ is ***nondecreasing*** if $a_n \le a_{n+1}$ for all *n*. That is $a_1 \le a_2 \le a_3 \le \ldots$

Which each term is greater than or equal to its predecessor $\left(a_{n+1} \ge a_n\right)$

Example: $\left\{1-\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots\right\}$

A sequence $\{a_n\}$ is ***nonincreasing*** if $a_n \ge a_{n+1}$ for all *n*, which each term is less than or equal to its predecessor $\left(a_{n+1} \le a_n\right)$

Example: $\left\{1+\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \ldots\right\}$

The sequence $\{a_n\}$ is ***monotonic*** if it is either nondecreasing or nonincreasing.

7.1–19 *Theorem*

If a sequence $\{a_n\}$ is both *bounded* and *monotonic*, then the sequence converges.

Example 21

The sequence $\{1, 2, 3, \ldots, n, \ldots\}$ is nondecreasing.

The sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$ is nondecreasing.

The sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\}$ is nonincreasing.

7.1–20 Adding or Deleting Terms

We can add finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

Example 22

Your drop a ball from *a* meter above a flat surface. Each time the ball hits the surface after falling a distance *h*, it rebounds a distance *rh*, where *r* is positive but less than 1. Find the total distance the ball travels up and down. $\left(a = 6\ m \quad and \quad r = \frac{2}{3}\right)$

Solution

The total distance is

$$s = a + 2ar + 2ar^2 + 2ar^3 + \ldots$$
$$= a + \frac{2ar}{1-r}$$
$$= a\left(1 + \frac{2r}{1-r}\right)$$
$$= a\frac{1+r}{1-r}$$

If $a = 6\ m \quad and \quad r = \frac{2}{3}$, the distance is:

$$s = 6\frac{1+\frac{2}{3}}{1-\frac{2}{3}}$$
$$= 30 \quad m$$

Exercises *Section* 7.1 – Sequences

1. Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \frac{1-n}{n^2}$

2. Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \frac{1}{n!}$

3. Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \frac{(-1)^{n+1}}{2n-1}$

4. Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = 2+(-1)^n$

5. Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \frac{2^n-1}{2^n}$

6. Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{2^n}$

7. Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

8. Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \frac{a_{n+1}}{a_n}$

9. Find a formula for the *n*th term of the sequence $-1,\ 1,\ -1,\ 1,\ -1, \cdots$

10. Find a formula for the *n*th term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25}, \cdots$

11. Find a formula for the *n*th term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21}, \cdots$

12. Find a formula for the *n*th term of the sequence $-3,\ -2,\ -1,\ 0,\ 1, \cdots$

13. Find a formula for the *n*th term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15{,}625}, \cdots$

14. Find a formula for the *n*th term of the sequence $0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ \cdots$

(15 – 19) Find a formula for the *n*th term partial sum of the series and use it to find the series' sum if the series converges

15. $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^n-1}+\cdots$

16. $\frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}+\cdots$

17. $1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$

18. $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$

19. $\frac{5}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots+\frac{5}{n(n+1)}+\cdots$

(20 – 23) Write out the first few terms of each series to show how the series starts. Then find the sum of the series

20. $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$

21. $\sum_{n=2}^{\infty}\frac{1}{4^n}$

22. $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$

23. $\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}$

(24 – 26) Determine if the geometric series converges or diverges. If a series converges, find its sum

24. $1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^4+\cdots$

25. $1+(-3)+(-3)^2+(-3)^3+(-3)^4+\cdots$

26. $\left(-\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^4+\left(-\frac{2}{3}\right)^5+\cdots$

(27 – 34) Use Divergence to determine if the series converges or diverges.

27. $\sum_{k=0}^{\infty}\frac{k}{2k+1}$

28. $\sum_{k=1}^{\infty}\frac{k}{k^2+1}$

29. $\sum_{k=2}^{\infty}\frac{k}{\ln k}$

30. $\sum_{k=1}^{\infty}\frac{k^2}{2^k}$

31. $\sum_{k=0}^{\infty}\frac{1}{k+100}$

32. $\sum_{k=0}^{\infty}\frac{k^3}{k^3+1}$

33. $\sum_{k=2}^{\infty}\frac{\sqrt{k}}{\ln^{10} k}$

34. $\sum_{k=1}^{\infty}\frac{\sqrt{k^2+1}}{k}$

(35 – 104) Determine if the sequence converge or diverge? Then find the limit of each convergent.

35. $a_n = \dfrac{n+(-1)^n}{n}$

36. $a_n = \dfrac{1-2n}{1+2n}$

37. $a_n = \dfrac{1-n^3}{70-4n^2}$

38. $a_n = \left(2-\dfrac{1}{2^n}\right)\left(3+\dfrac{1}{2^n}\right)$

39. $a_n = n\pi\cos(n\pi)$

40. $a_n = n-\sqrt{n^2-n}$

41. $a_n = \sqrt{\dfrac{2n}{n+1}}$

42. $a_n = \dfrac{\sin^2 n}{2^n}$

43. $a_n = \dfrac{n^2}{2^n-1}$

44. $a_n = \dfrac{\ln n}{\ln 2n}$

45. $a_n = \dfrac{3^n\cdot 6^n}{2^{-n}\cdot n!}$

46. $a_n = \sqrt{n}\ \sin\dfrac{1}{\sqrt{n}}$

47. $\{c_n\} = \left\{(-1)^n\dfrac{1}{n!}\right\}$

48. $a_n = \dfrac{5}{n+2}$

49. $a_n = 8+\dfrac{5}{n}$

50. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

51. $a_n = \dfrac{1+(-1)^n}{n^2}$

52. $a_n = \dfrac{10n^2+3n+7}{2n^2-6}$

53. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

54. $a_n = \dfrac{\ln\left(n^3\right)}{2n}$

55. $a_n = \dfrac{5^n}{3^n}$

56. $a_n = \dfrac{(n+1)!}{n!}$

57. $a_n = \dfrac{(n-2)!}{n!}$

58. $a_n = \dfrac{n^p}{e^n},\quad p>0$

59. $a_n = n\sin\dfrac{1}{n}$

60. $a_n = 2^{1/n}$

61. $a_n = -3^{-n}$

62. $a_n = \dfrac{\sin n}{n}$

63. $a_n = \dfrac{\cos\pi n}{n^2}$

64. $a_n = \dfrac{n^3}{n^4+1}$

65. $a_n = n^{1/n}$

66. $\left\{\dfrac{n^{12}}{3n^{12}+4}\right\}$

67. $\left\{\dfrac{2e^{n+1}}{e^n}\right\}$

68. $\left\{\dfrac{\tan^{-1}n}{n}\right\}$

69. $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$

70. $\left\{\left(\dfrac{n}{n+5}\right)^n\right\}$

71. $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

72. $\{\ln\sin(1/n)+\ln n\}$

73. $a_n = \dfrac{n!}{n^n}$

74. $a_n = \dfrac{n^2+4}{\sqrt{4n^4+1}}$

75. $a_n = \dfrac{8^n}{n!}$

76. $a_n = \left(1+\dfrac{3}{n}\right)^{2n}$

77. $a_n = \sqrt[n]{n}$

78. $a_n = n-\sqrt{n^2-1}$

79. $a_n = \left(\dfrac{1}{n}\right)^{1/\ln n}$

80. $a_n = \sin\dfrac{\pi n}{6}$

81. $a_n = \dfrac{(-1)^n}{0.9^n}$

82. $a_n = \tan^{-1}n$

83. $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

84. $\sum_{n=1}^{\infty}(-1)^{n+1}n$

85. $\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

86. $\sum_{n=0}^{\infty}e^{-2n}$

87. $\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$

88. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

89. $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$

90. $\sum_{n=0}^{\infty}\frac{e^{\pi n}}{\pi^{ne}}$

91. $\sum_{n=0}^{\infty}(1.075)^n$

92. $\sum_{n=0}^{\infty}\frac{3^n}{1000}$

93. $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

94. $\sum_{n=1}^{\infty}\frac{4n+1}{3n-1}$

95. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

96. $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

97. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$

98. $\sum_{n=0}^{\infty}\frac{3}{5^n}$

99. $\sum_{n=2}^{\infty}\frac{n}{\ln n}$

100. $\sum_{n=1}^{\infty}\ln\frac{1}{n}$

101. $\sum_{n=1}^{\infty}\left(1+\frac{k}{n}\right)^n$

102. $\sum_{n=1}^{\infty}e^{-n}$

103. $\sum_{n=1}^{\infty}\arctan n$

104. $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

(105 – 110) Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges.

105. $a_n = 0.3^n$

106. $a_n = 1.3^n$

107. $a_n = (-0.6)^n$

108. $a_n = (-1.01)^n$

109. $a_n = 2^n 3^{-n}$

110. $a_n = (-0.003)^n$

(111 – 123) Find the limit of the following sequences or state that they diverge

111. $a_n = \frac{\sin n}{2^n}$

112. $a_n = \frac{n\sin^3 n}{n+1}$

113. $a_n = \frac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}}$

114. $a_n = \dfrac{2\tan^{-1} n}{n^3+4}$

115. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k$

116. $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$

117. $\displaystyle\sum_{k=1}^{\infty} 3(1.001)^k$

118. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

119. $\displaystyle\sum_{k=2}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k-1}}\right)$

120. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{3k-2} - \frac{3}{3k+1}\right)$

121. $\displaystyle\sum_{k=1}^{\infty} 4^{-3k}$

122. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}}$

123. $\displaystyle\sum_{k=0}^{\infty} \left(\left(\frac{1}{3}\right)^k - \left(\frac{2}{3}\right)^{k+1}\right)$

124. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

a) Write the first four terms of the sequence of partial sums S_1, S_2, S_3, S_4.

b) Write the nth term of the sequence of partial sums S_n.

c) Find $\lim\limits_{n\to\infty} S_n$ and evaluate the series.

125. Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person take 80 *mg* of aspirin every 24 *hr*. Assume also that aspirin has a half-life of 24 *hrs.*; that is, every 24 *hrs.* half of the drug in the blood is eliminated.

a) Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the nth dose, where $d_1 = 80$

b) Find the limit of $\{d_n\}$

126. Suppose a tank is filled with 100 *L* of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 *L* of the solution from the tank and replace them with 2 *L* of 10% alcohol solution

a) Let C_n be the concentration of the solution in the tank after the nth replacement, where $C_0 = 40\%$. Write the first five terms of the sequence $\{C_n\}$

b) After how many replacements does the alcohol concentration reach 15%?

c) Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

127. The Greeks solved several calculus problems almost 2,000 *years* before the discovery of calculus. One example is Archimedes' calculation of the area of the region *R* bounded by a segment of a parabola, which he did using the "method of exhaustion".

$$A_2 = \frac{1}{8}A_1$$

The idea was to fill *R* with an infinite sequence of triangles. Archimedes began with an isosceles triangle inscribed in the parabola, with an area A_1, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2 = \frac{1}{8}A_1$, and so forth. Show, as Archimedes did, that the area of *R* is $\frac{4}{3}$ times the area of A_1.

128.

a) Evalute the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}$

b) For what values of a does the the series converge, and in those cases, what is its value?

$$\sum_{k=1}^{\infty} \frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}$$

129. An insulated windows consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction *p* of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the windows? Assume the amount of incoming light is 1.

130. Suppose you borrow \$20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of \$600/month, after how many

months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer using infinite series.

131. Suppose a rubber ball, when dropped from a given height, returns to a fraction p of that height. In the absence of air resistance, a ball dropped from a height h requires $\sqrt{\frac{2h}{g}}$ seconds to fall to the ground, where $g \approx 9.8\ m/s^2$ is the acceleration due to gravity. The time taken to bounce up to a given to fall from that height to the ground. How long does it take a ball dropped from 10 m to come to rest?

132. The fractal called the snowflake island (or Koch island) is constructed as flows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 with a new outward equilateral triangle with sides of length $\frac{1}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n with a new outward equilateral triangle with sides of length of $\frac{1}{3^{n+1}}$. The limiting figure as $n \to \infty$ is called the snowflake island.

a) Let L_n be the perimeter of I_n. Show that $\lim_{n\to\infty} L_n = \infty$

b) Let A_n be the area of I_n. Find $\lim_{n\to\infty} A_n$. It exists!

133. Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1,\ r_2,\ r_3, \ldots$ Let h_n be the distance between the diameters of bubble n and bubble $n+1$, and let H_n be the total height of the stack with n bubbles.

a) Use the Pythagorean theorem to show that in a stack with n bubbles $h_1^2 = r_1^2 - r_2^2$, $h_2^2 = r_2^2 - r_3^2$, and so forth. Note that for the last bubble $h_n = r_n$.

b) Use part (*a*) to show that the height of a stack with n bubbles is

$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

c) The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$ where $0 < a < 1$ is a fixed real number. In terms of a, find the height H_n of a stack with n bubbles.

d) Suppose the stack in part (*c*) is extended indefinitely $(n \to \infty)$. In terms of a, how high would the stack be?

134. The fractal called the *Sierpinski triangle* is the limit of a sequence of figures. Starting with the equilateral triangle with sides of length 1, an inverted equilateral triangle with sides of length $\frac{1}{2}$ is removed. Then, the three inverted equilateral triangles with sides of length $\frac{1}{4}$ are removed from this figure.

The process continues in this way. Let T_n be the total area of the removed triangles after stage n of the process. The area on equilateral triangle with side length L is $A = \frac{\sqrt{3}}{4}L^2$.

a) Find T_1 and T_2 the total area of the removed triangles after stages 1 and 2, respectively.

b) Find T_n for $n = 1, 2, 3, \ldots$

c) Find $\lim_{n\to\infty} T_n$

d) What is the area of the original triangle that remains as $n \to \infty$?

135. The sides of a ***square*** are 16 *inches* in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded.

Determine the area of the shaded regions.

a) When this process is continued five more times.

b) When this pattern of shading is continued infinitely.

136. A right triangle XYZ is shown below where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle.

a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \frac{\pi}{6}$.

137. The sphere flake is a computer–generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely.

Prove that the sphere flake has an infinite surface area.

Section 7.2 – Integral Test

7.2–1 Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$$

Corollary

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Example 1

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$

Solution

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

The sum of the first 2 terms is $\frac{3}{2}$.

The sum of the next 2 terms is $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

The sum of the next 4 terms is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$\vdots \qquad \vdots \qquad \vdots$

The sum of 2^n terms ending with 2^{n+1} is $> \frac{2^n}{2^{n+1}} = \frac{1}{2}$

The sequence of partial sums is not bounded from above: If 2^k, the partial sum $s_n > \frac{k}{2}$.

The harmonic series diverges.

7.2–2 The Integral Test

Theorem

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both *converge*, or both *diverge*.

Example 2

Does the following series converge? $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

$$= f(1) + f(2) + f(3) + \cdots + f(n)$$

$$< 1 + \int_1^n \frac{1}{x^2}\,dx$$

$$< 1 + \lim_{b \to \infty} \left(-\frac{1}{x}\right) \Big|_1^b$$

$$= 1 - \left(\frac{1}{\infty} - 1\right)$$

$$= 2$$

Thus, the partial sums are bounded from above by 2 and the series converges.

$$\sum_{n-N}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\approx 1.64493$$

Example 3

Use the ***Integral Test*** to determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution

$$\int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty}$$

$$= -(0-1)$$

$$= 1$$

$\therefore$ By the *Integral Test*, the given series ***converges***.

Example 4

Use the ***Integral Test*** to determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

$$\int_1^{\infty} \frac{1}{x(x+1)}\,dx = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \ln x - \ln(x+1)\Big|_1^{\infty}$$

$$= \ln \frac{x}{x+1}\Big|_1^{\infty}$$

$$= \lim_{x\to\infty} \left(\ln \frac{x}{x+1}\right) - \ln \frac{1}{2}$$

$$= \ln 1 + \ln 2$$

$$= \ln 2$$

∴ By the *Integral Test*, the given series ***converges***.

Example 5

Use the ***Integral Test*** to determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \int_1^{\infty} x^{-1/2}\,dx$$

$$= \frac{1}{2}\sqrt{x}\,\Big|_1^{\infty}$$

$$= \frac{1}{2}\left(\infty - 1\right)$$

$$= \infty$$

∴ By the *Integral Test*, the given series ***diverges***.

Example 6

Does the following series converge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan x\right]_1^b$$

$$= \lim_{b\to\infty} \left(\arctan b - \arctan 1\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

∴ By the *Integral Test*, the given series ***converges***.

But we do not know the value of its sum

7.2–3 *p*-series

Example 7

Show that the ***p**-series* $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left(\frac{x^{-p+1}}{-p+1} \right|_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(b^{-p+1} - 1 \right)$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1 \right) \qquad p > 1$$

$$= \frac{1}{1-p} (0-1)$$

$$= \frac{1}{p-1}$$

The series ***converges*** when $p > 1$.

if $p \le 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{b\to\infty} \left(b^{1-p} - 1 \right)$$

$$= \infty$$

The series ***diverges***.

Note:

$p-$series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	***Convergences*** $p > 1$
		Divergences $p \le 1$

Example 8

Use the ***p-series*** to determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$p = \frac{1}{2} \le 1$$

$\therefore$ By the *p-series Test*, the given series ***diverges***.

Example 9

Use the ***p-series*** to determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$

$$p = \frac{5}{2} > 1$$

$\therefore$ By the *p-series Test*, the given series ***converges***.

7.2–4 Bounds for the Remainder in the Integral Test

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \ge n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx$$

Example 10

Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

a) Approximate the sum of the given series by using the sum of the first 10 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n = 10$.
d) How many terms are required to ensure that the sum is accurate to within $.005$?

Solution

a) $$\int_n^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_n^{\infty}$$

$$= -\left(0 - \frac{1}{n}\right)$$

$$= \frac{1}{n}$$

$$s_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2}$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100}$$

$$\approx 1.5497677$$

b) $$R_{10} = \int_{10}^{\infty} \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\Big|_{10}^{\infty}$$

$$= \frac{1}{10}$$

$$Error = \frac{1}{10} = .1$$

c) $$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}$$

$$1.5497677 + \frac{1}{11} \le S \le 1.5497677 + \frac{1}{10}$$

$$1.64067679 \le S \le 1.6497677$$

If we approximate the sum S by the midpoint of this interval, then

$$S = \frac{1.6497677 + 1.64067679}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.64522245$$

However, the value of the sum is:

$S = \frac{\pi^2}{6}$ by using the trigonometric *Fourier Series*.

$$S = \frac{\pi^2}{6} \approx 1.6449340668$$

$$\frac{1.64522245 - 1.64493407}{1.64493407} \approx .000175 < .0005$$

The *error* in this approximation is less than half the length of the interval, so the error is less than 0.0005.

d) $R_n = \frac{1}{n} < .005$

$$n > \frac{1}{.005}$$
$$= \frac{1000}{5}$$
$$= 200$$

$n = 200$ terms needed to ensure accuracy within .005

Example 11

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}\,\ln\left(\sqrt{n}\right)}$

Solution

$$\int_2^{\infty} \frac{1}{2\sqrt{x}\ln\left(\sqrt{x}\right)}\,dx = \int_2^{\infty} \frac{1}{\ln\left(\sqrt{x}\right)}\,d\left(\sqrt{x}\right)$$

$$= \ln\left(\ln\sqrt{x}\right)\Big|_2^{\infty}$$

$$= \ln\left(\ln\infty\right) - \ln\left(\ln\sqrt{2}\right)$$

$$= \infty$$

$\therefore$ By the *Integral Test*, the given series ***diverges***.

7.2–5 Difference Between the Geometric Series and p–Series

The Geometric series has power $\boldsymbol{n}$ and p–series has $\boldsymbol{n}$ as a base.

Geometric Series	p–Series
$a_n = (r)^n$	$a_n = \frac{*}{n^p}$
$r \in \mathbb{R}$	$p \in \mathbb{R}^+$

Exercises *Section* 7.2 – Integral Test

$(\mathbf{1-30})$ Use the ***Integral Test*** to determine if the series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

3. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2-2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

9. $\displaystyle\sum_{k=1}^{\infty} k\,e^{-2k^2}$

10. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

11. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\ln(\ln k)}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{e^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

21. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k^2+1)^3}$

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+10}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+3)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+3)^{3/2}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+5}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+5}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

29. $\displaystyle\sum_{n=1}^{\infty} n\,e^{-n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

(31 – 36) Use the ***p-series Test*** to determine if the series converge or diverge.

31. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^9}$

32. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^6}$

33. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1/9}}$

34. $\displaystyle\sum_{k=1}^{\infty} k^{-2}$

35. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$

36. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}}$

(37 – 75) Determine if the series converge or diverge

37. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^8}$

38. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$

39. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

40. $\displaystyle\sum_{n=1}^{\infty} n\,e^{-n^2}$

41. $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n^2+9}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

43. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

47. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

49. $\displaystyle\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

51. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-2)^4}$

52. $\displaystyle\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}$

53. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

54. $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

55. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

56. $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k\ln^2 k}$

57. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

60. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

61. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(2n-3)}$

62. $\displaystyle\sum_{n=1}^{\infty} n\,2^{-n^2}$

63. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

65. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+4}$

66. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+5}-1}$

67. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{\ln n}}$

68. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{5^n}+\frac{1}{n}\right)$

69. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^{n-1}}+\frac{2}{n(n+1)}\right)$

70. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n+2}-\frac{3}{n+3}\right)$

71. $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{9n^2+3n-2}$

72. $\displaystyle\sum_{n=1}^{\infty} \left(\left(\frac{3}{2}\right)^n+\left(\frac{2}{3}\right)^n\right)$

73. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots$

74. $1+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots$

75. $\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\cdots+\frac{1}{(n+3)\cdot(n+4)}+\cdots$

76. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where p is a real number.

a) Use the Integral Test to determine the values of p for which this series converges.

b) Does this series converge faster for $p=2$ or $p=3$? Explain.

77. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\,\left(\ln(\ln k)\right)^p}$, where p is a real number.

a) For what values of p does this series converge?

b) Which he following series converge faster? Explain.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \quad \text{or} \quad \sum_{k=2}^{\infty} \frac{1}{k(\ln k)\,\left(\ln(\ln k)\right)^2}$$

78. Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\frac{1}{2}$, the third layer has a radius of $\frac{1}{3}$, and the nth layer has a radius of $\frac{1}{n}$.

a) To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

b) Determine the volume of the cake.

c) Comment on your answer to parts (*a*) and (*b*)

79. The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. Is its defined by $\zeta(x) = \sum_{k=1}^{\neq} \frac{1}{k^x}$, when x is a real number, the zeta function becomes a p-series. For even positive integers ρ, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\neq} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \textit{and} \quad \sum_{k=1}^{\neq} \frac{1}{k^6} = \frac{\pi^6}{945}, \quad \ldots$$

a) Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

b) Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x = 2)$ without changing the value of the series.

80. Consider a set of identical dominoes that are 2 *inches* long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

a) If there are n dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the nth domino beneath the previous $n-1$ dominoes.)

b) If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

81. A theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\lim_{k\to\infty} \frac{p_k}{k\ln k}$.

Show that $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges, which implies that the series $\sum_{k=1}^{\infty} \frac{1}{p_k}$

(A prime number is a positive integer number that is divisible only by 1 and itself).

82. Given the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

a) Approximate the sum of the given series by using the sum of the first 10 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n=10$.
d) How many terms are required to ensure that the sum is accurate to within 4 *decimals*?

83. Given the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$

a) Approximate the sum of the given series by using the sum of the first 10 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n=10$.
d) How many terms are required to ensure that the sum is accurate to within $.00005$?

84. Given the series $\sum_{n=1}^{\infty} (2n+1)^{-5}$

a) Approximate the sum of the given series by using the sum of the first 5 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n=5$.
d) How many terms are required to ensure that the sum is accurate to within $.00005$?

85. Given the series $\sum_{n=1}^{\infty} n e^{-2n}$

a) Approximate the sum of the given series by using the sum of the first 8 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n = 8$.
d) How many terms are required to ensure that the sum is accurate to within $.00005$?

86. Given the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

a) Approximate the sum of the given series by using the sum of the first 10 terms.
b) Estimate the error involved in this approximation.
c) Estimate the sum of the given series with $n = 10$.
d) How many terms are required to ensure that the sum is accurate to within $.01$?

Section 7.3 – Comparison Tests

7.3–1 *Theorem*

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms.

Suppose that for some integer N.

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N$$

a) If $\sum c_n$ ***converges***, then $\sum a_n$ also ***converges***.

b) If $\sum d_n$ ***diverges***, then $\sum a_n$ also ***diverges***.

Example 1

Use the comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converges or diverges.

Solution

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}}$$

$$> \frac{1}{n}$$

The series ***diverges*** because its nth term is greater than the nth term of the divergent harmonic series.

$$\frac{1}{n^1}$$

By $p-$series $(p=1)$, it *diverges.*

$\therefore$ By the *Comparison Test*, the given series ***diverges***.

Example 2

Use the comparison Test to determine if $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges or diverges.

Solution

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series $|r| = \frac{1}{2} < 1$

$$1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{1-\frac{1}{2}}$$

$$= 3$$

$\therefore$ By the *Comparison Test*, the given series ***converges***.

Example 3

Use the comparison Test to determine if $\sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ converges or diverges.

Solution

$$n^2 + 1 \geq n^2$$

$$\sqrt[3]{n^2+1} \geq \sqrt[3]{n^2}$$

$$\frac{1}{\sqrt[3]{n^2+1}} \leq \frac{1}{n^{2/3}}$$

$$\sum \frac{1}{n^{2/3}}$$

By $p-$series $\left(p = \frac{2}{3} \le 1\right)$, it *diverges*.

$\therefore$ By the *Comparison Test*, the given series ***diverges***.

7.3–2 Limit Comparison Test

Theorem

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer)

1. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ ***both*** converge or both diverge.

2. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ ***converges***, then $\sum a_n$ ***converges.***

3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ ***diverges***, then $\sum a_n$ ***diverges.***

Example 4

Does the series $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$ converge or diverge?

Solution

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

$$= \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

Let $a_n = \frac{2n+1}{n^2+2n+1}$

$$\frac{2n}{n^2} = \frac{2}{n}$$

$$\frac{2}{n} > b_n = \frac{1}{n}$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*

By $p-$series $(p=1)$, it *diverges*.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2}$$

$$= 2$$

∴ By the *Limit Comparison Test*, the given series ***diverges***.

Example 5

Does the series $\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge or diverge?

Solution

Let $a_n = \frac{1}{2^n-1} \quad \to \quad b_n = \frac{1}{2^n}$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

By the Geometric Series $\left(r = \frac{1}{2} < 1\right)$, *it converges*.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}}$$

$$= 1$$

∴ By the *Limit Comparison Test*, the given series ***converges***.

Example 6

Does the series $\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots = \sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$ converge or diverge?

Solution

Let $a_n = \dfrac{1+n\ln n}{n^2+5}$

$$b_n = \frac{n\ln n}{n^2}$$

$$= \frac{\ln n}{n} > \frac{1}{n}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \quad \textit{diverges}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+n\ln n}{n^2+5} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+5}$$

$$= \lim_{n\to\infty} \frac{n^2\ln n}{n^2}$$

$$= \lim_{n\to\infty} \ln n$$

$$= \infty$$

$\therefore$ By the *Limit Comparison Test*, the given series ***diverges***.

Example 7

Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution

Let $a_n = \dfrac{\ln n}{n^{3/2}} < \dfrac{n^{1/4}}{n^{3/2}}$

$$= \frac{1}{n^{5/4}} = b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{3/2}} \cdot \frac{n^{5/4}}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4}n^{-3/4}} \qquad \textit{L'hôpital Rule}$$

$$= \lim_{n\to\infty} \frac{4}{n^{1/4}}$$

$$= 0$$

∴ By the *Limit Comparison Test*, the given series ***converges***.

$\ln n < n$

$$\frac{\ln n}{n^{3/2}} < \frac{n}{n^{3/2}}$$

$$= \frac{1}{n^{1/2}}$$

$$\sum \frac{1}{n^{1/2}}$$

By $p-$series $\left(p = \frac{1}{2} < 1\right)$, it *diverges*.

We can conclude the result since the given series is below the $\sum \frac{1}{n^{1/2}}$, we may conclude it converges.

$\ln n < n^{1/4}$

For $n = 2$ $\quad \ln 2 < 2$ ✓ P_2 *is true*

Let $\quad \ln k < k^{1/4}$

$$\ln(k+1) < (k+1)^{1/4}$$

$$(k+1)^{1/4} > k^{1/4}$$

$$> \ln k$$

$$> \ln(k+1)$$

Exercises *Section* 7.3 – Comparison Tests

(1 – 30) Use the Comparison Test to determine if the series converges or diverges

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

17. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^2+1}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3+\sqrt{n}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+4n^3+1}{2n^8+n^4+2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{n^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n(n+1)}$

29. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{n-1}}$

(31 – 45) Use the Limit Comparison Test to determine if the series converges or diverges.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$

32. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

41. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

(46 – 81) Use any method to determine if the series converges or diverges

46. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

50. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

55. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{an+b}$

56. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

57. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

58. $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

60. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

61. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

62. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

63. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

65. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,2^n}{4n^3+1}$

66. $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

67. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

68. $\displaystyle\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

69. $\displaystyle\sum_{k=1}^{\infty} \sin \frac{1}{k}$

70. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{1}{k}$

71. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{1}{k}$

72. $\displaystyle\sum_{k=1}^{\infty} (-1)^k k \sin \frac{1}{k}$

73. $\displaystyle\sum_{k=1}^{\infty} \tan \frac{1}{k}$

74. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

75. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

76. $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\ln k}$

77. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{2n}$

78. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^3-5}}$

81. $\displaystyle\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \cdots$

Section 7.4 – The Ratio and Root Tests

7.4–1 *Theorem* – The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

a) the series ***converges*** if $\rho < 1$,

b) the series ***diverges*** if $\rho > 1$, or ρ is infinite

c) the test is ***inconclusive*** if $\rho = 1$,

The value ρ doesn't mean the sum of the series.

Example 1

Investigate the convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}$$

$$= \frac{1}{3}\frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$= \frac{1}{3}\cdot\frac{2}{1}$$

$$= \frac{2}{3} < 1$$

$\therefore$ By the *Ratio Test* $\left(\rho = \frac{2}{3} < 1\right)$, the given series ***converges***.

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \frac{2^n}{3^n} + \sum_{n=0}^{\infty} \frac{5}{3^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n}$$

$$= \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}}$$

$$= \frac{21}{2}$$

Example 2

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1))!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}}$$

$$= \frac{1}{(n+1)(n+1)} \cdot \frac{(2n+2)!}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$= \frac{2(n+1)(2n+1)}{(n+1)(n+1)}$$

$$= \frac{4n+1}{n+1}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{4n+1}{n+1}$$

$$= 4 > 1$$

$\therefore$ By the *Ratio Test* $(\rho = 4 > 1)$, the given series ***diverges***.

Example 3

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{4^n n!\,n!}{(2n)!}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{4(n+1)}{2(2n+1)}$$

$$= \frac{2(n+1)}{2n+1}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+1}$$

$$= 1$$

Because the limit is $\rho = 1$, we can't decide from the *Ratio Test* whether the series converges.

However, since a_{n+1} *always* $>$ a_n, then the series ***diverges***.

$$a_{n+1} \;?\; a_n$$

$$\frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \;?\; \frac{4^n n!n!}{(2n)!}$$

$$\frac{4^{n+1}(n+1)!(n+1)!}{4^n n!n!} \;?\; \frac{(2(n+1))!}{(2n)!}$$

$$4(n+1)(n+1) \;?\; (2n+1)(2n+2)$$

$$4(n+1)(n+1) \;?\; 2(n+1)(2n+1)$$

$$2n+2 > 2n+1$$

$$a_{n+1} > a_n$$

$\therefore$ By the *Ratio Test*, the given series ***diverges***.

7.4–2 *Theorem* – The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

a) the series ***converges*** if $\rho < 1$,

b) the series ***diverges*** if $\rho > 1$, or ρ is infinite

c) the test is ***inconclusive*** if $\rho = 1$,

Example 4

Determine if the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges using the Root Test

Solution

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}}$$

$$= \frac{\left(\sqrt[n]{n}\right)^2}{2}$$

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$$

$$= \lim_{n\to\infty} \frac{n^{2/n}}{2}$$

$$= \frac{\infty^0}{2}$$

$$= \frac{1}{2} < 1$$

$\therefore$ By the *Root Test* $\left(\rho = \frac{1}{2} < 1\right)$, the given series ***converges***.

Example 5

Determine if the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ converges or diverges using the Root Test.

Solution

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}}$$

$$= \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^3}}$$

$$= \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$\rho = \lim_{n\to\infty} \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$= \frac{2}{1}$$

$$= 2 > 1$$

$\therefore$ By the *Root Test* $(\rho = 2 > 1)$, the given series ***diverges***.

Example 6

Determine if the series $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges or diverges using the Root Test.

Solution

$$\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n}$$

$$\lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

$\therefore$ By the *Root Test* $(\rho = 0 < 1)$, the given series ***converges***.

Example 7

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{2n}}$ converges or diverges.

Solution

$$\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 < 1$$

$\therefore$ By the *Root Test* $(\rho = 0 < 1)$, the given series ***converges***.

Example 8

Determine if the series $\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n\left(n^2+1\right)}$ converges or diverges.

Solution

$$\frac{a_{n+1}}{a_n} = \frac{2^n}{2^{n+1}} \cdot \frac{3(n+1)^2+5(n+1)}{(n+1)^2+1} \cdot \frac{n^2+1}{3n^2+5n}$$

$$= \frac{1}{2} \cdot \frac{3n^2+11n+8}{n^2+2n+2} \cdot \frac{n^2+1}{3n^2+5n}$$

$$\rho = \frac{1}{2} \lim_{n\to\infty} \frac{3n^2+11n+8}{n^2+2n+2} \cdot \frac{n^2+1}{3n^2+5n}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{3n^4}{3n^4}$$

$$= \frac{1}{2}$$

$\therefore$ By the *Ratio Test* $\left(\rho = \frac{1}{2} < 1\right)$, the given series ***converges***.

Exercises *Section* 7.4 – The Ratio and Root Tests

(1 – 24) Use the ***Ratio Test*** to determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n\,3^{n-1}}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\,(n+2)!}{n!\,3^{2n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,5^n}{(2n+3)\,\ln(n+1)}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

15. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(n+2)}{n\,(n+1)}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,2^n}{n!}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n\,3^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{(2n)!}$

(25 – 45) Use the ***Root Test*** to determine if the series converges or diverges.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

27. $\displaystyle\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$

28. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

29. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

33. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

36. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-2}{5n+1}\right)^n$

37. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

39. $\displaystyle\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

40. $\displaystyle\sum_{n=0}^{\infty} e^{-3n}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

43. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

44. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

(46 – 96) Use any method to determine if the series converges or diverges.

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

47. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

53. $\displaystyle\sum_{n=1}^{\infty} \left|\sin\frac{1}{n^2}\right|$

54. $\displaystyle\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

55. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

56. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n-n^{\pi}}$

57. $\displaystyle\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

60. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)^2}$

61. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)\sqrt{\ln\ln n}}$

62. $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

63. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!\,6^n}{(3n)!}$

65. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

66. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{100}\,2^n}{\sqrt{n}!}$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$

69. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n\, n!}$

70. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

71. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\,2^{n+1}}{3^n}$

72. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{100}{n}$

74. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

75. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

77. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

78. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\,\frac{3^{n-2}}{2^n}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n\,2^n}$

81. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

82. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

83. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n\,7^n}$

84. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

85. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln k)^2}$

86. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$

87. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2\,(\ln k)^2}$

88. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\ln k}$

89. $\displaystyle\sum_{k=2}^{\infty} \frac{5\ln k}{k}$

90. $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$

91. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$

92. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$

93. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

94. $\displaystyle\frac{1+\sqrt{2}}{2} + \frac{1+\sqrt{3}}{4} + \frac{1+\sqrt{4}}{8} + \cdots$

95. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

96. $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n\,(2n-1)\,n!}$

97. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\frac{\pi}{2}$

98. Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges

99. Use the root test to test that $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges

100. Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happen?

Now observe that $\frac{2^{2n}(n!)^2}{(2n)!} = \frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times 4\times 2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times 2\times 1}$

$$= \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\frac{4}{3}\times\frac{2}{1}$$

Does the given series converge? Why or why not?

101. Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

$\left(a_n \geq \frac{K}{n}\ \textit{for some constant } K\right)$

102. Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where p is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

103. Complete the following steps to find the values of $p > 0$ for which the series

$$\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!} \text{ converges}$$

a) Use the Ratio Test to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!}$ converges for $p > 2$.

b) Use Stirling's formula, $k! = \sqrt{2\pi k}\, k^k e^{-k}$ for large *k*, to determine whether the series converges when $p = 2$.

$$\left(Hint: \ 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2k-1) 2k}{2 \cdot 4 \cdot 6 \cdots 2k} \right)$$

Section 7.5 – Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive, and negative is an alternating series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

7.5−1 *Theorem* – The Alternating Series Test (Leibniz's Test)

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

Converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) non-increasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$

Example 1

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

Solution

1. $\frac{1}{n} > 0$
2. $n < n+1 \to \frac{1}{n} > \frac{1}{n+1}$
3. $\frac{1}{n} \to 0$

$\therefore$ By the *Alternating Test*, the given series ***converges***.

Example 2

Determine if the alternating series converges or diverges $\sum_{k=1}^{\infty} (-1)^k \frac{k}{4k+1}$

Solution

$$\lim_{k\to\infty} \frac{k}{4k+1} = \lim_{k\to\infty} \frac{k}{4k}$$

$$= 4 \neq 0$$

∴ By the *Alternating Test*, the given series ***diverges***.

Example 3

The alternating harmonic series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Solution

$$n! < (n+1)!$$

$$\frac{1}{n!} > \frac{1}{(n+1)!} \quad \checkmark$$

$$\lim_{n\to\infty} \frac{1}{n!} = 0 \quad \checkmark$$

∴ By the *Alternating Test*, the given series ***converges***.

7.5–2 *Theorem* – The Alternating Series Estimation Theorem

IF the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions, then for $n \geq N$

$$s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1} u_n$$

Approximates the sum L of the series with an error whose absolute values is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum $\boldsymbol{L}$ lies between any two successive partial sums s_n and s_{n+1}, and the remainder $L - s_n$, has the same sign as the first unused term.

$$R_n = \left| s - s_n \right| \leq \left| s_{n+1} - s_n \right| = b_{n+1}$$

Example 4

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ accurate to 3 decimals.

Solution

By the *Alternating Test*, the given series ***converges.***

$$3 \text{ decimals } \Rightarrow \frac{1}{5000}$$

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \ldots$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \ldots$$

$$b_7 = -\frac{1}{5040} < \frac{1}{5000} = .0002$$

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

$$\approx 0.3680556$$

$$|s - s_6| \le |s_7 - s_6| = b_7$$

$$< .0002$$

$$s \approx 0.368$$

7.5–3 Absolute Convergence

Definition

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges.

7.5–4 Conditional Convergence

Definition

A series converges but does not converge absolutely ***converges conditionally***.

7.5–5 *Theorem*

If $\sum_{n=1}^{\infty} |a_n|$ *converges*, then $\sum a_n$ *converges*.

Example 5

For $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \left|(-1)^{n+1}\right| \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

By p–series $(p = 2 > 1)$, it *converges*

∴ The given series *converges* because it converges *absolutely*.

Example 6

For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \frac{|\sin 3|}{9} + \cdots$$

Which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ because $|\sin n| \le 1$ for every n.

∴ The given series converges *absolutely*.

7.5–6 Rearranging Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$,

then $\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Example 7

Determine if the series converges or diverges $\sum_{k=1}^{\infty} \frac{\sin k}{k^3 + k^2 + k + 1 + \cos k}$

Solution

$$\lim_{k\to\infty} \frac{\sin k}{k^3 + k^2 + k + 1 + \cos k} = \lim_{k\to\infty} \frac{\sin k}{k^3}$$

$$= \nexists$$

$$\lim_{k\to\infty} \left| \frac{\sin k}{k^3 + k^2 + k + 1 + \cos k} \right| = \lim_{k\to\infty} \frac{|\sin k|}{k^3 + k^2 + k + 1 + \cos k}$$

$$\frac{|\sin k|}{k^3 + k^2 + k + 1 + \cos k} \le \frac{1}{k^3 + k^2 + k}$$

$$< \frac{1}{k^3}$$

$\sum_{k=1}^{\infty} \frac{1}{k^3}$ By p–series $(p = 3 > 1)$, it *converges*.

∴ By the *Comparison Test*, the given series converges *absolutely*.

Exercises *Section* 7.5 – Alternating Series, Absolute and Conditional Convergence

$(\mathbf{1-24})$ Determine if the alternating series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

2. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n^2+5}{n^2+4}$

5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$

8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3n+2}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{e^n}$

11. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{5n-1}{4n+1}$

12. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$

13. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{\ln(n+1)}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$

17. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{\ln(n+1)}$

18. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$

19. $\displaystyle\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi}{2}$

20. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi$

21. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$

22. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}$

23. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+2}$

24. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{\sqrt[3]{n}}$

$(\mathbf{25-59})$ Determine if the series converge absolutely or conditionally or diverges.

25. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$

26. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

27. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

30. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2+\ln n}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}\left(n^2-1\right)}{n^2+1}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n}}{n!}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n\,\pi^n}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

42. $\displaystyle\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln\ln n}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

49. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{(n+1)^2}$

50. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{2n+3}{n+10}$

51. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

52. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n}\, e^{-n^2}$

53. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n}\,\frac{n}{n^3-5}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

55. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$

56. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

57. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n}$

(60 – 68) For what values of x does the series converge absolutely? Converge conditionally? Diverge?

60. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,2^n}$

61. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\,2^{2n}}$

62. $\displaystyle\sum_{n=1}^{\infty} (n+1)^2\left(\frac{x}{x+2}\right)^n$

63. $\displaystyle\sum_{n=0}^{\infty} (-1)^n\,\frac{(x-1)^n}{2n+3}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

65. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{2^n\,\ln n}$

66. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$

(69 – 140) Use ***any*** method to determine if the series converges or diverges.

69. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\,\frac{\sqrt{n}}{n+1}$

70. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

71. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{5}{n}$

72. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{3^{n-1}}{n!}$

73. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{3^n}{n\,2^n}$

74. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{1}{n}$

75. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

77. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

78. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

80. $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$

81. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

82. $\displaystyle\sum_{n=1}^{\infty} 100\,e^{-n/2}$

83. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$

84. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{4}{3n^2-1}$

85. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

86. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$

87. $\displaystyle\sum_{k=1}^{\infty} k^{-2/3}$

88. $\displaystyle\sum_{k=1}^{\infty} \frac{2k^2+1}{\sqrt{k^3+2}}$

89. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{e^k}$

90. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+3}\right)^{2k}$

91. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k k!}{k^k}$

92. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}\,\sqrt{k+1}}$

93. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{2+e^k}$

94. $\displaystyle\sum_{k=1}^{\infty}k\sin\frac{1}{k}$

95. $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt[k]{k}}{k^3}$

96. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1+\ln k}$

97. $\displaystyle\sum_{k=1}^{\infty}k^5e^{-k}$

98. $\displaystyle\sum_{k=4}^{\infty}\frac{1}{k^2-10}$

99. $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k^2}{k^2}$

100. $\displaystyle\sum_{k=1}^{\infty}ke^{-k}$

101. $\displaystyle\sum_{k=0}^{\infty}\frac{2\cdot 4^k}{(2k+1)!}$

102. $\displaystyle\sum_{k=0}^{\infty}\frac{9^k}{(2k)!}$

103. $\displaystyle\sum_{k=1}^{\infty}\frac{\coth k}{k}$

104. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sinh k}$

105. $\displaystyle\sum_{k=1}^{\infty}\tanh k$

106. $\displaystyle\sum_{k=0}^{\infty}\operatorname{sech} k$

107. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k^2-1}$

108. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2+4}{2k^2+1}$

109. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\,k\,e^{-k}$

110. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}$

111. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{10^k}{k!}$

112. $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^{k+1}}{k^2}$

113. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{e^k+e^{-k}}$

114. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$

115. $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k^2}$

116. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln^2 k}$

117. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$

118. $\displaystyle\sum_{k=2}^{\infty}3e^{-k}$

119. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k}{e^k-1}$

120. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^k}{(k+1)!}$

121. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{\left(k^2+1\right)^3}$

122. $\displaystyle\sum_{k=2}^{\infty}\frac{k^e}{k^\pi}$

123. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{(k-2)^4}$

124. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k}{k+1}\right)^k$

125. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

126. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$

127. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin\frac{\pi}{2k}$

128. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{3k^2+1}}$

129. $\displaystyle\sum_{k=1}^{\infty} \frac{k+\ln k}{k^3+k+1}$

130. $\displaystyle\sum_{k=1}^{\infty} \sin\frac{1}{k^2}$

131. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^3}$

132. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^3}{(k+1)!}$

133. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2^n}{n+3^n}$

134. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n^2+3^n}$

135. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3^n}{n^2+2^n}$

136. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3^n}{n+2^n}$

137. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,4^n}{(n+1)!}$

138. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^4}$

139. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$

140. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(n+2)^n}$

141. Use a Riemann sum argument to show that $\ln n! \geq \displaystyle\int_1^n \ln t\ dt = n\ln n - n + 1$

Then for what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{n^n}$ converge absolutely? Converge conditionally? Diverge? (*Use the ratio test first*)

142. Let S_n be the nth partial sum of $\displaystyle\sum_{k=1}^{\infty} a_k = 8$. Find the $\displaystyle\lim_{k\to\infty} a_k$ and $\displaystyle\lim_{n\to\infty} S_n$

143. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ accurate to 4 decimals.

144. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,3^n}$ with an error less than 0.0008.

145. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\,3^n}$ with an error less than 0.0005.

146. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$ with an error less than 0.0001.

147. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ with an error less than 0.00005.

148. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ with an error less than 0.00005.

149. Find the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{2n}}$ with an error less than 0.00005.

150. It can be proved that if a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if a series converges conditionally, then the value of the series depends on the order of summation. For example, the (conditionally convergent) alternating harmonic series has the value

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$

Show that by rearranging the terms (so the sign pattern is ++–),

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

151. A crew of workers is constructing a tunnel through a mountain. Understandably, the rate of construction decreases because rocks and earth must be removed a greater distance as the tunnel gets longer. Suppose that each week the crew digs 0.95 of the distance it dug the previous week. In the first week, the crew constructed 100 *m* of tunnel.

a) How far does the crew dig in 10 *weeks*? 20 *weeks*? *N* weeks?

b) What is the longest tunnel the crew can build at this rate?

c) The time required to dig 100 *m* increases by 10% each *week*, starting with 1 *week* to dig the first 100 *m*. Can the crew complete a 1.5 *km* tunnel in 10 *weeks*? Explain.

152. Consider the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad where \quad a_k = \begin{cases} \dfrac{4}{k+1} & if\ k\ is\ odd \\ \dfrac{2}{k} & if\ k\ is\ even \end{cases}$$

a) Write out the first ten terms of the series, group them in pairs, and show that the even partial sums of the series form the (divergent) harmonic series.

b) Show that $\lim_{k\to\infty} a_k = 0$

c) Explain why the series diverges even though the terms of the series approach zero.

153. The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. Doses may therefore need to be repeated periodically to keep the concentration from dropping below some particular level. One model for the effect of repeated doses gives the residual concentration just before the $(n+1)st$ does as

$$R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \cdots + C_0 e^{-nkt_0}$$

Where $C_0 =$ the change in concentration achievable by a single dose (mg/mL),

$k =$ the elimination constant $\left(h^{-1}\right)$, and

$t_0 =$ time between doses (h).

a) Write R_n in closed form as a single fraction, and find $R = \lim_{n\to\infty} R_n$

b) Calculate R_1 and R_{10} for $C_0 = 1\ mg/\text{mL}$, $k = 0.1\ h^{-1}$, and $t_0 = 10\ h$. How good as estimate of R is R_{10}

c) If $k = 0.01\ h^{-1}$ and $t_0 = 10\ h$, find the smallest n such that $R_n > \frac{1}{2}R$.

154. If a drug is known to be ineffective below a concentration C_L and harmful above some higher concentration C_H, one needs to find values of C_0 and t_0 that will produce a concentration that is safe (not above C_H) but effective (not below C_L).

We want to find values for C_0 and t_0 for which

$$R = C_L \quad and \quad C_0 + R = C_H$$

Thus, $C_0 = C_H - C_L$. The resulting equation simplifies to

$$t_0 = \frac{1}{k} \ln \frac{C_H}{C_L}$$

To reach an effective level rapidly, one might administer a "loading" dose that would produce a concentration of $C_H \ mg/mL$. This could be followed every t_0 hours by a dose that raises the concentration by $C_0 = C_H - C_L \ mg/mL$.

a) Verify the preceding equation for t_0.

b) If $k = 0.05\ h^{-1}$ and the highest safe concentration is e times the lowest effective concentration, find the length of time between doses that will assure safe and effective concentrations.

c) Given $C_H = 2\ mg/mL$, $C_L = 0.5\ mg/mL$, and $k = 0.02\ h^{-1}$, determine a scheme for administering the drug.

d) Suppose that $k = 0.2\ h^{-1}$ and the smallest effective concentration is 0.03 *mg/mL*. A single dose that produces a concentration of 0.1 *mg/mL* is administered. About how long will the drug remain effective?

Section 7.6 – Power Series

7.6−1 Power Series and Converge

Definitions

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

In which the ***center*** a and the ***coefficients*** $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

Example 1

Find the convergence of $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

Solution

This is the geometric series with first term 1 and ratio x. it converges to $\frac{1}{1-x}$ *for* $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1$$

Example 2

The power series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$

This is the geometric series with first term 1 and ratio $r=-\frac{x-2}{2}$. it converges to

$\left|\frac{x-2}{2}\right|<1 \quad for \quad 0<x<4$. The sum

$$\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}$$

$$=\frac{2}{x}$$

$$\frac{2}{x}=1-\frac{x-2}{2}+\frac{(x-2)^2}{4}-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$$

The series generates polynomial approximations of $f(x)=\frac{2}{x}$ for values of x near 2:

$$P_0(x)=1$$

$$P_1(x)=1-\frac{1}{2}(x-2)$$

$$=2-\frac{x}{2}$$

$$P_2(x)=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2$$

$$=3-\frac{3x}{2}+\frac{x^2}{4}$$

7.6–2 *Theorem*

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

7.6–3 Radius of Convergence of a Power Series

Corollary to Theorem

The convergence of the series $\sum a_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

R is called the ***radius of convergence*** of the power series, and the interval of radius R ***centered*** at $x = a$ is called the ***interval of convergence***.

7.6–4 *Definition*

Suppose that $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists or is ∞. Then the power series $\sum a_n (x-a)^n$ has radius of convergence $R = \frac{1}{L}$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$) and $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

7.6–5 How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges. Ordinarily, this is an open interval
$$|x-a|<R \quad or \quad a-R<x<a+R$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use the Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally) because the *n*th term does not approach zero for those values of *x*.

7.6–6 *Theorem*

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function f on $|f(x)|<R$

Example 3

Determine the centre, radius, and interval of convergence of
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Solution

The centre of convergence is $x=0$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x}\right|$$
$$= \frac{n}{n+1}|x| \to |x|$$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$.

At $x=1$, we get the alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \qquad \textit{Alternating series}$$

$n < n+1$

$\frac{1}{n} > \frac{1}{n+1}$

$u_n > u_{n+1}$ ✓

$\frac{1}{n} \to 0$ ✓

By alternating series, it converges at $x = 1$.

At $x = -1$, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$

we get the alternating harmonic series $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, the negative of the harmonic series.

It diverges at $x = -1$

The series ***converges*** for $-1 < x \le 1$ and ***diverges*** elsewhere.

Example 4

Determine the centre, radius, and interval of convergence of

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Solution

The ***centre*** of convergence is $x = 0$

$$R = \lim_{n\to\infty} \left|\frac{2n-1}{2n+1}\right| \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{2n}{2n}$$

$$= 1$$

$$\left|\frac{x^{2n+1}}{x^{2n-1}}\right| = x^2$$

The ***Radius*** of convergence is $R = 1$

The series converges absolutely for $x^2 < 1$. It diverges if $x^2 > 1$.

The series converges absolutely on ***interval*** :

$$(0-1,\ 0+1) = \underline{(-1,\ 1)} \qquad a - R < x < a + R$$

At $x = 1$, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ *Alternating series*

$n < n+1$

$2n < 2n+2$

$2n-1 < 2n+1$

$\dfrac{1}{2n-1} > \dfrac{1}{2n+1}$

$u_n > u_{n+1}$ ✓

$\dfrac{1}{2n-1} \to 0$ ✓

By alternating series, it converges at $x = 1$.

At $x = -1$, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{3n-2}}{2n-1}$ *Alternating series*

$n < n+1$

$2n < 2n+2$

$2n-1 < 2n+1$

$\dfrac{1}{2n-1} > \dfrac{1}{2n+1}$

$u_n > u_{n+1}$ ✓

$\dfrac{1}{2n-1} \to 0$ ✓

By alternating series, it converges at $x = -1$.

The series ***converges*** for $-1 \le x \le 1$ and ***diverges*** elsewhere.

Example 5

Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Solution

The ***centre*** of convergence is $x = 0$

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} \left| \frac{x^{n+1}}{x^n} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} (n+1) \quad |x|$$

$$= \infty$$

The ***Radius*** of convergence is $R = \infty$

The series converges on ***interval*** $(-\infty, \infty)$ $\qquad a - R < x < a + R$

The series ***converges absolutely*** for all x.

Example 6

Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$$

Solution

The ***centre*** of convergence is $x = 0$

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} \left| \frac{x^{n+1}}{x} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \quad |x|$$

$$= 0$$

The ***Radius*** of convergence is $R = 0$

The series ***diverges absolutely*** for all x except $x = 0$.

Example 7

Determine the centre, radius, and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n$$

The ***centre*** of convergence is

$$x+\frac{5}{2}=0 \quad \Rightarrow \quad \underline{x=-\frac{5}{2}}$$

$$R = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n \frac{1}{n^2+1}}{\left(\frac{2}{3}\right)^{n+1} \frac{1}{(n+1)^2+1}} \qquad \left|\frac{(2x+5)^{n+1}}{(2x+5)^n}\right| \qquad R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \left|\frac{(x-a)^{n+1}}{(x-a)^n}\right|$$

$$= \lim_{n\to\infty} \frac{3}{2} \frac{(n+1)^2+1}{n^2+1} \qquad |(2x+5)|$$

$$= \frac{3}{2} \lim_{n\to\infty} \frac{n^2}{n^2}$$

$$\underline{= \frac{3}{2}}$$

The ***Radius*** of convergence is $R = \frac{3}{2}$

The series converges absolutely on ***interval***

$$\left(-\frac{5}{2}-\frac{3}{2},\ -\frac{5}{2}+\frac{3}{2}\right) \underline{= (-4,\ -1)} \qquad a-R<x<a+R$$

At $x=-4 \quad \Rightarrow \quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

$$n^2 < (n+1)^2$$

$$n^2+1 < (n+1)^2+1$$

$$\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$$

$$u_n > u_{n+1} \quad \checkmark$$

$$\frac{1}{n^2+1} \to 0 \quad \checkmark$$

By *alternating* series, it converges at $x=-4$.

At $x=-1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{3^n}{\left(n^2+1\right)3^n} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$$\int_0^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_0^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} 0$$

$$= \frac{\pi}{2}$$

By *Integral Test*, it converges at $x=-1$

Therefore, the ***interval*** of convergence of the given power is $[-4,\ -1]$

7.6–7 *Theorem* – The Series Multiplication Theorem for Power Series

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

Then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Finding the coefficients c_n

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\right) = \left(1+x+x^2+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)$$

$$= \underbrace{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}_{by\ 1}+\underbrace{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)}_{by\ x}+\underbrace{\left(x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots\right)}_{by\ x^2}+\cdots$$

$$= x+\frac{x^2}{2}+\frac{5x^3}{6}-\frac{x^4}{6}\cdots$$

7.6–8 *Theorem* – The term-by-Term Differentiation Theorem

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, it defines a function.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \textit{on the interval} \quad a-R < x < a+R$$

This function f has derivatives of all order inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n (x-a)^{n-2}$$

And so on. Each of these derived series converges at every point of the interval $a-R < x < a+R$

Example 8

Find the series for $f'(x)$ and $f''(x)$ if

$$f(x)=\frac{1}{1-x}$$
$$=1+x+x^2+x^3+x^4+\cdots+x^n+\cdots$$
$$=\sum_{n=0}^{\infty}x^n,\quad -1<x<1$$

Solution

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots$$
$$=\sum_{n=1}^{\infty}n\,x^{n-1}$$

$$f''(x)=\frac{2}{(1-x)^3}=2+6x+12x^2+20x^3+\cdots+n(n-1)x^{n-1}+\cdots$$
$$=\sum_{n=2}^{\infty}n(n-1)\,x^{n-2}\qquad\qquad -1<x<1$$

7.6–9 *Theorem* – The term-by-Term Integration Theorem

Suppose that $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ converges for $a-R<x<a+R\quad(R>0)$. Then

$$\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}$$

Converges $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}+C\quad \textit{for}\quad a-R<x<a+R$$

Example 9

Identify the function $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots,\quad -1\le x\le 1$

Solution

$$f'(x)=1-x^2+x^4-x^6+\cdots,\quad -1\le x\le 1$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x)=\frac{1}{1-\left(-x^2\right)}$$

$$=\frac{1}{1+x^2}$$

$$\int f'(x)\,dx=\int\frac{dx}{1+x^2}$$

$$=\tan^{-1}x+C$$

The series for $f(x=0)=0$

$$\tan^{-1}0+C=0$$

$$C=0$$

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$$

$$=\tan^{-1}x,\quad -1<x<1$$

Example 7

Determine the centre, radius, and interval of convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n\,n}x^{2n+1}$

Solution

The *centre of convergence*: $x=0$

$$L=\lim_{n\to\infty}\frac{2^n\,n}{2^{n+1}\,(n+1)}\left|\frac{x^{2n+3}}{x^{2n+1}}\right|$$

$$L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\left|\frac{(x-a)^{n+1}}{(x-a)^n}\right|$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}x^2$$

$$= \frac{1}{2} x^2 < 1$$

$$x^2 < 2$$

$$-\sqrt{2} < x < \sqrt{2}$$

The *radius of convergence*: $\underline{R = \sqrt{2}}$

At $x = -\sqrt{2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\ n}\left(-2^{1/2}\right)^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{2^n\ n}\ 2^{n+\frac{1}{2}}$$

$$= \sqrt{2} \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n}$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$u_n > u_{n+1} \quad \checkmark$$

$$\frac{1}{n} \to 0 \quad \checkmark$$

By *alternating* series, it converges at $x = -\sqrt{2}$.

At $x = \sqrt{2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\ n}\ 2^{n+\frac{1}{2}} = \sqrt{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$u_n > u_{n+1} \quad \checkmark$$

$$\frac{1}{n} \to 0 \quad \checkmark$$

By *alternating* series, it converges at $x = \sqrt{2}$.

The *interval of convergence*: $\left[-\sqrt{2},\ \sqrt{2}\right]$

Exercises *Section* 7.6 – Power Series

(1 – 9) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

1. $\sum_{n=0}^{\infty} x^n$

2. $\sum_{n=0}^{\infty} (x+5)^n$

3. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

4. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

5. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

6. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

8. $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2x+5)^n$

9. $\sum_{n=1}^{\infty} \left(2+(-1)^n\right)\cdot(x+1)^{n-1}$

(10 – 18) Find the radius of convergence of the power series.

10. $\sum_{n=0}^{\infty} n!\,x^n$

11. $\sum_{n=0}^{\infty} 3(x-2)^n$

12. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

13. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

14. $\sum_{n=0}^{\infty} (3x)^n$

15. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

17. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

18. $\sum_{n=0}^{\infty} \frac{(2n)!\,x^{2n}}{n!}$

(19 – 27) Find the interval of convergence of the power series.

19. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

20. $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$

21. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

23. $\sum_{n=0}^{\infty} (2x)^n$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

25. $\sum_{n=0}^{\infty} (-1)^n (n+1)\,x^n$

26. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

27. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

(28 – 80) Determine the ***centre***, ***radius***, and ***interval*** of convergence of each of the power series.

28. $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

29. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n+1)(n+2)}$

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{6^n}$

31. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n!\,(x-5)^n}{3^n}$

32. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{n\,9^n}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

34. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(x-1)^{n+1}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^{n+1}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

37. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

39. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

40. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{(2n)!}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)\,x^n}{n!}$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

44. $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4\,2^{2n}}x^n$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

48. $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!}x^n$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\,x^n}{n!}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{4n}}{n^2}$

51. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^{2n}}{n!}$

52. $\displaystyle\sum_{n=1}^{\in} \frac{(x-1)^n}{n\cdot 5^n}$

53. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{9}\right)^{3n}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$

55. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^k}{2^k \ln k}$

56. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+4}$

57. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}x^n$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}x^n$

59. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2+1}x^n$

60. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}x^n$

61. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{4^n}(x-4)^n$

62. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{10^n} x^n$

63. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(-2)^n} x^{2n+1}$

64. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n}$

65. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}(x-2)^n$

66. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^{3n}}(x+4)^n$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{e^n}(x-e)^n$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,3^n}(2x-1)^n$

69. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{3\cdot 6\cdot 9\cdots(3n)} x^n$

70. $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$

71. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n\,n}(x-5)^n$

72. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\,n} x^{2n+1}$

74. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3n+1} x^{2n+1}$

75. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}} x^n$

76. $\displaystyle\sum_{n=0}^{\infty} n(x-3)^n$

77. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^5 (x-7)^n$

78. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$

79. $\displaystyle\sum_{n=4}^{\infty} e^n (x-2)^n$

80. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$

81. For what values of x does the series $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

82. The series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$ converges to sin*x* for all *x*.

a) Find the first six terms of a series for cos*x*. For what values of *x* should the series converge?

b) By replacing *x* by 2*x* in the series for sin*x*, find a series that converges to sin2*x* for all *x*.

c) Using the result in part (***a***) and series multiplication, calculate the first six term of a series for $2\sin x\cos x$. Compare your answer with the answer in part (***b***).

83. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series:

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

84. Find a series representation of $f(x) = \frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

85. Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ *and* $-x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

86. Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

$(\mathbf{77-79})$ Determine the interval of convergence and the sum of each of the series.

87. $1 - 4x + 16x^2 - 64x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$

88. $3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty} (n+3)\, x^n$

89. $\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$

Section 7.7 – Taylor and Maclaurin Series

7.7–1 *Definition*

The sum of a power series:

$$f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$$

$$=a_0+a_1(x-a)+a_2(x-a)^2+\cdots+a_n(x-a)^n+\cdots$$

$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+\cdots+na_n(x-a)^{n-1}+\cdots$$

$$f''(x)=2a_2+2\bullet 3a_3(x-a)+3\bullet 4a_4(x-a)^2+\cdots+(n-1)\bullet na_n(x-a)^{n-2}+\cdots$$

$$f'''(x)=1\bullet 2\bullet 3a_3+2\bullet 3\bullet 4a_4(x-a)++3\bullet 4\bullet 5a_5(x-a)^2\cdots+(n-2)\bullet(n-1)\bullet na_n(x-a)^{n-3}+\cdots$$

$$f^{(n)}(x)=n!a_n+\textit{a sum of terms with } (x-a) \textit{ as a factor}$$

In general: $\boxed{f^{(n)}(x)=n!\,a_n}$

$$a_n=\frac{f^{(n)}(a)}{n!}$$

If f has a series representation, then the series must be

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

7.7–2 Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing ***a*** as an interior point. Then the ***Taylor series generated by*** f at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

The ***Maclaurin series generated by*** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

The Taylor series generated by f at $x = 0$.

Example 1

Find the Taylor series generated by $f(x) = \frac{1}{x}$ *at* $a = 2$. Where, if anywhere, does the series converges to $\frac{1}{x}$.

Solution

$f(x) = x^{-1}$	$f(2) = 2^{-1} = \frac{1}{2}$
$f'(x) = -x^{-2}$	$f'(2) = -\frac{1}{2^2}$
$f''(x) = 2!x^{-3}$	$\frac{f''(2)}{2!} = 2^{-3} = \frac{(-1)^3}{2^3}$
$f'''(x) = -3!x^{-4}$	$\frac{f^{(3)}(2)}{3!} = 2^{-4} = \frac{(-1)^4}{2^4}$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$f^{(n)}(x) = (-1)^n\, n!\, x^{-(n+1)}$	$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$

The Taylor series is:

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

7.7–3 *Taylor* Polynomials

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 2

Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

Solution

$$f^{(n)}(x) = e^x \quad \rightarrow \quad f^{(n)}(0) = 1$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

This is also the Maclaurin series of e^x

$P_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$P_1 = 1 + x$

$P_2 = 1 + x + \frac{1}{2}x^2$

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

Example 3

Find the Taylor series and the Taylor polynomials generated by $f(x)=\cos x$ at $x=0$

Solution

$$f(x)=\cos x, \qquad f'(x)=-\sin x,$$
$$f''(x)=-\cos x, \qquad f''(x)=\sin x,$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(2n)}(x)=(-1)^n\cos x, \qquad f^{(2n+1)}(x)=(-1)^{2n+1}\sin x$$

$$f^{(2n)}(0)=(-1)^n, \qquad f^{(2n+1)}(0)=0$$

The Taylor series generated by f at $x=0$ is

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$
$$=1+0\cdot x-\frac{x^2}{2!}+0\cdot x^3+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$

$$\cos x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$$

$$P_{2n}(x)=P_{2n+1}(x)$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$$

Example 4

Find the Taylor series for $\cos x$ about $\frac{\pi}{3}$. Where is the series valid?

Solution

$$\begin{aligned}
\cos x &= \cos\left(x-\frac{\pi}{3}+\frac{\pi}{3}\right) \\
&= \cos\left(x-\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(x-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) \\
&= \frac{1}{2}\left[1-\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4-\cdots\right]-\frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{3}\right)-\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{5!}\left(x-\frac{\pi}{3}\right)^5-\cdots\right] \\
&= \frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2}\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2-\frac{\sqrt{3}}{2}\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{2}\frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4+\frac{1}{5!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^5-\cdots
\end{aligned}$$

This series representation is valid for all x.

Example 5

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

Solution

Let $t=\frac{x-2}{2}$, then

$$\begin{aligned}
\ln x &= \ln\left(2+(x-2)\right) \\
&= \ln\left[2\left(1+\frac{x-2}{2}\right)\right] \\
&= \ln 2+\ln(1+t)
\end{aligned}$$

$f(t)=\ln(1+t)$ $\qquad$ $f(0)=\ln(1)=0$

$f'(t)=\frac{1}{1+t}$ $\qquad$ $f'(0)=1$

$f''(t)=\frac{-1}{(1+t)^2}$ $\qquad$ $f''(0)=-1$

$f'''(t)=\frac{2}{(1+t)^3}$ $\qquad$ $f'''(0)=2$

$f^{(4)}(t)=\frac{-6}{(1+t)^4}$ $\qquad$ $f^{(4)}(0)=-6$

$f^{(n)}(t)=(-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$

$$\ln(1+t) = f(0) + f'(0)\,t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \cdots$$
$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$\ln x = \ln 2 + \ln(1+t)$$
$$= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$
$$= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots$$
$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \le 1$

$-2 < x-2 \le 2 \quad \rightarrow \quad \underline{0 < x \le 4}$

7.7–4 Error Estimation using *Taylor / Maclaurin* Approximations

To approximate the error of a function near $x = a$ is often called the remainder $R_n(x)$. It is the $(n+1)$ terms and is the difference between $f(x)$ and the approximation $P_n(x)$

$$R_n(x) = f(x) - P_n(x)$$

Where $P_n(x)$ is the nth degree Taylor polynomial of $f(x)$.

If $\lim_{n\to\infty} R_n(x) = 0$, then $f(x) = P_n(x)$.

$f(x)$ is equal to the sum of its Taylor series on the interval $|x-a| < R$.

For $n = 0$, then the remainder is:

$$R_0(x) = f(x) - P_0(x)$$
$$= f(x) - f(a) \qquad \textit{equation of a line}: \quad y - y_0 = m(x - x_0)$$
$$= f'(a)(x-a)$$

Therefore,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad f^{(n)}(x) = n!\,a_n$$

7.7–5 *Definition*

For a point c such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}|x-a|^{n+1} \quad where \quad a < c < x$$

Then ***Lagrange***'s formula (also known ***Taylor***'s Inequality) is derived by:

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

Where M is a positive real number such that $f^{(n+1)}(c) \le M$.

Example 5

Consider the function $f(x) = \sqrt[3]{x}$.

a) Approximate $f(x)$ by Taylor polynomial of degree 3 at $a = 8$?

b) How accurate is this approximation when $7 \le x \le 9$

Solution

a) $f(x) = x^{1/3}$ $\qquad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\qquad f''(8) = -\frac{1}{144}$

$f^{(3)}(x) = \frac{10}{27}x^{-8/3}$ $\qquad f^{(3)}(8) = \frac{5}{3,456}$

$f^{(4)}(x) = -\frac{80}{81}x^{-11/3}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{144}\cdot\frac{1}{2!}(x-8)^2 + \frac{5}{3,456}\cdot\frac{1}{3!}(x-8)^3$$

$$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

b) Using Taylor series since it is not an alternating series.

Given: $n = 3 \quad a = 8$

$$\left|R_3(x)\right| \le \frac{M}{4!}|x-8|^4$$

$$\left|f^{(4)}(x)\right| \le M$$

$$x \ge 7 \quad \rightarrow \quad x^{11/3} \ge 7^{11/3}$$

$$\left|f^{(4)}(x)\right| = \frac{80}{81}\frac{1}{x^{11/3}}$$

$$\le \frac{80}{81}\frac{1}{7^{11/3}}$$

$$< 0.787 \times 10^{-3}$$

That implies to $\underline{M = 7.87 \times 10^{-4}}$

$$7 \le x \le 9 \qquad -1 \le x - 8 \le 1$$

$$|x - 8| \le 1$$

$$\left|R_3(x)\right| \le \frac{7.87 \times 10^{-4}}{4!}\left(1^4\right)$$

$$< 3.3 \times 10^{-5}$$

This approximation is accurate to within .000033

Exercises *Section* 7.7 – Taylor and Maclaurin Series

$(\mathbf{1-25})$ Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a.

1. $f(x)=e^{2x}$, $a=0$

2. $f(x)=\sin x$, $a=0$

3. $f(x)=\ln(1+x)$, $a=0$

4. $f(x)=\dfrac{1}{x+2}$, $a=0$

5. $f(x)=\sqrt{1-x}$, $a=0$

6. $f(x)=x^3$, $a=1$

7. $f(x)=8\sqrt{x}$, $a=1$

8. $f(x)=\sin x$, $a=\dfrac{\pi}{4}$

9. $f(x)=\cos x$, $a=\dfrac{\pi}{6}$

10. $f(x)=\sqrt{x}$, $a=9$

11. $f(x)=\sqrt[3]{x}$, $a=8$

12. $f(x)=\ln x$, $a=e$

13. $f(x)=\sqrt[4]{x}$, $a=8$

14. $f(x)=\tan^{-1}x+x^2+1$, $a=1$

15. $f(x)=e^x$, $a=\ln 2$

16. $f(x)=e^{3x}$, $a=0$

17. $f(x)=\dfrac{1}{x}$, $a=1$

18. $f(x)=\cos x$, $a=\dfrac{\pi}{2}$

19. $f(x)=\dfrac{1}{x+1}$, $a=0$

20. $f(x)=\tan^{-1}4x$, $a=0$

21. $f(x)=\sin 2x$, $a=-\dfrac{\pi}{2}$

22. $f(x)=\cosh 3x$, $a=0$

23. $f(x)=\dfrac{1}{4+x^2}$, $a=0$

24. $f(x)=\dfrac{1}{x^2}$, $a=1$

25. $f(x)=\sqrt[3]{x^2}$, $a=1$

$(\mathbf{26-40})$ Find the *n*th Maclaurin polynomial for the function.

26. $f(x)=e^{4x}$, $n=4$

27. $f(x)=e^{-x}$, $n=5$

28. $f(x)=e^{-x/2}$, $n=4$

29. $f(x)=e^{x/3}$, $n=4$

30. $f(x)=\sin x$, $n=5$

31. $f(x)=\cos \pi x$, $n=4$

32. $f(x)=xe^x$, $n=4$

33. $f(x)=x^2e^{-x}$, $n=4$

34. $f(x)=\dfrac{1}{x+1}$, $n=5$

35. $f(x)=\dfrac{x}{x+1}$, $n=4$

36. $f(x)=\sec x$, $n=2$

37. $f(x)=\tan x$, $n=3$

38. $f(x)=\sqrt{x}$, $n=3$

39. $f(x)=\dfrac{1}{x^2}$, $n=3$

40. $f(x)=\sqrt[3]{x}$, $n=4$

$(41-44)$ Find out the ***third*** term of the Maclaurin series for the following function.

41. $f(x)=(1+x)^{1/3}$

42. $f(x)=(1+x)^{-1/2}$

43. $f(x)=\left(1+\frac{x}{2}\right)^{-3}$

44. $f(x)=(1+2x)^{-5}$

$(45-60)$ Find the Maclaurin series for:

45. $f(x)=xe^{x}$

46. $f(x)=5\cos\pi x$

47. $f(x)=\dfrac{x^2}{x+1}$

48. $f(x)=e^{3x+1}$

49. $f(x)=\cos\left(2x^3\right)$

50. $f(x)=\cos(2x-\pi)$

51. $f(x)=x^2\sin\left(\frac{x}{3}\right)$

52. $f(x)=\cos^2\left(\frac{x}{2}\right)$

53. $f(x)=\sin x\cos x$

54. $f(x)=\tan^{-1}\left(5x^2\right)$

55. $f(x)=\ln\left(2+x^2\right)$

56. $f(x)=\dfrac{1+x^3}{1+x^2}$

57. $f(x)=\ln\dfrac{1+x}{1-x}$

58. $f(x)=\dfrac{e^{2x^2}-1}{x^2}$

59. $f(x)=\cosh x-\cos x$

60. $f(x)=\sinh x-\sin x$

$(61-64)$ Finding Taylor and Maclaurin Series generated by f *at* $x=a$

61. $f(x)=x^3-2x+4,\quad a=2$

62. $f(x)=2x^3+x^2+3x-8,\quad a=1$

63. $f(x)=3x^5-x^4+2x^3+x^2-2,\quad a=-1$

64. $f(x)=\cos\left(2x+\frac{\pi}{2}\right),\quad a=\frac{\pi}{4}$

$(65-73)$ Find the Taylor series of the functions, where is each series representation valid?

65. $f(x)=e^{-2x}$ *about* -1

66. $f(x)=\sin x$ *about* $\frac{\pi}{2}$

67. $f(x)=\ln x$ *in powers of* $x-3$

68. $f(x)=\ln(2+x)$ *in powers of* $x-2$

69. $f(x)=e^{2x+3}$ *in powers of* $x+1$

70. $f(x)=\sin x-\cos x$ *about* $\frac{\pi}{4}$

71. $f(x)=\cos^2 x$ *about* $\frac{\pi}{8}$

72. $f(x)=\dfrac{x}{1+x}$ *in powers of* $x-1$

73. $f(x)=xe^{x}$ *in powers of* $x+2$

(74 – 86) Find the nth−order Taylor polynomial centered at c for the function.

74. $f(x)=\frac{2}{x},\quad n=3,\quad c=1$

75. $f(x)=\frac{1}{x^2},\quad n=4,\quad c=2$

76. $f(x)=\sqrt{x},\quad n=3,\quad c=4$

77. $f(x)=\sqrt[3]{x},\quad n=3,\quad c=8$

78. $f(x)=\ln x,\quad n=4,\quad c=2$

79. $f(x)=x^2\cos x,\quad n=2,\quad c=\pi$

80. $f(x)=\sin 2x;\quad n=3,\quad c=0$

81. $f(x)=\cos x^2;\quad n=2,\quad c=0$

82. $f(x)=e^{-x};\quad n=2,\quad c=0$

83. $f(x)=\cos x;\quad n=2,\quad c=\frac{\pi}{4}$

84. $f(x)=\ln x;\quad n=2,\quad c=1$

85. $f(x)=\sinh 2x;\quad n=4,\quad c=0$

86. $f(x)=\cosh x;\quad n=3,\quad c=\ln 2$

(87 – 89) Find the sums of the series.

87. $1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots$

88. $1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots$

89. $x^3-\frac{x^9}{3!\times 4}+\frac{x^{15}}{5!\times 16}-\frac{x^{21}}{7!\times 64}+\frac{x^{27}}{9!\times 256}-\cdots$

(90 – 102) Use the geometric series $\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following functions.

90. $f(x)=\frac{1}{1-x^2}$

91. $g(x)=\frac{3x}{1-x^2}$

92. $h(x)=\frac{1}{(1-x)^2}$

93. $f(x)=\frac{1}{1+x^3}$

94. $g(x)=\frac{2x^2}{1+x^3}$

95. $h(x)=\frac{3x^2}{\left(1+x^3\right)^2}$

96. $f(x)=\frac{1}{5-3x}$

97. $f(x)=\frac{1}{6+7x}$

98. $f(x)=\frac{x}{4-x}$

99. $f(x)=\frac{1}{1+5x}$

100. $f(x)=\frac{10}{1+x}$

101. $f(x)=\frac{1}{(1-10x)^2}$

102. $f(x)=\ln(1-4x)$

$(\mathbf{103-137})$ Find a formula (in Summation Notation if possible) for the given function (If Possible).

103. $f(x)=\sin x$

104. $f(x)=\tan x$

105. $f(x)=\cot x$

106. $f(x)=\sec x$

107. $f(x)=\csc x$

108. $f(x)=\cos^{-1} x$

109. $f(x)=\sin^{-1} x$

110. $f(x)=\tan^{-1} x$

111. $f(x)=\cot^{-1} x$

112. $f(x)=\csc^{-1} x$

113. $f(x)=\cosh x$

114. $f(x)=\cosh^{-1} x$

115. $f(x)=\sinh x$

116. $f(x)=\sinh^{-1} x$

117. $f(x)=\tanh x$

118. $f(x)=\tanh^{-1} x$

119. $f(x)=\coth x$

120. $f(x)=\coth^{-1} x$

121. $f(x)=\operatorname{sech} x$

122. $f(x)=\operatorname{sech}^{-1} x$

123. $f(x)=\operatorname{csch} x$

124. $f(x)=\operatorname{csch}^{-1} x$

125. $f(x)=\ln x$

126. $f(x)=\dfrac{1}{x}$

127. $f(x)=\dfrac{1}{1-x}$

128. $f(x)=\dfrac{1}{x+1}$

129. $f(x)=\sqrt{x}$

130. $f(x)=\sqrt{1-x}$

131. $f(x)=\sqrt{1+x}$

132. $f(x)=e^{-x^2}$

133. $f(x)=e^{\sin x}$

134. $f(x)=e^{\cos x}$

135. $f(x)=e^{\tan x}$

136. $f(x)=e^{x}\sin x$

137. $f(x)=e^{x}\cos x$

138. The limit $\displaystyle\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi}\,n^{n+1/2}e^{-n}}=1$ that is the relative error in the approximation

$n!\approx\sqrt{2\pi}\,n^{n+1/2}e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\,n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!)=\displaystyle\sum_{j=1}^{n}\ln j$ and the increasing nature of ln to show that if $n\geq 1$,

$$\int_0^n \ln x\,dx \;<\; \ln(n!) \;<\; \int_1^{n+1}\ln x\,dx$$

And hence that $n\ln n - n < \ln(n!) < (n+1)\ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n$, show that

$$c_n - c_{n+1} = \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

c) Use the Maclaurin series for $\ln\frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \cdots\right)$$

$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim_{n\to\infty} c_n = c$ exists, and that:

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

139. Consider the function $f(x) = \sqrt[3]{x}$.

a) Approximate $f(x)$ by Taylor polynomial of degree 3 at $a = 27$?

b) How accurate is this approximation when $26 \le x \le 28$

140. Consider the function $f(x) = \sqrt{x}$.

a) Approximate $f(x)$ by Taylor polynomial of degree 4 at $a = 3$?

b) How accurate is this approximation when $2.9 \le x \le 3.1$

141. Consider the function $f(x) = \sin x$.

a) Approximate $f(x)$ by Taylor polynomial to 6 decimals at $a = 12°$?

b) How accurate is this approximation when $-0.3 \le x \le 0.3$

c) Use the approximation to find $\sin 12°$ to 6 decimals.

d) For what value of x is the approximation accurate within 5×10^{-5}

e) Find the upper bound error involving estimation of $\sin 12°$

142. Consider the function $f(x) = e^x$.

a) Approximate $f(x)$ by Taylor polynomial of degree 6 at $a = 0$?

b) How accurate is this approximation when $2.716 \le x \le 2.72$

c) For what value of x is the approximation accurate within 5×10^{-5}

d) How large must n be to approximate accuracy by no more than 10^{-7} ?

143. Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

a) Use a Taylor polynomial for $f(x) = (125 + x)^{1/3}$ centered at 0.

b) Use a Taylor polynomial for $f(x) = x^{1/3}$ centered at 125.

c) Compare the two approaches. Are they equivalent?

144. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \ne 0 \\ 0 & if \quad x = 0 \end{cases}$$

a) Use the definition of the derivative to show that $f'(0) = 0$

b) Assume the fact that $f^k(0) = 0$ for $k = 1, 2, 3, \ldots$. (Prove using the definition of the derivative.) Write the Taylor series for f centered at 0.

c) Explain why the Taylor series for f does not converge to f for $x \ne 0$

145. Teams A and B go into sudden death overtime after playing to a tie. The team's alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team A having the ball first.

a) The probability that Team A ultimately wins is $\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

b) The expected number of rounds (possessions by either team) required for the overtime to end is $\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.

Summary

Series or Test	Form of series	Convergence/Divergence
Geometric series	$\sum_{n=0}^{\infty} a r^n$	***Convergence*** $\lvert r\rvert < 1$
		Divergence $\lvert r\rvert \geq 1$
Divergence Test	$\sum_{n=1}^{\infty} a_n$	***Divergence*** $\lim_{n\to\infty} a_n \neq 0$
Integral Test	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $\int_1^{\infty} f(x)\,dx$
		Divergence $\int_1^{\infty} f(x)\,dx = \infty$
$p-$series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	***Convergence*** $p > 1$
		Divergence $p \leq 1$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$
		Divergence $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$
		Divergence $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$
Comparison Test	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $0 < a_n \leq b_n$ $\sum_{n=1}^{\infty} b_n$ ***converges***
		Divergence $0 < b_n \leq a_n$ $\sum_{n=1}^{\infty} b_n$ ***diverges***
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $0 \leq \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ $\sum_{n=1}^{\infty} b_n$ ***converges***

		Divergence $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ $\sum_{n=1}^{\infty} b_n$ ***diverges***
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n a_n$	***Convergence*** **1.** $a_n > 0$ **2.** $a_n \geq a_{n+1}$ **3.** $\lim_{n\to\infty} a_n = 0$ ***Divergence*** $\lim_{n\to\infty} a_n \neq 0$
Absolute Convergence	$\sum_{n=1}^{\infty} a_n$	***Convergence*** $\sum_{n=1}^{\infty} \lvert a_n \rvert$

$$\sum_{n=0}^{\infty} u_n (x-a)^n$$

The ***centre*** of convergence is $x = a$

$$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} \left| \frac{(x-a)^{n+1}}{(x-a)^n} \right| < 1$$

The ***Radius*** of convergence is $R = \frac{1}{L}$ *or* $R = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} \left| \frac{(x-a)^{n+1}}{(x-a)^n} \right|$

If $R = 0$: The series ***diverges absolutely*** for all x except $x = 0$.

If $R = \infty$: The series converges on ***interval*** $(-\infty, \infty)$

The ***interval*** of convergence $a - R < x < a + R$

Taylor series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

***Maclaurin series*:** $$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

Exercises *Lecture 7* – Review

$(\mathbf{1-6})$ Determine if the sequence ***converges*** or ***diverges***. Give reasons for your answers.

1. $a_n = \dfrac{1-(-1)^n}{\sqrt{n}}$

2. $a_n = \sin n\pi$

3. $a_n = \dfrac{n+\ln n}{n}$

4. $a_n = \left(1+\dfrac{1}{n}\right)^n$

5. $a_n = \dfrac{1}{n^2}\sin\dfrac{\pi}{2n}$

6. $a_n = \dfrac{(2n+1)^{2n}}{\left(3n^2+1\right)^n}$

$(\mathbf{7-42})$ Determine if the series ***converges*** or ***diverges***. Give reasons for your answers.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\left(n^2+1\right)}{2n^2+n-1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\,3^n}{n^n}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n+1}}{2^n}$

16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\,\frac{1+\sqrt{n+1}}{2^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot 3^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n-1}$

23. $\displaystyle\sum_{n=1}^{\infty} e^{1/n^2}$

24. $\displaystyle\sum_{n=1}^{\infty} e^n$

25. $\displaystyle\sum_{n=1}^{\infty} \pi^{-n}$

26. $\displaystyle\sum_{n=1}^{\infty} n^{-\pi}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+6}}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^n+1}$

30. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+2)}{n+2}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+2)}{\sqrt{n+2}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

34. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2}n\right) e^{-n}$

35. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)\, e^{-n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n\, n!}$

37. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{\pi}}{1+n^e}$

38. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{1+n^2}$

39. $\displaystyle\sum_{n=1}^{\infty} n^{-\frac{n+1}{n}}$

40. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+2}}{1+n}$

41. $\displaystyle\sum_{n=1}^{\infty} (2)^{1/n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n-2}}{2^2 \cdot 4^2 \cdot \ldots \cdot (2n-2)^2}$

$(\mathbf{43-51})$ Find the ***centre***, ***radius***, and ***interval*** of convergence of the series.

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x)^n}{n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} x^{n+1}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} (x+1)^n$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} (x-1)^n$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1} x^n$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} x^n$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{4\cdot 9\cdot 14\cdot\cdots\cdot(5n-1)} (x-1)^n$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5^{n+1}\,(n+1)} (x-2)^{2n}$

$(\mathbf{52-57})$ Finding Taylor and Maclaurin Series generated by $f\ at\ \ x=0$.

52. $f(x) = \dfrac{1}{1-2x}$

53. $f(x) = \dfrac{1}{1+x^3}$

54. $f(x) = \sin\dfrac{2x}{3}$

55. $f(x) = e^{-x^2}$

56. $f(x) = (x-2)^2$

57. $f(x) = \dfrac{1}{(x-2)^2}$

$(\mathbf{58-66})$ Finding Taylor and Maclaurin Series generated by f at $x = a$.

58. $f(x)=\sqrt{3+x^2}$, $a=-1$

59. $f(x)=\dfrac{1}{1-x}$, $a=2$

60. $f(x)=\dfrac{1}{x+1}$, $a=3$

61. $f(x)=\sqrt{x^3}$, $a=1$

62. $f(x)=3-x$, $a=1$

63. $f(x)=x^3$, $a=-1$

64. $f(x)=\sin x$, $a=\dfrac{\pi}{2}$

65. $f(x)=\cos x$, $a=0$

66. $f(x)=e^x$, $a=-1$

Practice *Exam*

1. Use the ***geometric series*** to determine if the series converge (***find*** the value) or diverge

$$\sum_{n=0}^{\infty} e^{-2n}$$

2. Use the ***integral Test*** to determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

3. Use the ***Comparison Test*** to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

4. Use the ***Ratio Test*** to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\,(n+2)!}{n!\,3^{2n}}$

5. Use the ***Root Test*** to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

6. Determine if the ***alternating series*** converges or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$

Use ***any method*** to determine if the series converges or diverges. Give reasons for your answer

7. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{\sqrt{k}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,2^n(n+1)!}{3^n\,n!}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^{3/5}}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{5^n}$

13. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

15. Determine the ***centre***, the ***radius***, and the ***interval*** of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!}x^n$

16. Find the **Taylor** polynomials of orders 0, 1, 2, and 3 generated by f *at* a $\quad f(x)=x^3, \quad a=1$

17. Find the **Maclaurin *series*** for: $f(x)=xe^x \qquad n=3$

Lecture Eight

Parametric Equations and Polar Coordinates

Section 8.1 – Parameterization of Plane Curves

If you take one step, from a position of a point (x_0, y_0) to next position point (x_1, y_1). That traveling it take time to get to the next point. Therefore, x and y must be in function of time.

8.1−1 *Definition*

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

Over an interval I as t–values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ***parametric curve***. The equations are ***parametric equations*** for the curve

The variable t is a parameter for the curve, and its domain I is the parameter interval.

8.1–2 *Definition*

The direction in which a parametric curve is generated as the parameter increases is called the forward or positive orientation of the curve.

Example 1

Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$$

Then obtain an algebraic equation in x and y.

Solution

t	$x = t^2$	$y = t + 1$
–3	9	–2
–2	4	–1
–1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

$$y = t + 1$$
$$\underline{t = y - 1}$$

$$x = t^2$$
$$= (y-1)^2$$
$$\underline{= y^2 - 2y + 1}$$

It ***represents a parabola.***

Example 2

Graph the parametric curve $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

Solution

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

This parametric curve lies along the unit circle $x^2 + y^2 = 1$.

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at (1, 0) and traces the entire circle once counterclockwise.

$x^2 + y^2 = 1$

Example 3

Graph the parametric curve $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$

Solution

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2\left(\cos^2 t + \sin^2 t\right) = a^2$$

This parametric curve lies along the unit circle $x^2 + y^2 = a^2$.

As t increases from 0 to 2π, the point $(x, y) = (a\cos t, a\sin t)$ starts at $(a, 0)$ with a radius $r = a$ and traces the entire circle once counterclockwise.

Example 4

Graph the parametric curve $x = t, \quad y = t^2, \quad -\infty < t < \infty$

Solution

$y = x^2$

t	x	y
–2	–2	4
–1	–1	1
0	0	0
1	1	1
2	2	4

Example 5

Find a parameterization for the line through the point (a, b) having slope m.

Solution

A Cartesian equation of the line is $y - b = m(x - a)$

If $t = x - a \quad \Rightarrow \quad x = t + a$

$y - b = mt \quad \Rightarrow \quad y = mt + b$

That is, $x = t + a \quad y = mt + b, \quad -\infty < t < \infty$

Example 6

Sketch the curve defined by the parametric equations $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$

Then obtain an algebraic equation in x and y.

Solution

t	$x = t + \frac{1}{t}$	$y = t - \frac{1}{t}$
.2	5.2	–4.8
.5	2.5	–1.5
1	2	0
2	2.5	1.5
5	5.2	4.8

$$x + y = t + \frac{1}{t} + t - \frac{1}{t}$$

$$= 2t$$

$$x - y = t + \frac{1}{t} - t + \frac{1}{t}$$

$$= \frac{2}{t}$$

$$(x + y)(x - y) = 2t\left(\frac{2}{t}\right)$$

$$= 4$$

$$x^2 - y^2 = 4$$

8.1–3 Cycloids

The problem with a pendulum clocks whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing.

Cycloid

Example 7

A wheel of a radius *a* roll along a horizontal straight line. Find parametric equations for the path traced by a point *P* on the wheel's circumference. The path is called a cycloid.

Solution

We take the line to be on the *x*-axis, mark a point *P* on the wheel, start the wheel with *P* at the origin, and roll the wheel.

As parameter, we use the angle *t* through which the wheel turns, measured in radians.

The wheel's center *C* lies at $(at,\ a)$ and the coordinates of *P* are

$$x = at + a\cos\theta\,, \quad y = a + a\sin\theta$$

To express θ in terms of *t*, we observe that $t + \theta = \frac{3\pi}{2}$, so that $\theta = \frac{3\pi}{2} - t$

That makes $\cos\theta = \cos\left(\frac{3\pi}{2} - t\right)$

$$\underline{= -\sin t}$$

$$\sin\theta = \sin\left(\frac{3\pi}{2} - t\right)$$

$$\underline{= -\cos t}$$

$$x = at - a\sin t\,, \quad y = a - a\cos t$$

That implies to: $x = a(t - \sin t)\,, \quad y = a(1 - \cos t)$

Exercises *Section* 8.1 – Parameterization of Plane Curves

(1 – 40) Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

1. $x = 3t$, $y = 9t^2$, $-\infty < t < \infty$

2. $x = -\sqrt{t}$, $y = t$, $t \ge 0$

3. $x = 3 - 3t$, $y = 2t$, $0 \le t \le 1$

4. $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le \pi$

5. $x = 4\sin t$, $y = 5\cos t$, $0 \le t \le 2\pi$

6. $x = t^2$, $y = t^6 - 2t^4$, $-\infty < t < \infty$

7. $x = \dfrac{t}{t-1}$, $y = \dfrac{t-2}{t+1}$, $-1 < t < 1$

8. $x = \sqrt{t+1}$, $y = \sqrt{t}$, $t \ge 0$

9. $x = \sqrt{t} + 4$, $y = 3\sqrt{t}$; $0 \le t \le 16$

10. $x = t - 1$, $y = t^3$; $-4 \le t \le 4$

11. $x = e^{2t}$, $y = e^t + 1$; $0 \le t \le 25$

12. $x = e^t$, $y = e^{t+1}$; $t \ge 0$

13. $x = e^t$, $y = e^{-t}$; $t \ge 0$

14. $x = e^t$, $y = e^{1-t}$; $t \ge 0$

15. $x = e^t$, $y = 2\ln t$; $t > 0$

16. $x = 3^{2t}$, $y = 3^{t+1}$; $t \ge 0$

17. $x = 2^t$, $y = 2^{t-1}$; $t \ge 1$

18. $x = t^2 + 2$, $y = 4t$; $-4 \le t \le 4$

19. $x = t - 2$, $y = 4t$; $-\infty \le t \le \infty$

20. $x = 2t$, $y = 3t - 4$; $-10 \le t \le 10$

21. $x = -t + 6$, $y = 3t - 3$; $-5 \le t \le 5$

22. $x = t^3 - 1$, $y = 5t + 1$; $-3 \le t \le 3$

23. $x = \cos t$, $y = \sin^2 t$, $0 \le t \le \pi$

24. $x = 3\cos t$, $y = 3\sin t$; $\pi \le t \le 2\pi$

25. $x = -7\cos 2t$, $y = -7\sin 2t$; $0 \le t \le \pi$

26. $x = 1 - \sin^2 t$, $y = \cos t$, $\pi \le t \le 2\pi$

27. $x = \cos t$, $y = 1 + \sin t$; $0 \le t \le 2\pi$

28. $x = 2\sin t - 3$, $y = 5 + \cos 2t$; $0 \le t \le 2\pi$

29. $x = 4\cos t - 1$, $y = 4\sin t + 2$; $0 \le t \le 2\pi$

30. $x = 1 - 3\sin 4\pi t$, $y = 2 + 3\cos 4\pi t$; $0 \le t \le \frac{1}{2}$

31. $x = \cos(\pi - t)$, $y = \sin(\pi - t)$, $0 \le t \le \pi$

32. $x = 2\sinh t$, $y = 2\cosh t$, $-\infty < t < \infty$

33. $x = 1 + \sin t$, $y = \cos t - 2$, $0 \le t \le 2\pi$

34. $x = 4\cos 2\pi t$, $y = 4\sin 2\pi t$, $0 \le t \le 1$

35. $x = (t+1)^2$, $y = t + 2$; $-10 \le t \le 10$

36. $x = 4\tan 2t$, $y = 3\sec 2t$; $0 < t < \pi$

37. $x = 2\tan 2t$, $y = 4\sec 2t$; $0 < t < \pi$

38. $x = 1 + \sin^2 t$, $y = 2 - \cos t$; $0 \le t \le \pi$

39. $x = \frac{1}{2}\ln\left(t^2 - 1\right)$, $y = \sqrt{t^2 - 1}$; $-1 < t < 1$

40. $x = t + \frac{1}{t}$, $y = 2\ln t$; $t > 0$

$(\mathbf{41-44})$ For the given parametric equations:

a) Plot the following curve, indicating the positive orientation.
b) Eliminate the parameter to obtain an equation in x and y.
c) Identify or briefly describe the curve.
d) Evaluate $\frac{dy}{dx}$ at the specified point.

41. $x = t^2 + 4, \quad y = 6 - t; \quad -\infty \le t \le \infty \quad (5,\ 5)$

42. $x = e^t, \quad y = 3e^{-2t}; \quad -\infty \le t \le \infty \quad (1,\ 3)$

43. $x = 10\sin 2t, \quad y = 16\cos 2t; \quad 0 \le t \le \pi \quad (5\sqrt{3},\ 8)$.

44. $x = \ln t, \quad y = 8\ln t^2; \quad 1 \le t \le e^2 \quad (1,\ 16)$

$(\mathbf{45-59})$ Find parametric equations:

45. For the left half of the parabola $y = x^2 + 1$, originating at $(0,\ 1)$

46. A line that passes through the points $(1,\ 1)$ and $(3,\ 5)$, oriented in the direction of increasing x.

47. Lower half of the circle centered at $(-2,\ 2)$ with radius 6, oriented in the counterclockwise direction.

48. Upper half of the parabola $x = y^2$, originating at $(0,\ 0)$

49. The circle centered at the origin with radius 4, generated counterclockwise.

50. The circle centered at the origin with radius 12, generated clockwise with initial point $(0,\ 12)$

51. The circle centered at $(2,\ 3)$ with radius 1, generated counterclockwise.

52. The circle centered at $(2,\ 0)$ with radius 3, generated clockwise.

53. The circle centered at $(-2,\ -3)$ with radius 8, generated clockwise.

54. The circle $x^2 + y^2 = 9$, generated clockwise.

55. The upper half of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.

56. The right side of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.

57. The line $y = 4x + 11$

58. The line segment from $P(-1,\ 0)$ to $Q(1,\ 1)$ and the line segment from Q to P.

59. The segment of the curve $f(x) = x^3 + 2x$ from $(0,\ 0)$ to $(2,\ 12)$

60. What is the relationship among a, b, c, and d such that the equations?

$$\begin{cases} x(t) = a\cos t + b\sin t \\ y(t) = c\cos t + d\sin t \end{cases} \qquad \text{describe a circle?}$$

What is the radius of the circle?

61. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the x-*axis* and minor axis of length 3 on the y-*axis*, generated counterclockwise. Graph the ellipse and find a description in terms of x and y.

62. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the x-*axis* and minor axis of length 2 on the y-*axis*, generated clockwise. Graph the ellipse and find a description in terms of x and y.

63. Find parametric equations (not unique) of an ellipse centered at $(-2,\ -3)$ with major and minor axes of lengths 30 and 20, parallel to the x-*axis* and y-*axis*, respectively. Graph the ellipse and find a description in terms of x and y.

64. Find a parametric equation and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2 + y^2 = 4$ four times.

65. Find a parametrization for the line segment joining points $(0,\ 2)$ and $(4,\ 0)$ using the angle θ in the accompanying figure as the parameter.

66. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at (1, 0) and moving counterclockwise to the terminal point (0, 1), using the angle θ in the accompanying figure as the parameter.

67. A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.

a) The segment of the parabola $y = 9 - x^2$, *for* $-1 \le x \le 3$

b) The complete curve $x = (y-5)^2 + \sqrt{y}$

c) The piecewise linear path that connects $P(-2,\ 0)$ to $Q(0,\ 3)$ to $R(4,\ 0)$ (in that order), where the parameter varies over the interval $0 \le t \le 2$

68. A projectile launched from the ground with an initial speed of 20 *m/s* and a launch angle θ follows a trajectory approximated by

$$x = (20\cos\theta)t\,, \quad y = -4.9t^2 + (20\sin\theta)t$$

Where *x* and *y* are the horizontal and vertical positions of the projectile relative to the launch point $(0,\ 0)$.

a) Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.

b) Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?

69. Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling when is a cycloid, which has the parametric equations

$$\begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases} \quad \textit{for} \quad t \ge 0$$

Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

(70 – 71) Find parametric equations that describe the circular path of the objects. Assume $(x,\ y)$ denotes the position of the object relative to the origin at the center of the circle.

70. A go-cart moves counterclockwise with constant speed around a circular track of radius 400 *m*, completing in 1.5 *min*.

71. The tip of the 15-*in* second hand of a clock completes one revolution in 60 *sec*.

72. A Ferris wheel has a radius of 20 *m* and completes a revolution in the clockwise direction at constant speed in 3 *min*. Assume that *x* and *y* measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

73. A plane traveling horizontally at 80 *m/s* over flat ground at an elevation of 3000 *m* releases an emergency packet. The trajectory of the packet is given by

$$x = 80t, \quad y = -4.9t^2 + 3000, \quad for \;\; t \geq 0$$

Where the origin is the point on the ground directly beneath the plane, at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

74. A plane traveling horizontally at 100 *m/s* over flat ground at an elevation of 4,000 *m* must drop an emergency packet on a target on the ground. The trajectory of the packet is given by

$$x(t) = 100t\,, \quad y(t) = -4.9t^2 + 4{,}000\,; \quad for \;\; t \geq 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release.

How many horizontal meters before the target should the packet be released in order to hit the target?

75. The path of a point on circle *A* with radius $\frac{a}{4}$ that rolls on the inside of circle *B* with a radius *a* is an asteroid or hypocycloid. Its parametric equations are

$$x = a\cos^3 t\,, \quad y = a\sin^3 t\,, \quad 0 \le t \le 2\pi$$

Graph the asteroid with $a = 1$ and find its equation in terms of x and y.

Section 8.2 – Calculus with Parametric Curves

8.2–1 Definition

Tangents and *Areas*

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at *t* if *f* and *g* are differentiable at *t*.

8.2–2 Parametric Formula for $\frac{dy}{dx}$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

8.2–3 Parametric Formula for $\frac{d^2y}{dx^2}$

If the equations $x = f(t)$, $y = g(t)$ define *y* as a twice–differentiable function of *x*, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Example 1

Find the tangent to the curve $x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $\left(\sqrt{2},\ 1\right)$, where $t = \frac{\pi}{4}$

Solution

The slope of the curve at *t* is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sec^2 t}{\sec t \tan t}$$

$$= \frac{\sec t}{\tan t}$$

$$= \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t}$$

$$= \frac{1}{\sin t}$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}$$

$$= \frac{1}{\sin\frac{\pi}{4}}$$

$$= \sqrt{2}$$

The tangent line is

$$y = \sqrt{2}\left(x - \sqrt{2}\right) + 1 \qquad y = m\left(x - x_1\right) + y_1$$

$$= \sqrt{2}x - 2 + 1$$

$$= \sqrt{2}\,x - 1$$

$$1 + \tan^2 t = \sec^2 t$$

$$1 + y^2 = x^2$$

$$\frac{d}{dx}\left(1 + y^2\right) = \frac{d}{dx}\left(x^2\right)$$

$$2yy' = 2x$$

$$y' = \frac{x}{y}$$

$$m = \frac{\sec t}{\tan t}$$

Example 2

Find $\frac{d^2y}{dx^2}$ as a function of t if $x(t) = t - t^2, \quad y(t) = t - t^3$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{-3t^2+1}{-2t+1}\right)$$

$$= \frac{6t^2-6t+2}{(1-2t)^2}$$

$$\begin{matrix} -3 & 0 & 1 \\ 0 & -2 & 1 \end{matrix}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{6t^2-6t+2}{(1-2t)^2} \div (1-2t)$$

$$= \frac{6t^2-6t+2}{(1-2t)^3}$$

Example 3

Find $\frac{d^2y}{dx^2}$ as a function of t if $x(t) = t - \ln t, \quad y(t) = t + \ln t$

Solution

$$y' = \frac{1+\frac{1}{t}}{1-\frac{1}{t}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{t+1}{t-1}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{t+1}{t-1}\right)$$

$$= -\frac{2}{(t-1)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{2}{(t+1)^2} \cdot \frac{t}{t-1}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= -\frac{2t}{(t-1)^3}$$

8.2–4 Area under a Parametric Curve

The area under a plane curve is given by

$$\begin{aligned} Area &= \int_a^b |y\ dx| \\ &= \int_a^b \left| y(t)\ \frac{dx}{dt} \right| dt \end{aligned}$$

Example 4

Find the area enclosed by the asteroid: $x(t) = \cos^3 t\,,\quad y(t) = \sin^3 t\,,\quad 0 \le t \le 2\pi$

Solution

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \frac{\pi}{2}$.

$$A = 4\int_0^1 |y\ dx|$$

$$= 4\int_0^{\pi/2} \sin^3 t\ \left| d\left(\cos^3 t\right) \right| \qquad d\left(\cos^3 t\right) = -3\cos^2 t \sin t\ dt$$

$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t\ \sin t\ dt$$

$$= 12\int_0^{\pi/2} \sin^4 t \cdot \cos^2 t\ dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - 2\cos 2t + \cos^2 2t\right)\left(1 + \cos 2t\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - 2\cos 2t + \cos^2 2t + \cos 2t - 2\cos^2 2t + \cos^3 2t\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2}\left(1-\cos 2t-\cos^2 2t+\cos^3 2t\right)\,dt$$

$$= \frac{3}{2}\left(t-\frac{1}{2}\sin 2t\,\Big|_0^{\pi/2} - \frac{3}{2}\int_0^{\pi/2}\cos^2 2t\;dt + \frac{3}{2}\int_0^{\pi/2}\cos^3 2t\;dt\right. \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}-0\right) - \frac{3}{2}\int_0^{\pi/2}\frac{1+\cos 2t}{2}\;dt + \frac{3}{2}\int_0^{\pi/2}\left(1-\sin^2 2t\right)\cos 2t\;dt$$

$$= \frac{3\pi}{4}-\frac{3}{4}\left(t+\frac{1}{2}\sin 2t\,\Big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\left(1-\sin^2 2t\right)d\left(\sin 2t\right)\right.$$

$$= \frac{3\pi}{4}-\frac{3}{4}\left(\frac{\pi}{2}-0\right)+\frac{3}{4}\left(\sin 2t-\frac{1}{3}\sin^3 2t\,\Big|_0^{\pi/2}\right.$$

$$= \frac{3\pi}{4}-\frac{3\pi}{8}+\frac{3}{4}(0-0)$$

$$= \frac{3\pi}{8}\;unit^2$$

Example 4

Find the area: $x(t)=e^t\,,\quad y(t)=\sin t\,,\quad 0\le t\le\frac{\pi}{2}$

Solution

$$Area = \int_0^{\frac{\pi}{2}}\left|\sin t\;d\left(e^t\right)\right| \qquad A=\int_a^b |y\;dx|$$

$$= \int_0^{\frac{\pi}{2}}\sin t\;e^t\;dt$$

		$\int \sin t$
+	e^t	$-\cos t$
−	e^t	$-\sin t$
+	e^t	

$$\int \sin t\, e^t\, dt = (-\cos t + \sin t)\, e^t - \int \sin t\, e^t\, dt$$

$$2\int \sin t\, e^t\, dt = (-\cos t + \sin t)\, e^t$$

$$\int \sin t\, e^t\, dt = \frac{1}{2}(-\cos t + \sin t)\, e^t$$

$$= \frac{1}{2}(-\cos t + \sin t)\, e^t \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(e^{\pi/2} + 1\right) \quad unit^2$$

Example 5

Find the area: $x(t) = 1 - \cos t\,, \quad y(t) = t - \sin t\,, \quad 0 \le t \le 2\pi$

Solution

$$Area = \int_0^{2\pi} (t - \sin t)\, d(1 - \cos t) \qquad A = \int_a^b |y\, dx|$$

$$= \int_0^{2\pi} (t - \sin t)\, \sin t\, dt$$

$$= \int_0^{2\pi} \left(t\, \sin t - \sin^2 t\right) dt$$

$$= \int_0^{2\pi} \left(t\, \sin t - \frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$$

		$\int \sin t$
+	t	$-\cos t$
−	1	$-\sin t$

$$= \left| -t\cos t + \sin t - \frac{1}{2}t + \frac{1}{4}\sin 2t \right| \Big|_0^{2\pi}$$

$$= |-2\pi - \pi|$$

$$= 3\pi \quad unit^2$$

8.2–5 Length of a Parametrically Defined Curve

8.2–6 *Definition*

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Example 6

Find the length of the circle of radius r defined parametrically by

$x(t) = r\cos t$, $y(t) = r\sin t$; $0 \le t \le 2\pi$

Solution

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\, dt$$

$$= \int_0^{2\pi} r\, dt$$

$$= rt \Big|_0^{2\pi}$$

$$= 2\pi r \ \ unit$$

Example 7

Find the length of the asteroid: $x(t)=\cos^3 t\,,\quad y(t)=\sin^3 t\,;\quad 0\le t\le 2\pi$

Solution

Because of the curve's symmetry with respect to the coordinate axes, its length is 4 times the length of the first quadrant.

$$\left(\frac{dx}{dt}\right)^2=\left[3\cos^2 t(-\sin t)\right]^2$$
$$=9\cos^4 t\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2=\left[3\sin^2 t(\cos t)\right]^2$$
$$=9\sin^4 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}$$
$$=3|\cos t\sin t|\sqrt{\cos^2 t+\sin^2 t} \qquad \cos^2 t+\sin^2 t=1$$
$$=3\cos t\sin t \qquad \cos t\sin t\ge 0,\quad 0\le t\le \frac{\pi}{2}$$

$$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=4\int_0^{\pi/2}3\cos t\sin t\,dt \qquad \sin 2t=2\cos t\sin t$$
$$=\frac{12}{2}\int_0^{\pi/2}\sin 2t\,dt$$
$$=-\frac{6}{2}\cos 2t\,\Big|_0^{\pi/2}$$
$$=-3(-1-1)$$
$$=-3(-2)$$
$$=6\ \ unit$$

8.2–7 Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x–axis $(y \ge 0)$:**

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y–axis $(x \ge 0)$:**

$$S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 8

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy–plane is

$$x(t) = \cos t, \quad y(t) = 1 + \sin t, \quad 0 \le t \le 2\pi$$

Use the parametrization to find the area of the surface swept out by revolving the circle about the x–axis.

Solution

$$x = \cos t \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2$$

$$y = 1 + \sin t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (\cos t)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t)\ dt$$

$$= 2\pi \left[t - \cos t\right]_0^{2\pi}$$

$$= 2\pi\left(2\pi - 1 - (0-1)\right)$$

$$= 4\pi^2 \ \ unit^2$$

Example 9

Find the area of the surface of $x(t) = t^2$, $y(t) = \ln t$, $1 \le t \le 2$ by revolving about the y–axis.

Solution

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t)^2 + \left(\frac{1}{t}\right)^2}$$

$$= \sqrt{4t^2 + \frac{1}{t^2}}$$

$$= \frac{1}{t}\sqrt{4t^4+1}$$

$$S = 2\pi \int_1^2 \left(t^2\right) \frac{1}{t} \sqrt{4t^4+1}\ dt \qquad S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi \int_1^2 t\sqrt{4t^4+1}\ dt$$

$$2t^2 = \tan\theta \qquad \sqrt{4t^4+1} = \sec\theta$$

$$4t\ dt = \sec^2\theta\ d\theta$$

$$= 2\pi \int_1^2 \sqrt{\frac{1}{2}\tan\theta}\ \sec\theta\ \frac{1}{4\sqrt{\frac{1}{2}\tan\theta}}\ \sec^2\theta\ d\theta$$

$$= \frac{\pi}{2} \int_1^2 \sec^3\theta\ d\theta$$

$$u = \sec\theta \qquad v = \int \sec^2\theta\, d\theta$$

$$du = \sec\theta\tan\theta\, d\theta \qquad = \tan\theta$$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \tan\theta\left(\sec\theta\tan\theta\, d\theta\right)$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\,\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|$$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\,2t^2\sqrt{4t^4+1} + \frac{1}{2}\ln\left|\sqrt{4t^4+1} + 2t^2\right|\right)\Bigg|_1^2$$

$$= \frac{\pi}{2}\left(t^2\sqrt{4t^4+1} + \frac{1}{2}\ln\left|\sqrt{4t^4+1} + 2t^2\right|\right)\Bigg|_1^2$$

$$= \frac{\pi}{2}\left(4\sqrt{65} + \frac{1}{2}\ln\left(\sqrt{65}+8\right) - \sqrt{5} - \frac{1}{2}\ln\left(\sqrt{5}+2\right)\right)$$

$$= \frac{\pi}{2}\left(4\sqrt{65} - \sqrt{5} + \frac{1}{2}\ln\left(\frac{\sqrt{65}+8}{\sqrt{5}+2}\right)\right) \quad unit^2$$

Exercises *Section* 8.2 – Calculus with Parametric Curves

(1 – 4) Find all the points at which the curve has the given slope.

1. $x = 4\cos t$, $y = 4\sin t$; $slope = \frac{1}{2}$

2. $x = 2\cos t$, $y = 8\sin t$; $slope = -1$

3. $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$; $slope = 1$

4. $x = 2 + \sqrt{t}$, $y = 2 - 4t$; $slope = -8$

(5 – 15) Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

5. $x(t) = \sin t$, $y(t) = \cos t$, $t = \frac{\pi}{4}$

6. $x(t) = t^2 - 1$, $y(t) = t^3 + t$, $t = 2$

7. $x(t) = e^t$, $y(t) = \ln(t+1)$, $t = 0$

8. $x(t) = \cos t + t\sin t$, $y(t) = \sin t - t\cos t$, $t = \frac{\pi}{4}$

9. $x(t) = 6t$, $y(t) = t^2 + 4$, $t = 1$

10. $x(t) = t - 2$, $y(t) = \frac{1}{t} + 3$, $t = 1$

11. $x(t) = t^2 - t + 2$, $y(t) = t^3 - 3t$, $t = -1$

12. $x = -t^2 + 3t$, $y = 2t^{3/2}$, $t = \frac{1}{4}$

13. $x(t) = 3t^2 - t$, $y(t) = t^2$, $t = \frac{1}{4}$

14. $x(t) = t^2 - t$, $y(t) = t^2 + 1$, $t = 1$

15. $x(t) = \ln t$, $y(t) = t^2$, $t = 1$

(16 – 40) Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point

16. $x(t) = \sin 2\pi t$, $y(t) = \cos 2\pi t$, $t = -\frac{1}{6}$

17. $x(t) = \cos t$, $y(t) = \sqrt{3}\cos t$, $t = \frac{2\pi}{3}$

18. $x(t) = t$, $y(t) = \sqrt{t}$, $t = \frac{1}{4}$

19. $x(t) = \sec^2 t - 1$, $y(t) = \tan t$ $t = -\frac{\pi}{4}$

20. $x(t) = \frac{1}{t+1}$, $y(t) = \frac{t}{t-1}$ $t = 2$

21. $x(t) = t + e^t$, $y(t) = 1 - e^t$ $t = 0$

22. $x(t) = 4t$, $y(t) = 3t - 2$, $t = 3$

23. $x(t) = \sqrt{t}$, $y(t) = 3t - 1$, $t = 1$

24. $x(t)=t+1$, $y(t)=t^2+3t$, $t=-1$

25. $x(t)=t^2+5t+4$, $y(t)=4t$, $t=0$

26. $x(t)=4\cos t$, $y(t)=4\sin t$, $t=\frac{\pi}{4}$

27. $x(t)=\cos t$, $y(t)=3\sin t$, $t=0$

28. $x(t)=2+\sec t$, $y(t)=1+2\tan t$, $t=\frac{\pi}{6}$

29. $x(t)=\sqrt{t}$, $y(t)=\sqrt{t-1}$, $t=2$

30. $x(t)=\cos^3 t$, $y(t)=\sin^3 t$, $t=\frac{\pi}{4}$

31. $x(t)=t-\sin t$, $y(t)=1-\cos t$, $t=\pi$

32. $x(t)=5t+2$, $y(t)=1-4t$ $t=3$

33. $x(t)=t-6$, $y(t)=t^2$ $t=5$

34. $x(t)=\frac{1}{t}$, $y(t)=2t+3$ $t=-1$

35. $x(t)=\frac{1}{t}$, $y(t)=t^2$ $t=-2$

36. $x(t)=e^t$, $y(t)=e^{-t}$ $t=1$

37. $x(t)=\ln t$, $y(t)=e^t$ $t=1$

38. $x(t)=\ln t$, $y(t)=t^2$ $t=1$

39. $x(t)=\cosh t$, $y(t)=\sinh t$, $t=0$

40. $x(t)=\tan t$, $y(t)=\sec t$, $t=\frac{\pi}{4}$

(41 – 44) Find the equations of the tangent lines at the point where the curve crosses itself

41. $x(t)=2\sin 2t$, $y(t)=3\sin t$

42. $x(t)=2-\pi\cos t$, $y(t)=2t-\pi\sin t$

43. $x(t)=t^2-t$, $y(t)=t^3-3t-1$

44. $x(t)=t^3-6t$, $y(t)=t^2$

(45 – 47) Find the slope of the curve $x=f(t)$, $y=g(t)$ at the given value of t. Define x and y as differentiable functions.

45. $x^3+2t^2=9$, $2y^3-3t^2=4$, $t=2$

46. $x+2x^{3/2}=t^2+t$, $y\sqrt{t+1}+2t\sqrt{y}=4$, $t=0$

47. $t=\ln(x-t)$, $y=te^t$, $t=0$

(48 – 57) Find $\frac{d^2y}{dx^2}$

48. $x(t)=t-t^2$ $y(t)=t-t^3$

49. $x(t)=2\sec t$ $y(t)=4\tan t+2$

50. $x(t)=t^2+1$ $y(t)=2t-1$

51. $x(t)=2t^2-1$ $y(t)=2t^3+t$

52. $x(t)=t^2$ $y(t)=t^2+1$

53. $x(t)=t^4$ $y(t)=t^2+1$

54. $x(t)=e^t$ $y(t)=t^2e^t$

55. $x(t)=e^{4t}$ $y(t)=\cos 2t$

56. $x(t)=e^{3t}$ $y(t)=\sin 3t$

57. $x(t)=a\cosh t$ $y(t)=b\sinh t$

58. Find an equation of the line tangent to cycloid $x(t) = t - \sin t, \quad y(t) = 2 - \cos t$ at the points corresponding to $t = \frac{\pi}{6}$ and $t = \frac{2\pi}{3}$.

59. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x(t) = \sin 2t, \quad y(t) = 2\sin t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line
b) A vertical tangent line.

60. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x(t) = \sin 4t, \quad y(t) = \sin 3t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line.
b) A vertical tangent line.

(61 – 66) Find the ***area*** of the region

61. $\begin{cases} x(t) = b\cos t \\ y(t) = a\sin t \end{cases} \quad 0 \le t \le 2\pi$

Ellipse

62. $\begin{cases} x(t) = a\cos^3 t \\ y(t) = a\sin^3 t \end{cases} \quad 0 \le t \le 2\pi$

Astroid

63. $\begin{cases} x(t) = 2a\cos t - a\cos 2t \\ y(t) = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Cardioid

64. $\begin{cases} x(t) = 2a\cos t + a\cos 2t \\ y(t) = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Deltoid

65. $\begin{cases} x(t) = a\sin 2t \\ y(t) = b\sin t \end{cases}$ $0 \le t \le 2\pi$

Hourglass

66. $\begin{cases} x(t) = 2a\cos t - a\sin 2t \\ y(t) = b\sin t \end{cases}$ $0 \le t \le 2\pi$

Teardrop

$(\mathbf{67-78})$ Find the ***area*** under the curve

67. $x(t) = 2\sin^2 t$, $y(t) = 2\sin^2 t\tan t$, $0 \le t < \frac{\pi}{2}$

68. $x(t) = 2\cot t$, $y(t) = 2\sin^2 t$, $0 \le t < \pi$

69. $x(t) = t^4 + 1$, $y(t) = t^2$, $0 \le t \le 1$

70. $x(t) = t^2 + t + 1$, $y(t) = (t+1)^2$, $0 \le t \le 1$

71. $x(t) = t^2 + 1$, $y(t) = (t+1)^3$, $0 \le t \le 1$

72. $x(t) = \frac{1}{2}t^2$, $y(t) = \frac{1}{3}t^3$, $0 \le t \le 1$

73. $x(t) = e^t \sin t$, $y(t) = e^t \cos t$, $0 \le t \le \frac{\pi}{2}$

74. $x(t) = t - \sin t$, $y(t) = 1 + \cos t$, $0 \le t \le \frac{\pi}{4}$

75. $x(t) = \sin 2t$, $y(t) = \cos 2t$, $0 \le t \le \frac{\pi}{2}$

76. $x(t) = \tan t$, $y(t) = \sec^2 t$, $0 \le t \le \frac{\pi}{6}$

77. $x(t)=\tan^{-1}t$, $y(t)=t^3$, $0\le t\le 1$

78. $x(t)=\sin^{-1}t$, $y(t)=t$, $0\le t\le \frac{1}{2}$

79. Find the area under one arch of the cycloid $x(t)=a(t-\sin t)$, $y(t)=a(1-\cos t)$

80. Find the area enclosed by the y–axis and the curve $x(t)=t-t^2$, $y(t)=1+e^{-t}$

81. Find the area enclosed by the ellipse $x(t)=a\cos t$, $y(t)=b\sin t$, $0\le t\le 2\pi$

(82 – 104) Find the ***length*** of the curves

82. $x(t)=\cos t$, $y(t)=t+\sin t$, $0\le t\le \pi$

83. $x(t)=t^3$, $y(t)=\frac{3}{2}t^2$, $0\le t\le \sqrt{3}$

84. $x(t)=8\cos t+8t\sin t$, $y(t)=8\sin t-8t\cos t$, $0\le t\le \frac{\pi}{2}$

85. $x(t)=\ln(\sec t+\tan t)-\sin t$, $y(t)=\cos t$, $0\le t\le \frac{\pi}{3}$

86. Hypocycloid perimeter curve: $x(t)=a\cos t$, $y(t)=a\sin t$

87. Circle circumference: $x(t)=a\cos^3 t$, $y(t)=a\sin^3 t$

88. Cycloid arch: $x(t)=a(t-\sin t)$, $y(t)=a(1-\cos t)$

89. Involute of a circle: $x(t)=\cos t+t\sin t$, $y(t)=\sin t-t\cos t$

90. $x(t)=t^2$, $y(t)=t^3$, $0\le t\le 2$

91. $x(t)=5\sin t$, $y(t)=5\cos t$, $-\frac{\pi}{3}\le t\le \frac{\pi}{2}$

92. $x(t)=3\sin t$, $y(t)=3\cos t$, $0\le t\le \frac{\pi}{2}$

93. $x(t)=\sqrt{t}$, $y(t)=t^{1/4}$, $0\le t\le 4$

94. $x(t)=2t^2$, $y(t)=t^3$, $1\le t\le 2$

95. $x(t)=\ln\left(t^2-1\right)$, $y(t)=\sqrt{t^2-1}$, $3\le t\le 7$

96. $x(t)=t\sin t+\cos t$, $y(t)=\sin t-t\cos t$, $0\le t\le \frac{\pi}{2}$

97. $x(t)=\sin 5t$, $y(t)=\cos 5t$, $0\le t\le \pi$

98. $x(t)=2\arcsin t$, $y(t)=\ln\left(1-t^2\right)$, $0\le t\le \frac{\sqrt{3}}{2}$

99. $x(t)=\frac{1}{2}t^2$, $y(t)=\frac{1}{3}t^3$, $0\le t\le 1$

100. $x(t)=t^2\sin t$, $y(t)=t^2\cos t$, $0\le t\le 2\pi$

101. $x(t)=e^{2t}\sin t$, $y(t)=e^{2t}\cos t$, $0\le t\le 2\pi$

102. $x(t)=2\ln t$, $y(t)=t+\frac{1}{t}$, $1\le t\le 4$

103. $x(t)=2e^t$, $y(t)=\frac{1}{2}e^{2t}+e^{-t}$, $0\le t\le 1$

104. $x(t)=4t+3$, $y(t)=3t-2$, $0\le t\le 2$

(105 – 120) Find the areas of the ***surfaces*** generated by revolving the curves

105. $x(t)=\frac{1}{3}t^3$, $y(t)=t+1$, $1\le t\le 2$, *y-axis*

106. $x(t)=\frac{2}{3}t^{3/2}$, $y(t)=2\sqrt{t}$, $0\le t\le\sqrt{3}$; $x-axis$

107. $x(t)=t+\sqrt{2}$, $y(t)=\frac{t^2}{2}+\sqrt{2}\,t$, $-\sqrt{2}\le t\le\sqrt{2}$; $y-axis$

108. $x(t)=2t$, $y(t)=3t$; $0\le t\le 3$ *x-axis*

109. $x(t)=2t$, $y(t)=3t$; $0\le t\le 3$ *y-axis*

110. $x(t)=t$, $y(t)=4-2t$; $0\le t\le 2$ *x-axis*

111. $x(t)=t$, $y(t)=4-2t$; $0\le t\le 2$ *y - axis*

112. $x(t)=5\cos t$, $y(t)=5\sin t$, $0\le t\le\frac{\pi}{2}$, *y - axis*

113. $x(t)=a\cos^3 t$, $y(t)=a\sin^3 t$, $0\le t\le\pi$, *x-axis*

114. $x(t)=a\cos t$, $y(t)=b\sin t$, $0\le t\le 2\pi$

a) *x-axis* *b)* *y-axis*

115. $x(t)=2t$, $y(t)=3t$, $0\le t\le 3$

a) *x-axis* *b)* *y-axis*

116. $x(t)=t$, $y(t)=4-2t$, $0\le t\le 2$

a) *x-axis* *b)* *y-axis*

117. $x(t)=t^3$, $y(t)=t^2$, $0\le t\le 1$

a) *x-axis* *b)* *y-axis*

118. $x(t)=t+t^3$, $y(t)=1-\frac{1}{t^2}$, $1\le t\le 2$

a) *x-axis* *b)* *y-axis*

119. $x(t)=t^2$, $y(t)=2t$, $0\le t\le 4$

a) *x-axis* *b)* *y-axis*

120. $x(t)=2t^2$, $y(t)=t^2$, $0\le t\le 1$

a) *x-axis* *b)* *y-axis*

121. Use the parametric equations $x(t)=t^2\sqrt{3}$ *and* $y(t)=3t-\frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3\le t\le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

c) Find the equation of the tangent line at the point $\left(\sqrt{3},\ \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the *x-axis*

122. Use the parametric equations $x=a(\theta-\sin\theta)$ *and* $y=a(1-\cos\theta)$ $a>0$

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

b) Find the equation of the tangent line at the point where $\theta=\frac{\pi}{6}$

c) Find all points (if any) of horizontal tangency.

d) Determine where the curve is concave upward or concave downward.

e) Find the length of one arc of the curve

123. The involute of a circle is described by the endpoint *P* of a string that is held taut as it is unwound from a spool that does not turn.

Show that a parametric representation of the involute is

$$x=r(\cos\theta+\theta\sin\theta) \quad \textit{and} \quad y=r(\sin\theta-\theta\cos\theta)$$

124. The figure shows a piece of string tied to a circle with a radius of *one* unit. The string is just long enough to reach the opposite side if the circle.

Find the area that is covered when the string is unwounded counterclockwise.

125. An Air traffic controller spots two planes at the same altitude flying toward each other.

Their flight paths are 20° and 315°. One plane is 150 *miles* from point *P* with a speed of 375 *miles per hour*. The other is 190 *miles* from point *P* with a speed of 450 *miles per hour*.

a) Find parameteric equations for the path of each plane where *t* is the time in *hours*, with $t = 0$ corresponding to the time at which the air traffic controller spots the planes.

b) Use part (*a*) to write the distance between the planes as a function of *t*.

c) Graph the function in part (*b*).

d) When the distance between the planes be minimum?

e) If the planes must keep a separation of at least 3 *miles*, is the requirement met?

Section 8.3 – Polar Coordinates and Graphs

8.3–1 *Definition* of Polar Coordinates

To define polar coordinates, let an ***origin*** O (called the ***pole***) and an ***initial ray*** from O. Then each point P can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from O to P and θ gives the directed angle from the initial ray to ray OP.

8.3–2 Polar Coordinates

$$P\,(r,\ \theta)$$

Distance from O to P — Angle from Initial ray to OP

Drawing a polar coordinate point:

1. Draw the angle
   - ***a.*** Positive direction: Counterclockwise.
   - ***b.*** Negative direction: Clockwise.

2. Measure radius
   - ***a.*** Positive direction: straight
   - ***b.*** Negative direction: backward

Example 1

Find all the polar coordinates of the point $P\left(2,\ \frac{\pi}{4}\right)$

Solution

For $r = 2 \quad \Rightarrow \quad \theta = \frac{\pi}{4},\ \frac{\pi}{4} \pm 2\pi,\ \frac{\pi}{4} \pm 4\pi,\ \ldots$

$$\left(2,\ \frac{\pi}{4} + 2n\pi\right), \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

For $r = -2 \quad \Rightarrow \quad \theta = -\frac{3\pi}{4},\ -\frac{3\pi}{4} \pm 2\pi,\ -\frac{3\pi}{4} \pm 4\pi,\ \ldots$

$$\left(-2,\ -\frac{3\pi}{4} + 2n\pi\right), \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

8.3–3 Polar Equations and Graphs

Equation	***Graph***
$r = a$	Circle of radius $\lvert a \rvert$ centered at O.
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

Example 2

Graph the polar coordinate $\quad 1 \le r \le 2 \quad$ *and* $\quad 0 \le \theta \le \frac{\pi}{2}$

Solution

Example 3

Graph the polar coordinate $-3 \le r \le 2$ *and* $\theta = \frac{\pi}{4}$

Solution

Example 4

Graph the polar coordinate $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$ $\left(\textit{no restriction on } r\right)$

Solution

8.3–4 Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive *x*-axis.

The ray $\theta = \frac{\pi}{2}$, $r > 0$ becomes the positive *y*-axis.

The two coordinate systems are then related by the following equations

8.3–5 Equations Relating Polar and Cartesian Coordinates

$$\begin{cases} x = r\cos\theta\,, & y = r\sin\theta \\ r^2 = x^2 + y^2\,, & \tan\theta = \dfrac{y}{x} \end{cases}$$

Polar equation	*Cartesian equation*
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$

Example 5

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

Solution

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0 \qquad x^2 + y^2 = r^2$$

$$r^2 - 6r\sin\theta = 0$$

$$r(r - 6\sin\theta) = 0$$

$$\Rightarrow \quad \underline{r = 0} \quad \underline{r = 6\sin\theta}$$

Example 6

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r\cos\theta = -4$

Solution

$r\cos\theta = -4 \quad \Rightarrow \quad x = -4$

The graph: Vertical line through $x = -4$

Example 7

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r^2 = 4r\cos\theta$

Solution

$r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 4$

The ***graph***: Circle with center (2, 0) and radius 2.

Example 8

Replace the polar equation by equivalent Cartesian equation: $r = 1 - \cos\theta$

Solution

$r = 1 - \cos\theta \qquad \times r$

$r^2 = r - r\cos\theta$

$x^2 + y^2 = \sqrt{x^2 + y^2} - x$

$x^2 + y^2 + x = \sqrt{x^2 + y^2}$

$\left(x^2 + y^2 + x\right)^2 = \left(\sqrt{x^2 + y^2}\right)^2$

$x^4 + y^4 + x^2 + 2x^2y^2 + 2x^3 + 2xy^2 = x^2 + y^2$

$\underline{x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0}$

Example 9

Find the point(s) of intersection of the given curves $r = 2\cos\theta$ & $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta$$
$$\cos\theta = \sin\theta$$

$$\begin{cases} \theta = \dfrac{\pi}{4} + 2n\pi \\ \theta = \dfrac{5\pi}{4} + 2n\pi \end{cases}$$

$$\theta = \frac{\pi}{4} + n\pi$$

8.3–6 *Symmetry* Test for Polar Graphs

1. ***Symmetry about the x-axis***: If the point $(r,\ \theta)$ lies on the graph, then the point $(r,\ -\theta)$ or $(-r,\ \pi-\theta)$ lies on the graph.

2. ***Symmetry about the y-axis***: If the point $(r,\ \theta)$ lies on the graph, then the point $(r,\ \pi-\theta)$ or $(-r,\ -\theta)$ lies on the graph.

3. ***Symmetry about the origin***: If the point $(r,\ \theta)$ lies on the graph, then the point $(-r,\ \theta)$ or $(r,\ \theta+\pi)$ lies on the graph.

8.3–7 A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of $(r,\ \theta)$ values, plot the corresponding points, and connect them in order of increasing.

Another method of graphing more reliable is

1. First graph $r = f(\theta)$ in the *Cartesian* $r\theta - plane$,
2. Then use the *Cartesian* graph as a table and guide to sketch the *polar coordinate* graph.

Example 10

Graph the curve $r = 1 - \cos\theta$

Solution

The curve is symmetric about the *x*-axis:

$$1 - \cos(-\theta) = 1 - \cos\theta$$
$$= r$$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

Example 11

Graph the curve $r^2 = 4\cos\theta$

Solution

The curve is symmetric about the *x*-axis:

$$4\cos(-\theta) = 4\cos\theta = r^2$$

$$(r, -\theta)$$

The curve is symmetric about the *origin*:

$$(-r)^2 = 4\cos\theta$$

$$(-r, \theta)$$

Therefore, the curve is also symmetric about the *y*-axis.

$$r^2 = 4\cos\theta$$

$$r = \pm 2\sqrt{\cos\theta}$$

θ	$r = \pm 2\sqrt{\cos\theta}$
0	± 2
$\pm\frac{\pi}{6}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0

Example 12

Graph the *lemniscate* curve $r^2 = \sin 2\theta$

Solution

θ	$r = \pm 2\sqrt{\cos\theta}$
0	0
$\frac{\pi}{6}$	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm .93$
$\frac{\pi}{4}$	± 1
$\frac{\pi}{3}$	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm .93$
$\frac{\pi}{2}$	0
π	0
$\frac{7\pi}{6}$	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm .93$

Exercises *Section* 8.3 – Polar Coordinates and Graphs

1. Find the Cartesian coordinates of the following points (given in polar coordinates)

$a)\ \left(\sqrt{2},\ \frac{\pi}{4}\right)\quad b)\ (1,\ 0)\quad c)\ \left(0,\ \frac{\pi}{2}\right)\quad d)\ \left(-\sqrt{2},\ \frac{\pi}{4}\right)$

2. Find the polar coordinates, $0 \le \theta < 2\pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (1,\ 1)\quad b)\ (-3,\ 0)\quad c)\ \left(\sqrt{3},\ -1\right)\quad d)\ (-3,\ 4)$

3. Find the polar coordinates, $-\pi \le \theta < \pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (-2,\ -2)\quad b)\ (0,\ 3)\quad c)\ \left(-\sqrt{3},\ 1\right)\quad d)\ (5,\ -12)$

(4 – 20) Replace the polar equation with equivalent Cartesian equation and identify the graph

4. $r\cos\theta = 2$

5. $r\sin\theta = -1$

6. $r = -3\sec\theta$

7. $r\cos\theta + r\sin\theta = 1$

8. $r^2 = 4r\sin\theta$

9. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

10. $r = 4\tan\theta\sec\theta$

11. $r\sin\theta = \ln r + \ln\cos\theta$

12. $\cos^2\theta = \sin^2\theta$

13. $r = 2\cos\theta + 2\sin\theta$

14. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

15. $r = \dfrac{4}{2\cos\theta - \sin\theta}$

16. $r = \cot\theta\csc\theta$

17. $r\cos\theta = \sin 2\theta$

18. $r = \sin\theta\sec^2\theta$

19. $r = \dfrac{1}{2\cos\theta + 3\sin\theta}$

20. $r^2 = 4\cos 2\theta$

(21 – 29) Replace the Cartesian equation with equivalent polar equation

21. $x = y$

22. $x^2 - y^2 = 1$

23. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

24. $xy = 1$

25. $x^2 + xy + y^2 = 1$

26. $x^2 + (y-2)^2 = 4$

27. $(x+2)^2 + (y-5)^2 = 16$

28. $(x+2)^2 + (y-3)^2 = 13$

29. $y^2 - x^2 = 4$

30. ***a)*** Show that every vertical line in the *xy*–plane has a polar equation of the form $r = a\sec\theta$
b) Find the analogous polar equation for horizontal lines in the *xy*–plane.

$(\mathbf{31-46})$ Find the point(s) of intersection of the given curves

31. $r = 2\cos\theta$ & $r = -2\sin\theta$

32. $r = 2$ & $r = -2\sin\theta$

33. $r = \cos\theta$ & $r = -2$

34. $r = 2\cos\theta$ & $r = 1$

35. $r = 2(1+\cos\theta)$ & $r = 2(1-\cos\theta)$

36. $r = 2(1+\sin\theta)$ & $r = 2(1-\sin\theta)$

37. $r = 2\sqrt{2}\,\sin\theta$ & $r^2 = 4\cos 2\theta$

38. $r = 2+\sin\theta$ & $r = 4\cos\theta$

39. $r = 2(1+\cos\theta)$ & $r = -4\sin\theta$

40. $r = 2(1-\cos\theta)$ & $r = 4\sin\theta$

41. $r = 2$ & $r = \dfrac{2}{1+\cos\theta}$

42. $r = 2$ & $r^2 = 4\cos 2\theta$

43. $r = \sqrt{2}\,\sin\theta$ & $r^2 = \sin 2\theta$

44. $r = \cos\theta$ & $r = \dfrac{2}{1-\cos\theta}$

45. $r = \dfrac{2}{1-\cos\theta}$ & $r = \dfrac{1}{1+\cos\theta}$

46. $r = \dfrac{1}{1-\cos\theta}$ & $r = \dfrac{2}{1+\cos\theta}$

$(\mathbf{47-52})$ Identify the symmetries of the curves. Then sketch the curves.

47. $r = 2-2\cos\theta$

48. $r = 1+\sin\theta$

49. $r = 2+\sin\theta$

50. $r^2 = \sin\theta$

51. $r^2 = -\sin\theta$

52. $r^2 = -\cos\theta$

$(\mathbf{53-55})$ Graph the lemniscate. What symmetries do these curves have?

53. $r^2 = 4\cos 2\theta$

54. $r^2 = 4\sin 2\theta$

55. $r^2 = -\cos 2\theta$

$(\mathbf{56-61})$ Graph the limaçons is Old French for "snail". Equations for limaçons have the form

$r = a \pm b\cos\theta$ *or* $r = a \pm b\sin\theta$

56. $r = \frac{1}{2}+\cos\theta$

57. $r = \frac{1}{2}+\sin\theta$

58. $r = 1-\cos\theta$

59. $r = 1+\cos\theta$

60. $r = \frac{3}{2}-\sin\theta$

61. $r = 2+\cos\theta$

$(\mathbf{62-82})$ Graph the equation

62. $1 \le r \le 2$

63. $0 \le \theta \le \frac{\pi}{6}, \quad r \ge 0$

64. $\theta = \frac{\pi}{2}, \quad r \le 0$

65. $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, \quad 0 \le r \le 1$

66. $0 \le \theta \le \frac{\pi}{2}, \quad 1 \le |r| \le 2$

67. $r = 2$

68. $r = \theta^2$

69. $r = e^{2\theta}$

70. $r = 1-2\sin 3\theta$

71. $r = \sin^2\frac{\theta}{2}$

72. $r = 1-\sin\theta$

73. $r^2 = 4\sin\theta$

74. $r = 4\sin\theta$

75. $r = 4-\sin\theta$

76. $r = \sin 4\theta$

77. $r = \sin 3\theta$

78. $r = 4\cos 2\theta$

79. $r = 2\cos 3\theta$

80. $r = 3\cos\frac{\theta}{2}$

81. $r = 1+2\cos\theta$

82. $r^2 = \cos 2\theta$

83. Graph the ***nephroid*** of ***Freeth*** equation $r = 1 + 2\sin\frac{\theta}{2}$

84. Water flows in a shallow semicircular channel with inner and outer radii of 1 *m* and 2 *m*. At a point $P(r,\ \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on *r*, the distance from the center of the semicircles.

a) Express the region formed by the channel as set-in polar coordinates.
b) Express the inflow and outflow regions of the channel as sets in polar coordinates.
c) Suppose the tangential velocity of the water in *m/s* is given by $v(r) = 10r$, for $1 \le r \le 2$. Is the velocity greater at $\left(1.5,\ \frac{\pi}{4}\right)$ or $\left(1.2,\ \frac{3\pi}{4}\right)$? Explain.
d) Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for. Is the velocity greater $\left(1.8,\ \frac{\pi}{6}\right)$ or $\left(1.3,\ \frac{2\pi}{3}\right)$? Explain.
e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r)\,dr$. Is the total flow through the channel greater for the flow in part (*c*) or (*d*)?

85. A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t = 0$. Earth is at $(2,\ 0)$ and Mars is at $(3,\ 0)$; both orbit the Sum (at $(0,\ 0)$) in the counterclockwise direction.
The position of Mars relative to Earth is given by the parametric equations

$$x = (3 - 4\cos\pi t)\cos\pi t + 2, \quad y = (3 - 4\cos\pi t)\sin\pi t$$

a) Graph the parametric equations, for $0 \le t \le 2$
b) Letting $r = 3 - 4\cos\pi t$, explain why the path of Mars relative to Earth is a limaçon.

Section 8.4 – Calculus in Polar Coordinates

8.4–1 *Slope*

The slope of a polar curve $r = f(\theta)$ in the *xy*-plane is still given by $\frac{dy}{dx}$, which is not $r' = \frac{df}{d\theta}$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta}\\ &= \frac{\frac{d}{d\theta}\left(f(\theta)\bullet\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\bullet\cos\theta\right)}\\ &= \frac{\frac{df}{d\theta}\bullet\sin\theta + f(\theta)\bullet\cos\theta}{\frac{df}{d\theta}\bullet\cos\theta + f(\theta)\bullet\sin\theta}\end{aligned}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\,\sin\theta + f(\theta)\,\cos\theta}{f'(\theta)\,\cos\theta - f(\theta)\,\sin\theta}}$$

Example 1

Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = f(\theta) = 1 - \cos\theta$ has a vertical or horizontal tangent line.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\\ &= \frac{\sin\theta\sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}\\ &= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{\sin\theta\cos\theta - \sin\theta + \cos\theta\sin\theta}\\ &= \frac{1 - \cos^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - \sin\theta}\\ &= \frac{-2\cos^2\theta + \cos\theta + 1}{\sin\theta(2\cos\theta - 1)}\end{aligned}$$

$$= \frac{-(2\cos\theta+1)(\cos\theta-1)}{\sin\theta(2\cos\theta-1)}$$

The points with a horizontal tangent line: $\dfrac{dy}{dx}=0$

$$(2\cos\theta+1)(\cos\theta-1)=0$$

$$\rightarrow \begin{cases} \cos\theta=-\frac{1}{2} & \rightarrow \ \theta=\pm\frac{2\pi}{3} \\ \cos\theta=1 & \rightarrow \ \cancel{\theta=0} \end{cases} \quad \textit{numerator is } 0$$

The points with a Vertical tangent line: $\sin\theta(2\cos\theta-1)=0$

$$\rightarrow \begin{cases} \sin\theta=0 & \rightarrow \ \theta=\cancel{0},\pm\pi \\ \cos\theta=\frac{1}{2} & \rightarrow \ \theta=\pm\frac{\pi}{3} \end{cases} \quad \textit{denominator is } 0$$

$$\frac{dy}{dx}=\lim_{\theta\to 0^+}\left(\frac{-2\cos^2\theta+\cos\theta+1}{2\cos\theta\sin\theta-\sin\theta}\right)=\frac{0}{0}$$

$$=\lim_{\theta\to 0^+}\left(\frac{4\cos\theta\sin\theta-\sin\theta}{-2\sin^2\theta+2\cos^2\theta-\cos\theta}\right)$$

$$=\frac{0}{1}$$

$$=0$$

Therefore, the curve has a slope of 0 at origin.

8.4–2 Area in the plane

The region OTS is bounded by the rays $\theta=\alpha$ *and* $\theta=\beta$ and the curve $r=f(\theta)$.

We approximate the region with n non-overlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f\left(\theta_k\right)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\frac{\Delta\theta_k}{2\pi}$ times the area of a circle of radius r_k, or

$$\begin{aligned} A_k &= \tfrac{1}{2} r_k^2 \Delta\theta_k \\ &= \tfrac{1}{2}\left(f\left(\theta_k\right)\right)^2 \Delta\theta_k \end{aligned}$$

8.4–3 Area of the Fan-Shaped Region between the Origin and the curve $r = f(\theta),\ \alpha \le \theta \le \beta$

$$Area = \int_{\alpha}^{\beta} \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta$$

This is the integral of the ***area differential***

$$\begin{aligned} dA &= \tfrac{1}{2} r^2 d\theta \\ &= \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta \end{aligned}$$

Example 2

Find the area of the region in the plane enclosed by the cardioid $r = 2(1+\cos\theta)$

Solution

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2\, d\theta \\ &= \int_0^{2\pi} 2\left(1+2\cos\theta+\cos^2\theta\right) d\theta \\ &= \int_0^{2\pi} \left(2+4\cos\theta+2\frac{1+\cos 2\theta}{2}\right) d\theta \end{aligned}$$

$$= \int_0^{2\pi} \left(3 + 4\cos\theta + \cos 2\theta\right) d\theta$$

$$= 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \Big|_0^{2\pi}$$

$$= 3(2\pi) + 4\sin(2\pi) + \frac{1}{2}\sin(2\pi) - \left(3(0) + 4\sin(0) + \frac{1}{2}\sin(0)\right)$$

$$= 6\pi \quad unit^2$$

Example 3

Find the area of one leaf of a rose given by: $r = 2\cos(3\theta)$

Solution

$$2\cos(3\theta) = 0$$

$$3\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

$$2\cos(-3\theta) = 2\cos(3\theta) = r$$

Symmetric about $x-$axis

$$Area = 2 \times \frac{1}{2} \int_0^{\frac{\pi}{6}} 4\cos^2(3\theta)\, d\theta$$

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \left(1 + \cos(6\theta)\right) d\theta$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= 2\left(\theta + \frac{1}{6}\sin(6\theta)\right) \Big|_0^{\frac{\pi}{6}}$$

$$= 2\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} \quad unit^2$$

8.4–4 Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r_2^2\, d\theta - \frac{1}{2}\int_{\alpha}^{\beta} r_1^2\, d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} \left(r_2^2 - r_1^2\right) d\theta$$

Example 4

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$

Solution

$$Area = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\frac{1}{2}\int_{0}^{\pi/2} \left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= \int_{0}^{\pi/2} \left(1 - \left(1 - 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_{0}^{\pi/2}$$

$$= 2\sin\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\sin 2\frac{\pi}{2} - 0$$

$$= 2 - \frac{\pi}{4} \quad unit^2$$

Example 5

Find the area of the region that lies inside the circle $r = 6\sin\theta$ and outside the cardioid $r = 2 + 2\sin\theta$

Solution

$$r = 6\sin\theta = 2 + 2\sin\theta$$

$$4\sin\theta = 2$$

$$\sin\theta = \frac{1}{2} \quad \rightarrow \quad \theta = \frac{\pi}{6},\ \frac{5\pi}{6}$$

$$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$$

For testing: $\theta = \frac{\pi}{2} \quad \rightarrow \quad \begin{cases} r_1 = 6 \\ r_2 = 4 \end{cases}$

$$Area = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36\sin^2\theta - (2 + 2\sin\theta)^2\right) d\theta \qquad Area = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_2^2 - r_1^2\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(36\sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(32\sin^2\theta - 4 - 8\sin\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(16 - 16\cos 2\theta - 4 - 8\sin\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(12 - 16\cos 2\theta - 8\sin\theta\right) d\theta$$

$$= \frac{1}{2}\left(12\theta - 8\sin 2\theta + 8\cos\theta\right)\Big|_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(10\pi - 4 - 4\sqrt{3} - 2\pi + 4 + 4\sqrt{3}\right)$$

$$= \frac{1}{2}\left(8\pi\right)$$

$$= 4\pi \quad unit^2$$

8.4–5 Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r,\ \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Example 6

Find the length of the cardioid $r = 1 - \cos\theta$

Solution

$$r = 1 - \cos\theta \quad \Rightarrow \quad \frac{dr}{d\theta} = \sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= 2 - 2\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\ d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_{0}^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\ d\theta$$

$$= \int_{0}^{2\pi} 2\sin\frac{\theta}{2}\ d\theta$$

$$= -4\cos\frac{\theta}{2}\Big|_{0}^{2\pi}$$

$$= -4\left(\cos\frac{2\pi}{2} - \cos 0\right)$$

$$= -4(-1-1)$$

$$= 8 \quad unit$$

Example 7

Find the length of the cardioid $r = 2 + 2\sin\theta$

Solution

$$2 + 2\sin(\pi - \theta) = 2 + 2\sin\theta \quad = r$$

$$(r,\ \pi - \theta) \quad \rightarrow \quad (r,\ \theta)$$

Symmetric about $y-axis$

$$\sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2} = 2\sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} \qquad \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= 2\sqrt{2(1+\sin\theta)}$$

$$= 2\sqrt{2}\sqrt{1+\sin\theta\,\frac{1-\sin\theta}{1-\sin\theta}}$$

$$= 2\sqrt{2}\sqrt{\frac{1-\sin^2\theta}{1-\sin\theta}}$$

$$= 2\sqrt{2}\sqrt{\frac{\cos^2\theta}{1-\sin\theta}}$$

$$L = 2\left(2\sqrt{2}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta \qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= -4\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin\theta)^{-1/2}\, d(1-\sin\theta)$$

$$= -8\sqrt{2}\,\sqrt{1-\sin\theta}\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= -8\sqrt{2}\left(0 - \sqrt{2}\right)$$

$$= 16 \quad unit$$

8.4−6 Area of a surface of Revolution

Theorem

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$.

The area of the surface formed by revolving the graph of $r = f(\theta)$ about the indicated line

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\ \sin\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta \qquad \textit{About the polar axis}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\ \cos\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta \qquad \textit{About the line } \theta = \frac{\pi}{2}$$

Example 8

Find the area of the surface formed by revolving the circle $f(\theta) = \cos\theta$ about the line $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\ \cos\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta$$

$$= 2\pi \int_{0}^{\pi} \cos^2\theta\ d\theta$$

$$= \pi \int_{0}^{\pi} (1 + \cos 2\theta)\ d\theta$$

$$= \pi \left(\theta + \frac{1}{2}\sin 2\theta\right|_{0}^{\pi}$$

$$= \pi^2 \ \ unit^2$$

Example 9

Find the area of the surface formed by revolving $f(\theta) = 3 + 3\cos\theta$ about polar axis.

Solution

$$3 + 3\cos(-\theta) = 3 + 3\cos\theta \quad = r$$

$$(r,\ -\theta) \quad \rightarrow \quad (r,\ \theta)$$

Symmetric about $y-axis$

$$\sqrt{r^2 + (r')^2} = \sqrt{(3 + 3\cos\theta)^2 + 9\sin^2\theta}$$

$$= 3\sqrt{2 + 2\cos\theta}$$

$$S = 2\pi \int_0^{\pi} (3 + 3\cos\theta)\ \sin\theta\ \left(3\sqrt{2}\right) \sqrt{1 + \cos\theta}\, d\theta$$

$$S = 2\pi \int_{\alpha}^{\beta} r\sin\theta\ \sqrt{r^2 + (r')^2}\ d\theta$$

$$= -18\sqrt{2}\,\pi \int_0^{\pi} (1 + \cos\theta)^{3/2}\ d(1 + \cos\theta)$$

$$= -\frac{36\pi\sqrt{2}}{5}\ (1 + \cos\theta)^{5/2} \Big|_0^{\pi}$$

$$= -\frac{36\pi\sqrt{2}}{5}\ \left(0 - 2^{5/2}\right)$$

$$= \frac{288\pi}{5} \quad unit^2$$

Exercises *Section* 8.4 – Calculus in Polar Coordinates

(1 – 12) Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

1. $r = 2$; $\left(2, \frac{\pi}{4}\right)$

2. $f(\theta) = \theta$; $\theta = \frac{\pi}{2}$

3. $f(\theta) = \sec\theta$; $\theta = \frac{\pi}{4}$

4. $r = 1 - \cos\theta$; $\left(\frac{2-\sqrt{2}}{2}, \frac{\pi}{4}\right)$

5. $f(\theta) = 2\cos\theta \sin^2\theta$; $\theta = \frac{\pi}{6}$

6. $f(\theta) = 4\cos^2\theta \sin^2\theta$; $\theta = \frac{\pi}{4}$

7. $f(\theta) = \dfrac{4}{3\sin\theta - 4\cos\theta}$; $\theta = \pi$

8. $f(\theta) = \dfrac{3}{3\sin\theta + 2\cos\theta}$; $\theta = \pi$

9. *Cardioid* $r = -1 + \cos\theta$; $\theta = \pm\frac{\pi}{2}$

10. *Cardioid* $r = -1 + \sin\theta$; $\theta = 0, \pi$

11. *Four – leaved rose* $r = \sin 2\theta$; $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

12. *Four – leaved rose* $r = \cos 2\theta$; $\theta = 0, \pm\frac{\pi}{2}, \pi$

13. Find the area of the region bounded by the spiral $r = \theta$ *for* $0 \le \theta \le \pi$

14. Find the area of the region bounded by the circle $r = 2\sin\theta$ *for* $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$(\mathbf{15-107})$ Find the area of the region

15. Enclosed by $r=3$

16. Enclosed by $r=3\sin\theta$

17. Enclosed by $r=3\cos\theta$

18. Enclosed by $r=\cos 3\theta$

19. Enclosed by $r=\sin 3\theta$

20. Enclosed by $r=\cos 4\theta$

21. Enclosed by $r=\sin 4\theta$

22. Enclosed by $r=\cos 5\theta$

23. Enclosed by $r=\sin 5\theta$

24. Enclosed by $r=\sin\left(\frac{\theta}{2}\right)$

25. Enclosed by $r=\cos\left(\frac{\theta}{2}\right)$

26. $f(\theta)=\sin\theta$, $0\le\theta\le\frac{5\pi}{6}$

27. $f(\theta)=\cos\theta$, $0\le\theta\le\frac{\pi}{6}$

28. $f(\theta)=\sqrt{\sin\theta}$, $0\le\theta\le\frac{\pi}{4}$

29. $f(\theta)=\sqrt{\cos\theta}$, $\frac{\pi}{6}\le\theta\le\frac{\pi}{3}$

30. $f(\theta)=\sin\theta$, $-\frac{\pi}{3}\le\theta\le\frac{\pi}{4}$

31. $f(\theta)=e^{\theta}$, $0\le\theta\le 2\pi$

32. $f(\theta)=\frac{1}{\pi}\theta$, $0\le\theta\le 2\pi$

33. $f(\theta)=\frac{1}{\pi}\theta^2$, $0\le\theta\le 2\pi$

34. $f(\theta)=\sin\theta+\cos\theta$, $0\le\theta\le\frac{\pi}{2}$

35. $f(\theta)=2\sec\theta$, $0\le\theta\le\frac{\pi}{4}$

36. Inside oval limaçon $r=4+2\sin\theta$

37. Inside Cardioid $r=a(1+\cos\theta)$, $a>0$

38. Inside Six-leaved rose $r^2=2\sin 3\theta$

39. Inside curve $r=\sqrt{\cos\theta}$

40. Inside right lobe of $r=\sqrt{\cos 2\theta}$

41. Inside Cardioid $r=4+4\sin\theta$

42. Inside Limaçon $r=2+\cos\theta$

43. Inside One leaf: $r=\cos 3\theta$

44. Inside One leaf: $r=\sin 3\theta$

45. Inside One leaf: $r=\cos 4\theta$

46. Inside One leaf: $r=\sin 4\theta$

47. Inside one leaf: $r=2\sin 6\theta$

48. Inside inner loop $r=\cos\theta-\frac{1}{2}$

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49. Inside circle $r=6$ above the line $r=3\csc\theta$

50. Shared by the circles $r=2\cos\theta$ *and* $r=2\sin\theta$

51. Shared by the circle $r=2$ and the cardioid $r=2(1-\cos\theta)$

52. Enclosed by the four-leaf rose $r=f(\theta)=2\cos 2\theta$

53. Lies inside the circle $r=1$ and outside the cardioid $r=1+\cos\theta$

54. Outside the circle $r=\frac{1}{2}$ and inside the circle $r=\cos\theta$

55. Outside the circle $r=\frac{1}{\sqrt{2}}$ and inside the curve $r=\sqrt{\cos\theta}$

56. Inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

57. Inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

58. Inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$

59. Inside all the leaves of the rose $r = 3\sin 2\theta$

60. Inside one leaf of the rose $r = \cos 5\theta$

61. A complete three-leaf rose $r = 2\cos 3\theta$

62. Inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

63. Bounded by the lemniscate $r^2 = 6\sin 2\theta$

64. Bounded by the limaçon $r = 2 - 4\sin\theta$

65. Bounded by the limaçon $r = 4 - 2\cos\theta$

66. Between inner and outer: $r = 3 - 6\cos\theta$

67. Inner loop of $r = 1 + 2\cos\theta$

68. Inner loop of $r = 2 - 4\cos\theta$

69. Inner loop of $r = 1 + 2\sin\theta$

70. Inner loop of $r = 4 - 6\sin\theta$

71. Inner loop of $r = 2 - 3\cos\theta$

72. Inner loop of $r = 3 - 4\cos\theta$

73. Between the loops $r = 1 + 2\cos\theta$

74. Between the loops $r = 2(1 + 2\sin\theta)$

75. Between the loops $r = 3 - 6\sin\theta$

76. Between the loops $r = \frac{1}{2} + \cos\theta$

77. Inside $r = 2\cos\theta$ and outside $r = 1$

78. Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$

79. Inside $r = 3$ and outside $r = 1 + \cos\theta$

80. Common interior of $r = 2\sin\theta$ and $r = 2\cos\theta$

81. Common interior of $r = 2$ and $r = 2\cos\theta$

82. Common interior of $r = 1$ and $r = \sin\theta$

83. Common interior of $r = 1$ and $r = 2\sin 2\theta$

84. Common interior of $r = 2$ and $r = 2\sin 2\theta$

85. Common interior of $r = 2$ and $r = 2\cos 4\theta$

86. Common interior of $r = 2\sin\theta$ and $r = 2 + 2\cos\theta$

87. Common interior of $r = 4\sin 2\theta$ and $r = 2$

88. Common interior of $r = 4\sin\theta$ and $r = 2$

89. Common interior of $r = 2\cos\theta$ and $r = 2\sin\theta$

90. Common interior of $r = 2(1+\cos\theta)$ *and* $r = 2(1-\cos\theta)$

91. Common interior of $r = 3 - 2\sin\theta$ *and* $r = -3 + 2\sin\theta$

92. Common interior of $r = 5 - 3\sin\theta$ *and* $r = 5 - 3\cos\theta$

93. Inside $r = a(1+\cos\theta)$ and outside $r = a\cos\theta$

94. Inside $r = 2a\cos\theta$ and outside $r = a$

95. Common interior of $r = a(1+\cos\theta)$ and $r = a\sin\theta$

96. Common interior of $r = a\cos\theta$ and $r = a\sin\theta$, where $a > 0$.

97. Enclosed by all the leaves of the rose $r = 3\sin 4\theta$

98. Enclosed by the limaçon $r = 2 - \cos\theta$

99. Inside limaçon $r = 2 + \cos\theta$ and outside the circle $r = 2$

100. Inside lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \frac{1}{2}$

101. Inside both cardioids $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$

102. Inside the cardioid $r = 1 + \cos\theta$ and outside the cardioid $r = 1 - \cos\theta$

103. Inside both cardioids $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

104. common interior of $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

105. Inside both cardioids $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

106. Common interior $r = 1$ and $r = \sqrt{2}\ \cos 2\theta$

107. Outside $r = 1$ and inside $r = \sqrt{2}\ \cos 2\theta$

(108 – 144) Find the length of

108. $r = 4$, $0 \le \theta \le \frac{\pi}{2}$

109. $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

110. $r = \frac{e^\theta}{\sqrt{2}}$, $0 \le \theta \le \pi$

111. $r = a\sin^2\left(\frac{\theta}{2}\right)$, $0 \le \theta \le \pi$, $a > 0$

112. $r = \frac{6}{1+\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$

113. $r = \cos^3\left(\frac{\theta}{3}\right)$, $0 \le \theta \le \frac{\pi}{4}$

114. $r = \sqrt{1+\sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

115. $r = 8$ $0 \le \theta \le 2\pi$

116. $r = a$ $0 \le \theta \le 2\pi$

117. $r = 4\sin\theta \quad 0 \le \theta \le \pi$

118. $r = 2a\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

119. $r = 1 + \sin\theta \quad 0 \le \theta \le 2\pi$

120. $r = 8(1+\cos\theta)$, $\quad 0 \le \theta \le 2\pi$

121. $r = 2\theta$, $\quad 0 \le \theta \le \frac{\pi}{2}$

122. $r = \frac{1}{\theta}$, $\quad \pi \le \theta \le 2\pi$

123. $r = e^{\theta}$, $\quad 0 \le \theta \le \pi$

124. $r = e^{3\theta}$, $\quad 0 \le \theta \le 2$

125. $r = 5\cos\theta$, $\quad \frac{\pi}{2} \le \theta \le \pi$

126. $r = 6\cos\theta$, $\quad 0 \le \theta \le \pi$

127. $r = 3(1-\cos\theta)$, $\quad 0 \le \theta \le \pi$

128. $r = 2\sin 6\theta$ one petal

129. Inner loop $\quad r = 3 - 6\cos\theta$

130. $r = e^{2\theta} \quad 0 \le \theta \le 2\theta$

131. $r = \cos\theta$

132. $r = a(1-\cos\theta)$

133. $r = \sec\theta$, $\quad 0 \le \theta \le \frac{\pi}{3}$

134. $f(\theta) = 2\sec\theta$, $\quad 0 \le \theta \le \frac{\pi}{4}$

135. $f(\theta) = 1 - \cos\theta$, $\quad 0 \le \theta \le 2\pi$

136. $f(\theta) = 1 + \cos\theta$, $\quad 0 \le \theta \le 2\pi$

137. $f(\theta) = 2 + 2\sin\theta$, $\quad 0 \le \theta \le 2\pi$

138. $f(\theta) = 2 - 2\sin\theta$, $\quad 0 \le \theta \le 2\pi$

139. $f(\theta) = 4 - 4\sin\theta$, $\quad 0 \le \theta \le \pi$

140. $f(\theta) = \cos\theta + \sin\theta$, $\quad 0 \le \theta \le \pi$

141. $f(\theta) = \cos\theta - \sin\theta$, $\quad 0 \le \theta \le \frac{\pi}{2}$

142. $f(\theta) = \cos 4\theta$, $\quad 0 \le \theta \le \frac{\pi}{2}$

143. $f(\theta) = \sin 4\theta$, $\quad 0 \le \theta \le 2\pi$

144. $r = e^{\cos\theta} - 2\cos 4\theta + \sin^5\frac{\theta}{12}$, $\quad 0 \le \theta \le 2\pi$

(145 – 160) Find the surface area bounded by

145. $r = 6\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

146. $r = a\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

147. $r = e^{a\theta} \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

148. $r = a(1+\cos\theta) \quad 0 \le \theta \le \pi$ *revolving about polar axis*

149. $r = 1 + 4\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

150. $r = 2\sin\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

151. $r = \sin 4\theta \quad 0 \le \theta \le \frac{\pi}{4}$ *revolving about* $\theta = \frac{\pi}{2}$

152. $r = \cos 4\theta \quad 0 \le \theta \le \frac{\pi}{4}$ *revolving about Polar axis*

153. $r = 3 + 3\cos\theta \quad 0 \le \theta \le \pi$ *revolving about Polar axis*

154. $r = 3 + 3\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

155. $r = 4 + 4\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

156. $r = 4 + 4\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

157. $r = 5 - 4\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about polar axis*

158. $r = \sin^2\theta \quad 0 \le \theta \le \frac{\pi}{3}$ *revolving about* $\theta = \frac{\pi}{2}$

159. $r = \sin^3\theta, \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

160. $r = \sin^3\frac{\theta}{2}, \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about polar axis*

161. Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5\sec\theta$

162. Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

163. Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

164. Assume m is a positive integer

a) Even number of leaves: what is the relationship between the total area enclosed by the $4m$-leaf rose $r = \cos(2m\theta)$ and m?

b) Odd number of leaves: what is the relationship between the total area enclosed by the $(2m+1)$–leaf rose $r = \cos((2m+1)\theta)$ and m?

165. Let R_n be the region bounded by the nth turn and the $(n+1)st$ turn of the spiral $r = e^{-\theta}$ in the first and second quadrants, for $\theta \ge 0$

a) Find the area A_n of R_n.

b) Evaluate $\lim_{n\to\infty} A_n$

c) Evaluate $\lim_{n\to\infty} \frac{A_{n+1}}{A_n}$

166. The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$. $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

167. The larger circle in the figure is the graph of $r = 1$.

Find the polar equation of the smaller circle such that the shaded regrions are equal.

168. Find equations of the circles in the figure.

Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

169. A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \le a \le 2$.

What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a = 0$ and $a = 2$

170. A circular corral of unit radius is enclosed by a fence. A goat outside the corral is tied to the fence with a rope of length $0 \le a \le \pi$. What is the area of the grassy region (outside the corral) that the goat can reach?

171. A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0 \le a \le 2$.

What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a = 0$ and $a = 2$

172. When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r if measured in meters and the microphone is at the pole.

The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.

173. The curve given by the parametric equations

$$x(t)=\frac{1-t^2}{1+t^2} \quad and \quad y(t)=\frac{t\left(1-t^2\right)}{1+t^2}$$

a) Find the rectangular equation of the strophoid.
b) Find a polar equation of the strophoid.
c) Sketch a graph of the strophoid.
d) Find the equations of the two tangent lines at the origin.
e) Find the points on the graph at which the tangent lines are horizontal.

174. Let a polar curve be described by $r = f(\theta)$ and let ℓ be the line tangent to the curve at the point $P(x, y) = P(r, \theta)$

a) Explain why $\tan\alpha = \dfrac{dy}{dx}$

b) Explain why $\tan\theta = \dfrac{y}{x}$

c) Let ϕ be the angle between ℓ and the line O and P. Prove that $\tan\phi = \dfrac{f(\theta)}{f'(\theta)}$

d) Prove that the value of θ for which ℓ is parallel to the x-axis satisfy $\tan\theta = -\dfrac{f(\theta)}{f'(\theta)}$

e) Prove that the value of θ for which ℓ is parallel to the y-axis satisfy $\tan\theta = \dfrac{f'(\theta)}{f(\theta)}$

Summary

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$Area = \int_a^b |y\,dx| \qquad = \int_a^b \left| y(t)\ \frac{dx}{dt} \right| dt$$

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\ dt$$

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \qquad x-axis$$

$$S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \qquad y-axis$$

$$\begin{cases} x = r\cos\theta\ , & y = r\sin\theta \\ r^2 = x^2 + y^2\ , & \tan\theta = \dfrac{y}{x} \end{cases}$$

Symmetry about the x-axis $(r,\ \theta) \ \rightarrow\ (-r,\ \pi-\theta)\ \ (r,\ -\theta)$

Symmetry about the y-axis $(r,\ \theta) \ \rightarrow\ (r,\ \pi-\theta)\ \ (-r,\ -\theta)$

Symmetry about the origin $(r,\ \theta) \ \rightarrow\ (-r,\ \theta)\ \ (r,\ \theta+\pi)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\ \sin\theta + f(\theta)\ \cos\theta}{f'(\theta)\ \cos\theta - f(\theta)\ \sin\theta}$$

$$Area = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta \qquad\qquad Area = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\ \sin\theta \sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\ d\theta$$ *About the polar axis*

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\ \cos\theta \sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\ d\theta$$ *About the line* $\theta = \frac{\pi}{2}$

Exercises *Lecture* 8 – Review

$(\mathbf{1-5})$ Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

1. $x(t)=\frac{t}{2}$, $y(t)=t+1$, $-\infty<t<\infty$

2. $x=\sqrt{t}$, $y=1-\sqrt{t}$, $t\geq 0$

3. $x(t)=-2\cos t$, $y(t)=2\sin t$, $0\leq t\leq \pi$

4. $x(t)=-\cos t$, $y(t)=\cos^2 t$, $0\leq t\leq \pi$

5. $x(t)=e^t$, $y(t)=e^{-2t}$, $-1\leq t\leq 1$

6. Find a parametric equation and a parameter interval for the motion of a particle in the *xy*-plane that traces the ellipse $16x^2+9y^2=144$ once counterclockwise.

7. Find a parametric equation and a parameter interval for the motion of a particle starting at the point $(-2,\ 0)$ and tracing the circle $x^2+y^2=4$ three times clockwise.

8. Find the tangent to the curve at the point defined by the given value of *t*.
$x(t)=t-\sin t$, $y(t)=1-\cos t$, $t=\frac{\pi}{6}$ *and* $t=\frac{2\pi}{3}$

$(\mathbf{9-10})$ Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point

9. $x(t)=2\cos t$, $y(t)=2\sin t$, $t=\frac{\pi}{4}$

10. $x(t)=-\sqrt{t+1}$, $y(t)=\sqrt{3t}$, $t=3$

$(\mathbf{12-18})$ Find the ***area*** of the curve(s)

11. Enclosed by $x-axis$ *and* $\begin{cases} x(t)=t^3+1 \\ y(t)=2t-t^2 \end{cases}$

12. Enclosed by $y-axis$ *and* $\begin{cases} x(t)=t^2-t \\ y(t)=\sqrt{t} \end{cases}$

13. Enclosed by $x-axis$ *and* $\begin{cases} x(t)=\cos t \\ y(t)=\cos 3t \end{cases}$

14. Enclosed by $y-axis$ *and* $\begin{cases} x(t)=\cos 3t \\ y(t)=\cos t \end{cases}$

15. Enclosed by $y-axis$ *and* $\begin{cases} x(t)=\sin 3t \\ y(t)=\sin t \end{cases}$

16. Enclosed by $x-axis$ *and* $\begin{cases} x(t)=\ln t \\ y(t)=t \end{cases}$ $1 \le t \le 3$

17. Enclosed by $x(t)=\cos 3t$, $y(t)=\sin 3t$

18. Enclosed by $x(t)=e^{\sin t}$, $y(t)=e^{\cos t}$, $0 \le t \le 2\pi$

$(\mathbf{19-24})$ Find the ***length*** of the curve

19. $x(t)=5\cos t-\cos 5t$, $y(t)=5\sin t-\sin 5t$, $0 \le t \le \frac{\pi}{2}$

20. $x(t)=t^3-6t^2$, $y(t)=t^3+6t^2$, $0 \le t \le 1$

21. $x(t)=t^2$, $y(t)=\frac{1}{3}t^3-t$, $-\sqrt{3} \le t \le \sqrt{3}$

22. $x(t)=e^t-t$, $y(t)=e^t+t$, $0 \le t \le 2$

23. $x(t)=1-2\sin t$, $y(t)=1-2\cos t$, $0 \le t \le 2\pi$

24. $x(t)=t+\sqrt{t}$, $y(t)=t-\sqrt{t}$, $0 \le t \le 1$

$(\mathbf{25-29})$ Find the ***surface*** area of the curve

25. $x-axis$ *and* $\begin{cases} x(t)=\ln t \\ y(t)=t \end{cases}$ $1 \le t \le 2$

26. $x-axis$ *and* $\begin{cases} x(t)=1+e^t \\ y(t)=e^t \end{cases}$ $0 \le t \le 1$

27. $y-axis$ *and* $\begin{cases} x(t)=1+e^t \\ y(t)=e^t \end{cases}$ $0 \le t \le 1$

28. $x-axis$ *and* $\begin{cases} x(t)=t^2-t^3 \\ y(t)=t^2+t^3 \end{cases}$ $0 \le t \le 1$

29. $y-axis$ *and* $\begin{cases} x(t)=t^2-t^3 \\ y(t)=t^2+t^3 \end{cases}$ $0 \le t \le 1$

$(\mathbf{30-33})$ Replace the polar equation with equivalent Cartesian equation

30. $r=2\sec\theta$

31. $r=3\sqrt{3}\ \sec\theta$

32. $r=-4\sin\theta$

33. $r^2=4\sin 2\theta$

$(\mathbf{34-37})$ Replace the Cartesian equation with equivalent polar equation

34. $x^2+y^2+5y=0$

35. $x^2+y^2-3x=0$

36. $x^2+y^2+4x=0$

37. $y^2=6x$

$(\mathbf{38-43})$ Find the ***area*** of the region in the polar coordinate plane

38. $r=3\theta$, $0 \le \theta \le 2\pi$

39. Between $r=2a\cos\theta$ *and* $r=a$

40. Enclosed by one loop of the Lemniscate $r^2=a\cos 2\theta$

41. Inside the cardioid $r=2(1+\sin\theta)$ and outside the circle $r=2\sin\theta$

42. Between the parabola $\dfrac{4}{1+\cos\theta}$ and the vertical line.

43. Between the parabola $\dfrac{4}{1-\cos\theta}$ and its latus rectum (*latus rectum is the chord through the focus, and parallel to the directrix*.)

$(\mathbf{44-50})$ Find the ***length*** of the curve given by the polar coordinate equation

44. $r=-1+\cos\theta$

45. $r=2\sin\theta+2\cos\theta$, $0 \le \theta \le \frac{\pi}{2}$

46. $r=8\sin^3\left(\frac{\theta}{3}\right)$, $0 \le \theta \le \frac{\pi}{4}$

47. $r=\sqrt{1+\cos 2\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

48. $r = \dfrac{4}{1-\cos\theta}$, $\dfrac{\pi}{6} \le \theta \le \pi$

49. $r = e^{2\theta}$, $0 \le \theta \le 2\pi$

50. $r = \theta^2$, $0 \le \theta \le 2\pi$

$(\mathbf{51-59})$ Find the ***surface*** area of the region in the polar coordinate plane

51. $r = 1 + a\cos\theta$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

52. $r = a\sin\theta + b\cos\theta$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

53. $r = a\sin\theta + b\cos\theta$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

54. $r = \dfrac{8}{1+\cos\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

55. $r = \dfrac{8}{1+\cos\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

56. $r = \sqrt{1+\cos\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

57. $r = \sqrt{1+\cos\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

58. $r = \sqrt{1+\cos 2\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

59. $r = \sqrt{1+\cos 2\theta}$ $\quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

60. Consider a circle sector with radius a $\left(r(\theta) = a\right)$ and an angle $0 \le \theta \le \dfrac{5\pi}{12}$. For the given parts, try to use *different method* to solve (Polar, Cartesian, parametric, Formula if possible).

a) Determine the area of the sector for $0 \le \theta \le \dfrac{5\pi}{12}$.

b) Find the length of the arc.

c) Find the volume of the solid sector about $x-axis$

d) Find the volume of the solid sector about $y-axis$

e) Find the surface area of the sector about the $polar-axis$ and about $x-axis$

f) Find the surface area of the sector about the line $\theta = \dfrac{\pi}{2}$ and about $y-axis$

Practice *Exam*

1. Find the rectangular equation in x and y and ***simplify***.

a) $x(t)=t^2,\quad y(t)=2\ln t,\quad t>0$

b) $x(t)=1+\sin t,\quad y(t)=\cos t-2,\quad 0\le t\le 2\pi$

2. Find the value of $\dfrac{d^2y}{dx^2}$ at this point $x(t)=t+e^t,\quad y(t)=1-e^t,\quad t=0$

3. Find the ***length*** of the curves $x(t)=\cos t,\quad y(t)=t+\sin t,\quad 0\le t\le \pi$

4. Find the ***area*** enclosed by the y–axis and the curve $x(t)=t-t^2,\quad y(t)=1+e^{-t}$

5. Find the ***areas of the surfaces*** generated by revolving the curves

$x(t)=2t,\quad y(t)=3t;\quad 0\le t\le 3\quad y-axis$

6. Find the ***area*** of the region in the polar coordinate plane of the enclosed by the *limaçon*

$r=2-\cos\theta$

7. Find the ***area*** of the region inside one leave of $r=\cos 3\theta$

8. Find the ***area*** of the given region common interior of $r=4\sin\theta$ *and* $r=2$

9. Find the ***length*** of the curve given by the polar coordinate equation

$r=2\sin\theta+2\cos\theta,\quad 0\le\theta\le\frac{\pi}{2}$

10. Find the ***surface area*** bounded by $r=a\cos\theta\quad 0\le\theta\le\frac{\pi}{2}$ *revolving about* $\theta=\frac{\pi}{2}$

$$A=\int_a^b x\,dy \qquad A=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$S=2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad S=2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f)^2+(f')^2}\,d\theta \quad \text{About the line } \theta=\frac{\pi}{2}$$

Lecture Nine

Vectors

Section 9.1 – Introduction to Vectors

9.1−1 Coordinate Systems in Linear Algebra

The important of coordinate system that it will let us map the point and/or vector into $\mathbb{R}^2$ *or* $\mathbb{R}^3$. In addition, it will helps view the point coordinate location.

There are different types of graphs to provide the coordinate for the plane.

1- Perpendicular axes or rectangular coordinates and a unit of measurement for each axis. To graph a point, is to graph a rectangle from the origin with given coordinate.

2- Tilted axes or nonrectangular coordinates. To graph a point, by plotting a parallelogram.

In ***analytic geometry***, we use rectangular coordinate systems to create a point either in 2-space or 3-space.

Coordinates of P in a rectangular coordinate system in 2-space

Coordinates of P in a rectangular coordinate system in 3-space

Coordinates of P in a nonrectangular coordinate system in 2-space

Coordinates of P in a nonrectangular coordinate system in 3-space

9.1–2 Midpoint of a Line Segment

The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example 1

Find the midpoint of the segment $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$

Solution

$$M = \left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right)$$
$$= (5, 1, 2)$$

9.1–3 Distance in Space

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof

$$|P_1A| = |x_2 - x_1|$$
$$|AB| = |y_2 - y_1|$$
$$|BP_2| = |z_2 - z_1|$$

From the right triangles P_1AB and P_1BP_2:

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$
$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

$$\begin{aligned}
|P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\
&= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\
&= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\
&= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \checkmark
\end{aligned}$$

Example 2

Find the distance between $P_1(2,\ 1,\ 5)$ and $P_2(-2,\ 3,\ 0)$

Solution

$$\left|P_1P_2\right| = \sqrt{(-2-2)^2+(3-1)^2+(0-5)^2}$$

$$= \sqrt{16+4+25}$$

$$= \sqrt{45}$$

9.1–4 Standard Equation for the Sphere of Radius *a* and Center (x_0, y_0, z_0)

The standard equation for a sphere center at $P(x_0, y_0, z_0)$ with a radius $R = a$ is given by

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2$$

Example 3

Find the center and radius of the sphere $x^2+y^2+z^2+3x-4z+1=0$

Solution

$$\left(x^2+3x\right)+y^2+\left(z^2-4z\right)=-1$$

$$\left(x^2+3x+\left(\frac{3}{2}\right)^2\right)+y^2+\left(z^2-4z+\left(\frac{-4}{2}\right)^2\right)=-1+\left(\frac{3}{2}\right)^2+\left(\frac{-4}{2}\right)^2$$

$$\left(x+\frac{3}{2}\right)^2+y^2+(z-2)^2=-1+\frac{9}{4}+4$$

$$\left(x+\frac{3}{2}\right)^2+y^2+(z-2)^2=\frac{21}{4}$$

Therefore, the center is $\left(-\frac{3}{2},\ 0,\ 2\right)$ and the radius is $\frac{\sqrt{21}}{2}$

9.1– 5 Vector Definition

A vector is directed line segment and is representing by its direction of the arrowhead specifies the ***direction*** and the ***length*** of the arrow specifies the *magnitude*. A vector is used to describe physical quantities, for example, force, velocity, acceleration, and current.

Geometrically, the vector represents as an arrow that has magnitude and appropriate direction.

The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.

Vectors in two dimensions are also called **2–*space.***

Vectors in three dimensions are also called **3–*space*** by arrow.

9.1–6 Parallelogram Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\vec{w}+\vec{v}$ is the vector represented by the arrow from the common initial point of $\vec{v}$ and $\vec{w}$ to the opposite vertex of the parallelogram.

The magnitude of the resultant force is represented by $\vec{w}+\vec{v}$ on the scale as that is used in $\vec{v}$ and $\vec{w}$, and the direction of the resultant force is that $\vec{w}+\vec{v}$.

9.1–7 Triangle Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2–space or 3–space that are positioned so the initial point of $\vec{w}$ is at the terminal point of $\vec{v}$, then the sum $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.

$$\boxed{\vec{v}+\vec{w}=\vec{w}+\vec{v}}$$

9.1–8 Rule for Scalar Multiplication Vector

Quantities such as mass, length, volume, pressure, temperature, voltage, and time characterized by a single real number , scaled to a suitable unit of measurement. Such quantities are called ***scalar quantities***, and the real number associate with the magnitude of the quantity is called ***scalar***.

A scalar multiplication is a real number multiplies a vector and has the effect of changing the length of the vector.

9.1–9 Vector Algebra Operations

Definitions

Let $\vec{u}=\langle u_1, u_2, u_3\rangle$ and $\vec{v}=\langle v_1, v_2, v_3\rangle$ be vectors with ***k*** a scalar

Addition: $\vec{u}+\vec{v}=\langle u_1+v_1, u_2+v_2, u_3+v_3\rangle$

Scalar multiplication: $k\vec{u}=\langle ku_1, ku_2, ku_3\rangle$

Example 4

Let $\vec{v} = (1,\ 2)$.

Find and graph the following:

a) $\frac{1}{2}\vec{v}$ *b)* $-2\vec{v}$

Solution

a) $\frac{1}{2}\vec{v} = \frac{1}{2}(1,\ 2)$

$= \left(\frac{1}{2},\ 1\right)$

b) $-2\vec{v} = -2(1,\ 2)$

$= (-2,\ -4)$

9.1–10 Graphing Vector in 3-dim

In 3-dimensional space, the arrow starts at the origin $(0,\ 0,\ 0)$, where the *xyz* axis meet.

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is also *written* as $(1,\ 2,\ 2)$

9.1–11 Vectors in Coordinate Systems

It is sometimes necessary to consider vectors whose initial are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ \ y_2 - y_1)$$

If $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ \ y_2 - y_1,\ \ z_2 - z_1)$$

Example 5

The components of the vector $\vec{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$, find $\vec{v}$?

Solution

$$\vec{v} = (7-2,\ 5-(-1),\ -8-4)$$

$$= (5,\ 6,\ -12)$$

Example 6

Let $\vec{u} = \langle -1,\ 3,\ 1\rangle$ and $\vec{v} = \langle 4,\ 7,\ 0\rangle$. Find the components of

a) $2\vec{u} + 3\vec{v}$ *b)* $\vec{u} - \vec{v}$ *c)* $\left|\frac{1}{2}\vec{u}\right|$

Solution

a) $2\vec{u} + 3\vec{v} = 2\langle -1,\ 3,\ 1\rangle + 3\langle 4,\ 7,\ 0\rangle$

$$= \langle -2,\ 6,\ 2\rangle + \langle 12,\ 21,\ 0\rangle$$

$$= \langle 10,\ 27,\ 2\rangle$$

b) $\vec{u} - \vec{v} = \langle -1,\ 3,\ 1\rangle - \langle 4,\ 7,\ 0\rangle$

$$= \langle -5,\ -4,\ 1\rangle$$

c) $\left|\frac{1}{2}\vec{u}\right| = \left|\left\langle -\frac{1}{2},\ \frac{3}{2},\ \frac{1}{2}\right\rangle\right|$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{11}{4}}$$

$$= \frac{\sqrt{11}}{2}$$

9.1–12 *Theorem*

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$, and if k and m are scalars, then

a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

d) $\vec{u} + (-\vec{u}) = \vec{0}$

e) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

f) $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

g) $k(m\vec{u}) = (km)\vec{u}$

h) $1\vec{u} = \vec{u}$

i) $0\vec{v} = 0$

j) $k\vec{0} = \vec{0}$

9.1–13 *Definition*

If $\vec{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the ***component form*** of $\vec{v}$ is:

$$\vec{v} = \langle v_1, v_2 \rangle$$

If $\vec{v}$ is a **three-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the ***component form*** of $\vec{v}$ is:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

The magnitude or length of the vector $\vec{v} = \overrightarrow{PQ}$ is the nonnegative number.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the ***zero vector*** $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$

Example 7

Find the length of the vector $\vec{v} = (1,\ 2,\ 3)$

Solution

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Example 8

Find the component form and the length of the vector with initial point $P(-3,\ 4,\ 1)$ and terminal point $Q(-5,\ 2,\ 2)$

Solution

The component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle -5-(-3),\ 2-4,\ 2-1 \rangle$$

$$= \langle -2,\ -2,\ 1 \rangle$$

The length is

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2 + (-2)^2 + 1^2}$$

$$= 3$$

9.1–14 *Definition:* Unit Vectors

A ***unit vector*** $\vec{u}$ is a vector whose length equals to ***one***. Then $\vec{u} \bullet \vec{u} = 1$
Divide any nonzero vector $\vec{v}$ by its length.

Then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the same direction as $\vec{v}$.

Example 9

Find the unit vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 2, -1)$

Solution

$$\|\vec{v}\| = \sqrt{2^2 + 2^2 + (-1)^2}$$

$$= 3$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$
$$= \frac{1}{3}(2,\ 2,\ -1)$$
$$= \left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right)$$

$$\|\vec{u}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}$$
$$= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}}$$
$$= \sqrt{\frac{9}{9}}$$
$$= 1 \quad \checkmark$$

9.1–15 Examples of *unit vectors*

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\mathbb{R}^3$

$$\hat{i} = (1,\ 0,\ 0) \quad \hat{j} = (0,\ 1,\ 0) \quad \textit{and} \quad \hat{k} = (0,\ 0,\ 1)$$

Example 10

Consider the vectors $\hat{\boldsymbol{i}} = (1,\ 0,\ 0)$ $\hat{\boldsymbol{j}} = (0,\ 1,\ 0)$ $\hat{\boldsymbol{k}} = (0,\ 0,\ 1)$

These vectors each have length of 1 and lie along the coordinate axes. They are called the ***standard unit vectors*** in 3-space.

Example 11

$$(7,\ 3,\ -4) = 7\,\hat{i} + 3\,\hat{j} - 4\hat{k}$$

Any vector $\vec{v} = \langle v_1,\ v_2,\ v_3 \rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\begin{aligned}
\vec{v} &= \langle v_1,\ v_2,\ v_3 \rangle \\
&= \langle v_1,\ 0,\ 0 \rangle + \langle 0,\ v_2,\ 0 \rangle + \langle 0,\ 0,\ v_3 \rangle \\
&= v_1 \langle 1,\ 0,\ 0 \rangle + v_2 \langle 0,\ 1,\ 0 \rangle + v_3 \langle 0,\ 0,\ 1 \rangle \\
&= v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}
\end{aligned}$$

Example 12

Find a unit vector $\vec{u}$ in the direction of the vector from $P_1(1,\ 0,\ 1)$ to $P_2(3,\ 2,\ 0)$.

Solution

$$\begin{aligned}
\overrightarrow{P_1P_2} &= (3-1)\hat{i} + (2-0)\,\hat{j} + (0-1)\hat{k} \\
&= 2\hat{i} + 2\,\hat{j} - \hat{k}
\end{aligned}$$

$$\begin{aligned}
\left|\overrightarrow{P_1P_2}\right| &= \sqrt{2^2 + 2^2 + (-1)^2} \\
&= \sqrt{9} \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\vec{u} &= \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} \\
&= \frac{2\hat{i} + 2\,\hat{j} - \hat{k}}{3} \\
&= \frac{2}{3}\hat{i} + \frac{2}{3}\,\hat{j} - \frac{1}{3}\hat{k}
\end{aligned}$$

Example 13

If $\vec{v} = 3\hat{i} - 4\,\hat{j}$ is a velocity vector, express $\vec{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution

Speed is the magnitude (length) of $\vec{v}$:

$$|\vec{v}| = \sqrt{3^2 + (-4)^2}$$
$$= 5$$

The unit vector has the same direction as $\vec{v}$:

$$\frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{i} - 4\hat{j}}{5}$$
$$= \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\vec{v} = 3\hat{i} - 4\hat{j}$$
$$= \underbrace{5}_{\text{Length (Speed)}}\underbrace{\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)}_{\text{Direction of motion}}$$

Note:

If $\vec{v} \neq 0$, then

1. $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\vec{v}$;
2. The equation $\vec{v} = |\vec{v}|\frac{\vec{v}}{|\vec{v}|}$ expresses $\vec{v}$ as its length times its direction.

Example 14

A force of 6 *Newton* is applied in the direction of the vector $\vec{v} = 2\hat{i} + 2\hat{j} - \hat{k}$. Express the force $\vec{F}$ as a product of its magnitude and direction.

Solution

$$|\vec{v}| = \sqrt{2^2 + 2^2 + (-1)^2}$$
$$= 3$$

$$\vec{F} = |\vec{v}|\,\frac{\vec{v}}{|\vec{v}|}$$
$$= 3\,\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$$
$$= 3\left(\frac{2}{3}\hat{\boldsymbol{i}} + \frac{2}{3}\hat{\boldsymbol{j}} - \frac{1}{3}\hat{\boldsymbol{k}}\right)$$

9.1–16 Applications

Example 15

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution

$\vec{u}$ = the velocity of the airplane.

$\vec{v}$ = the velocity of the tailwind.

Given: $|\vec{u}| = 500 \quad |\vec{v}| = 70$

$$\vec{u} = \langle 500,\ 0 \rangle$$

$$\vec{v} = \langle 70\cos 60°,\ 70\sin 60° \rangle$$

$$= \langle 35,\ 35\sqrt{3} \rangle$$

$$\vec{u} + \vec{v} = \langle 535,\ 35\sqrt{3} \rangle \quad = 535\hat{i} + 35\sqrt{3}\hat{j}$$

$$|\vec{u} + \vec{v}| = \sqrt{535^2 + \left(35\sqrt{3}\right)^2}$$

$$\approx 538.4$$

$$\theta = \tan^{-1}\frac{35\sqrt{3}}{535}$$

$$\approx 6.5°$$

The ground speed of the airplane is about 538.4 *mph*, and its direction is about 6.5° north of east.

Example 16

A small cart is being pulled along a 20-*lb* smooth horizontal floor with a force $\vec{F}$ making a 45° angle to the floor. What is the effective force moving the cart forward?

Solution

$$a = |\vec{F}|\cos 45°$$

$$= (20)\left(\frac{\sqrt{2}}{2}\right)$$

$$\equiv 14.14\ \ lb$$

Example 17

A 75-*N* weight is suspended by two wires.

Find the forces $\vec{F}_1$ and $\vec{F}_2$ acting both wires

Solution

$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 55°,\ \left|\vec{F}_1\right|\sin 55° \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 40°,\ \left|\vec{F}_2\right|\sin 40° \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 75 \rangle$$

$$-\left|\vec{F}_1\right|\cos 55° + \left|\vec{F}_2\right|\cos 40° = 0$$

$$\Rightarrow \quad \left|\vec{F}_2\right| = \left|\vec{F}_1\right| \frac{\cos 55°}{\cos 40°}$$

$$\left|\vec{F}_1\right|\sin 55° + \left|\vec{F}_2\right|\sin 40° = 75$$

$$\left|\vec{F}_1\right|\sin 55° + \left|\vec{F}_1\right|\frac{\cos 55°}{\cos 40°}\sin 40° = 75$$

$$\left|\vec{F}_1\right|\left(\sin 55° + \cos 55° \tan 40°\right) = 75$$

$$\left|\vec{F}_1\right| = \frac{75}{\sin 55° + \cos 55° \tan 40°}$$

$$\approx 57.67\ N$$

$$\left|\vec{F}_2\right| = 57.67 \frac{\cos 55°}{\cos 40°}$$

$$\approx 43.18\ N$$

The force vectors are then:

$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 55°,\ \left|\vec{F}_1\right|\sin 55° \right\rangle$$

$$= \langle -57.67\cos 55°,\ 57.67\sin 55° \rangle$$

$$\approx \langle -33.08,\ 47.24 \rangle \quad N$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right| \cos 40°,\ \left|\vec{F}_2\right| \sin 40° \right\rangle$$

$$= \langle 43.18 \cos 40°,\ 43.18 \sin 40° \rangle$$

$$\simeq \langle 33.08,\ 27.76 \rangle \quad N$$

Exercises *Section* 9.1 – Introduction to Vectors

$(\mathbf{1-4})$ Sketch the following vectors with initial points located at the origin

1. $P_1(4,\ 8),\quad P_2(3,\ 7)$

2. $P_1(0,\ -2),\quad P_2(-3,\ 5)$

3. $P_1(-1,\ 0,\ 2),\quad P_2(0,\ -1,\ 0)$

4. $P_1(3,\ -7,\ 2),\quad P_2(-2,\ 5,\ -4)$

$(\mathbf{5-9})$ Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations:

5. $x^2+z^2=4,\quad y=0$

6. $x^2+y^2=4,\quad z=-2$

7. $x^2+y^2+z^2=1,\quad x=0$

8. $x^2+(y-1)^2+z^2=4,\quad y=0$

9. $x^2+y^2+z^2=4,\quad y=x$

$(\mathbf{10-13})$ Find the distance between points P_1 *and* P_2

10. $P_1(1,\ 1,\ 1),\quad P_2(3,\ 3,\ 0)$

11. $P_1(-1,\ 1,\ 5),\quad P_2(2,\ 5,\ 0)$

12. $P_1(1,\ 4,\ 5),\quad P_2(4,\ -2,\ 7)$

13. $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

$(\mathbf{14-16})$ Find the center and radii of the spheres.

14. $x^2+y^2+z^2+4x-4z=0$

15. $x^2+y^2+z^2-6y+8z=0$

16. $2x^2+2y^2+2z^2+x+y+z=9$

17. Find a formula for the distance from the point $P(x,\ y,\ z)$ to x-axis.

18. Find a formula for the distance from the point $P(x,\ y,\ z)$ to xz-plane.

$(\mathbf{19-20})$ Find the sum of the vectors and illustrate the sum geometrically:

19. $\vec{u}=(1,\ 3),\quad \vec{v}=(2,\ -2)$

20. $\vec{u}=(2,\ -3),\quad \vec{v}=(-3,\ -1)$

21. Let $\vec{u}=\langle -3,\ 4\rangle$ *and* $\vec{v}=\langle 2,\ -5\rangle$. Find the component form and the magnitude if the vector:

a) $3\vec{u}-4\vec{v}$ *b)* $-2\vec{u}$ *c)* $\vec{u}+\vec{v}$

22. Let $\vec{u} = \langle 3,\ -2 \rangle$ *and* $\vec{v} = \langle -2,\ 5 \rangle$. Find the component form and the magnitude if the vector:

a) $3\vec{u}$
b) $\vec{u} - \vec{v}$
c) $2\vec{u} - 3\vec{v}$
d) $-2\vec{u} + 5\vec{v}$
e) $-\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v}$

23. Find the vector $\vec{v}$ and verify the specified vector operations geometrically, when:

$\vec{u} = (-2,\ 3),\quad \vec{w} = (-3,\ -2)$

a) $\vec{v} = \frac{3}{2}\vec{u}$
b) $\vec{v} = \vec{u} + 2\vec{w}$
c) $\vec{v} = \frac{1}{2}(3\vec{u} + \vec{w})$

24. Given $\vec{u} = (1,\ 2,\ 3),\quad \vec{v} = (2,\ 2,\ -1),\quad \vec{w} = (4,\ 0,\ -4)$. Find

a) $\vec{u} - \vec{v}$
b) $\vec{v} - \vec{u}$
c) $2\vec{u} + 4\vec{v} - \vec{w}$
d) $\vec{z}:\ 3\vec{u} - 4\vec{z} = \vec{w}$

25. Let $\vec{u} = (-3,\ 1,\ 2)$, $\vec{v} = (4,\ 0,\ -8)$, and $\vec{w} = (6,\ -1,\ -4)$. Find the components

a) $\vec{v} - \vec{w}$
b) $6\vec{u} + 2\vec{v}$
c) $5(\vec{v} - 4\vec{u})$
d) $-3(\vec{v} - 8\vec{w})$
e) $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$
f) $-\vec{u} + (\vec{v} - 4\vec{w})$

26. Find scalars a, b, and c such that $\langle 2,\ 2,\ 2 \rangle = a\langle 1,\ 1,\ 0 \rangle + b\langle 0,\ 1,\ 1 \rangle + c\langle 1,\ 0,\ 1 \rangle$

27. Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where:

$A = (1,-1),\quad B = (2,0),\quad C = (-1,3),\quad and\ \ D = (-2,2)$

28. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$
b) $(3, -3, 3)$

(29 – 33) Find the component form of the vector:

29. The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

30. The unit vector obtained by rotating the vector $\langle 0,\ 1 \rangle$ $120°$ counterclockwise about the origin.

31. The unit vector obtained by rotating the vector $\langle 1,\ 0 \rangle$ $135°$ counterclockwise about the origin.

32. The unit vector that makes an angle $\theta = \frac{\pi}{6}$ with the positive x-axis

33. The vector 5 units long in the direction opposite to the direction of $\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$

34. Express the velocity vector $\vec{v} = \left(e^t \cos t - e^t \sin t\right)i + \left(e^t \cos t + e^t \sin t\right)\hat{j}$ *when* $t = \ln 2$ in terms of its length and direction.

35. Sketch the indicated vector.

a) $\vec{u}-\vec{v}$

b) $2\vec{u}-\vec{v}$

c) $\vec{u}-\vec{v}+\vec{w}$

36. An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

37. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have for the resultant vector to be 500 *mph* due east?

38. Consider a 100-*N* weight suspended by two wires. Find the magnitudes and components of the force vectors $\overrightarrow{F_1}$ *and* $\overrightarrow{F_2}$.

39. Consider a 50-*N* weight suspended by two wires, If the magnitude of vector $\left|\overrightarrow{F_1}\right| = 35\ N$, find the angle α and the magnitude of vector $\overrightarrow{F_2}$.

40. Consider a ***W***-*N* weight suspended by two wires, If the magnitude of vector $\left|\overrightarrow{F_2}\right| = 100\ N$, find ***W*** and the magnitude of vector $\overrightarrow{F_1}$

41. Consider a 25-N weight suspended by two wires, If the magnitude of vector $\overrightarrow{F_1}$ *and* $\overrightarrow{F_2}$ are both 75 N, then angles α and β are equal. Find α.

42. Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\overrightarrow{F_1}$ *and* $\overrightarrow{F_2}$

43. Consider a $W = 50\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\overrightarrow{F_1}$ *and* $\overrightarrow{F_2}$

44. Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\overrightarrow{F_1}$ *and* $\overrightarrow{F_2}$

45. A bird flies from its nest 5 *km* in the direction 60° northeast, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

46. Suppose that *A*, *B*, and *C* are the corner points of the thin triangular plate of constant density.

a) Find the vector from *C* to the midpoint *M* of side *AB*.

b) Find the vector from *C* to the point that lies two-thirds of the way from *C* to *M* on the median *CM*.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

47. Show that a unit vector in the plane can be expressed as $\vec{u} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$, obtained by rotating $\hat{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

48. Assume the positive *x-axis* points east and the positive *y-axis* points north.

a) An airliner flies northeast at a constant altitude at 550 *mi/hr* in calm air. Find *a* and *b* such that it velocity may be expressed in the form $\vec{v} = a\hat{i} + b\hat{j}$

b) An airliner flies northeast at a constant altitude at 550 *mi/hr* relative to the air in a southerly crosswind $\vec{w} = \langle 0,\ 40 \rangle$. Find the velocity of the airliner relative to the ground.

49. Let $\overrightarrow{PQ}$ extended from $P(2,\ 0,\ 6)$ to $Q(2,\ -8,\ 5)$

a) Find the position vector equal to $\overrightarrow{PQ}$.

b) Find the midpoint *M* of the line segment PQ. Then find the magnitude of $\overrightarrow{PM}$

c) Find a vector of length 8 with direction opposite that of $\overrightarrow{PQ}$

50. An object at the origin is acted on by the forces $\overrightarrow{F_1} = -10\hat{i} + 20\hat{k}$, $\overrightarrow{F_2} = 40\hat{j} + 10\hat{k}$, and $\overrightarrow{F_3} = -50\hat{i} + 20\hat{j}$. Find the magnitude of the combined force and use a sketch to illustrate the direction of the combined force.

51. A remote sensing probe falls vertically with a terminal of 60 *m/s* when it encounters a horizontal crosswind blowing north at 4 *m/s* and an updraft blowing vertically at 10 *m/s*. find the magnitude and direction of the resulting velocity relative to the ground.

52. A small plane is flying north in calm air at 250 *mi/hr* when it is hit by a horizontal crosswind blowing northeast at 40 *mi/hr* and a 25 *mi/hr* downdraft. Find the resulting velocity and speed of the plane.

53. A structural unit is being lifted by three cables, *AB*, *AC*, and *AD*. The unit has a mass of 78.5 *kg*.

Assume the point *A* as the origin of the coordinates.

The other points coordinates are given in meters (*m*):

$$B(-0.12,\ -0.12,\ -1)\quad C(-0.12,\ 0.32,\ -1)\quad D(0.32,\ -0.12,\ -1)$$

Find the tension force in each cable in Newtons.

Section 9.2 – Dot Products

9.2–1 *Definition*

If a force $\vec{F}$ is applied to a particle moving along a path, we often need to know the magnitude of the force and the direction of motion.

To calculate the angle between two vectors directly from their component, called the ***dot product***, also called *inner* or *scalar* products.

9.2–2 Angle between Vectors

Consider a triangle with sides *a*, *b*, and *c*.

Using Law of Cosine, then:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$v_1^2 - 2v_1u_1 + u_1^2 + u_2^2 - 2v_2u_2 + v_2^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$-2v_1u_1 - 2v_2u_2 = -2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$v_1u_1 + v_2u_2 = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\cos\theta = \frac{v_1u_1 + v_2u_2}{\|\vec{u}\|\,\|\vec{v}\|}$$

The angle θ between two nonzero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}|\,|\vec{v}|}\right)$$

We identify $\|\vec{v}\| \cos\theta$ as the component of $\vec{v}$ parallel to $\vec{u}$, and the length of the orthogonal projection of $\vec{v}$ in the direction of $\vec{u}$, with appropriate sign.

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

9.2–3 *Definition*

The dot product $\vec{u} \bullet \vec{v}$ of vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example 1

Find the dot product:

a) $\langle 1, -2, -1 \rangle \bullet \langle -6, 2, -3 \rangle$ *b)* $\left(\frac{1}{2}\hat{i} + 3\hat{j} + \hat{k}\right) \bullet \left(4\hat{i} - \hat{j} + 2\hat{k}\right)$

Solution

a) $\langle 1, -2, -1 \rangle \bullet \langle -6, 2, -3 \rangle = 1(-6) + (-2)(2) + (-1)(-3)$

$= -7$

b) $\left(\frac{1}{2}\hat{i} + 3\hat{j} + \hat{k}\right) \bullet \left(4\hat{i} - \hat{j} + 2\hat{k}\right) = \frac{1}{2}(4) + 3(-1) + 1(2)$

$= 1$

Example 2

Find the angle between $\vec{u} = \hat{i} - 2\hat{j} - 2\hat{k}$ *and* $\vec{v} = 6\hat{i} + 3\hat{j} + 2\hat{k}$

Solution

$\vec{u} \bullet \vec{v} = 1(6) + (-2)(3) + (-2)(2)$

$= -4$

$\vec{u} = \sqrt{1^2 + (-2)^2 + (-2)^2}$

$= 3$

$$\vec{v} = \sqrt{6^2 + 3^2 + 2^2}$$

$$= 7$$

$$\theta = \cos^{-1} \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\left(\frac{-4}{(3)(7)}\right)$$

$$= \cos^{-1}\left(-\frac{4}{21}\right)$$

$$\approx 1.76 \ \ rad$$

Example 3

Find the angle θ of the triangle ABC determined by the vertices:

$$A = (0,\ 0), \quad B = (3,\ 5), \quad and \quad C = (5,\ 2)$$

Solution

$$\overrightarrow{CA} = \langle -5,\ -2 \rangle \quad \overrightarrow{CB} = \langle -2,\ 3 \rangle$$

$$\overrightarrow{CA} \bullet \overrightarrow{CB} = (-5)(-2) + (-2)(3)$$

$$= 4$$

$$\left|\overrightarrow{CA}\right| = \sqrt{(-5)^2 + (-2)^2}$$

$$= \sqrt{29}$$

$$\left|\overrightarrow{CB}\right| = \sqrt{(-2)^2 + (3)^2}$$

$$= \sqrt{13}$$

$$\theta = \cos^{-1} \frac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\ \left|\overrightarrow{CB}\right|}$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{377}}\right)$$

$$\approx 1.36 \ \ rad \quad or \quad 78.1°$$

9.2–4 Perpendicular (*Orthogonal*) Vectors

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (*or* perpendicular) *iff* $\vec{u} \bullet \vec{v} = 0$

Example 4

Determine if the two vectors are orthogonal:

a) $\vec{u} = \langle 3,\ -2 \rangle$ *and* $\vec{v} = \langle 4,\ 6 \rangle$

b) $\vec{u} = 3\hat{i} - 2\hat{j} + \hat{k}$ *and* $\vec{v} = 2\hat{j} + 4\hat{k}$

Solution

a) $\vec{u} \bullet \vec{v} = 3(4) + (-2)(6)$

$= 0$

∴ The two vectors are orthogonal.

b) $\vec{u} \bullet \vec{v} = 3(0) + (-2)(2) + 1(4)$

$= 0$

∴ The two vectors are orthogonal.

9.2–5 Dot Product Properties and Vector Projection

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are any vectors and c is a scalar, then:

a) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

b) $\vec{u} \bullet \vec{u} = |\vec{u}|^2$

c) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

d) $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

e) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$

f) $(\vec{u} - \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} - \vec{v} \bullet \vec{w}$

g) $c(\vec{u} \bullet \vec{v}) = (c\vec{u}) \bullet \vec{v} = \vec{u} \bullet (c\vec{v})$

h) $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

The vector projection of $\vec{u} = \overrightarrow{PQ}$ onto a nonzero vector $\vec{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.

The notation for this vector is

$$proj_{\vec{v}}\ \vec{u} \qquad \text{(The vector projection of } \vec{u} \text{ onto } \vec{v}\text{)}$$

$$proj_{\vec{v}}\ \vec{u} = \left(|\vec{u}|\cos\theta\right)\frac{\vec{u}}{|\vec{v}|}$$

$$= \left(\frac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is the scalar:

$$|\vec{u}|\ \cos\theta = \frac{\vec{u} \bullet \vec{v}}{|\vec{v}|}$$

$$= \vec{u} \bullet \frac{\vec{v}}{|\vec{v}|}$$

Example 5

Find the vector projection of $\vec{u} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ *onto* $\vec{v} = \hat{i} - 2\hat{j} - 2\hat{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Solution

$$proj_{\vec{v}}\vec{u} = \frac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{6(1) + 3(-2) + 2(-2)}{1^2 + (-2)^2 + (-2)^2}\left(\hat{i} - 2\hat{j} - 2\hat{k}\right)$$

$$= \frac{-4}{9}\left(\hat{i} - 2\hat{j} - 2\hat{k}\right)$$

$$= -\frac{4}{9}\hat{i} + \frac{8}{9}\hat{j} + \frac{8}{9}\hat{k}$$

$$\vec{u}\cos\theta = \vec{u} \bullet \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right) \bullet \frac{\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{1^2 + (-2)^2 + (-2)^2}}$$

$$= \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right) \bullet \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= 6\left(\frac{1}{3}\right) + 3\left(-\frac{2}{3}\right) + 2\left(-\frac{2}{3}\right)$$

$$= 2 - 2 - \frac{4}{3}$$

$$= -\frac{4}{3}$$

Example 6

Find the vector projection of a force $\vec{F} = 5\hat{i} + 2\hat{j}$ *onto* $\vec{v} = \hat{i} - 3\hat{j}$ and the scalar component of $\vec{F}$ in the direction of $\vec{v}$.

Solution

$$proj_{\vec{v}}\vec{F} = \frac{5(1) + 2(-3)}{1^2 + (-3)^2}\left(\hat{i} - 3\hat{j}\right)$$

$$proj_{\vec{v}}\vec{F} = \frac{\vec{F} \bullet \vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= -\frac{1}{10}\left(\hat{i} - 3\hat{j}\right)$$

$$= -\frac{1}{10}\hat{i} + \frac{3}{10}\hat{j}$$

$$\vec{F}\cos\theta = \vec{F} \bullet \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{\left(5\hat{i} + 2\hat{j}\right) \bullet \left(\hat{i} - 3\hat{j}\right)}{\sqrt{1^2 + (-3)^2}}$$

$$= \frac{5 - 6}{\sqrt{10}}$$

$$= -\frac{1}{\sqrt{10}}$$

9.2–6 Work

The work is done by a constant force of magnitude $\vec{F}$ in moving an object through a distance $\boldsymbol{d}$ as $W = \left|\vec{F}\right| d$.

$$\begin{aligned} Work &= \begin{pmatrix} scalar\ component\ of\ \vec{F} \\ \text{in}\ the\ direction\ of\ \overrightarrow{PQ} \end{pmatrix} (distance\ \boldsymbol{d}) \\ &= \left(\left|\vec{F}\right| \cos\theta\right) \left|\overrightarrow{PQ}\right| \\ &= \vec{F} \bullet \overrightarrow{PQ} \end{aligned}$$

9.2–7 *Definition*

The work done by a constant force $\vec{F}$ acting through a displacement $\vec{D} = \overrightarrow{PQ}$ is

$$W = \vec{F} \bullet \vec{D}$$

Example 7

If $\left|\vec{F}\right| = 40\ N,\quad \left|\vec{D}\right| = 3\ m,\quad and\quad \theta = 60°$ find the work done by $\vec{F}$ in acting from P to Q.

Solution

$$\begin{aligned} Work &= \vec{F} \bullet \vec{D} \\ &= \left|\vec{F}\right| \bullet \left|\vec{D}\right| \cos\theta \\ &= (40)(3)\cos 60° \\ &= 60\quad J\quad (joules) \end{aligned}$$

Exercises *Section* 9.2 – Dot Products

$(\mathbf{1-5})$ Find

a) $\vec{v}\bullet\vec{u},\ |\vec{v}|,\ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

1. $\vec{v}=2\hat{i}-4\hat{j}+\sqrt{5}\hat{k},\quad \vec{u}=-2\hat{i}+4\hat{j}-\sqrt{5}\hat{k}$

2. $\vec{v}=\frac{3}{5}\hat{i}+\frac{4}{5}\hat{k},\quad \vec{u}=5\hat{i}+12\hat{j}$

3. $\vec{v}=2\hat{i}+10\hat{j}-11\hat{k},\quad \vec{u}=2\hat{i}+2\hat{j}+\hat{k}$

4. $\vec{v}=-\hat{i}+\hat{j},\quad \vec{u}=2\hat{i}+\sqrt{17}\hat{j}$

5. $\vec{v}=\left\langle\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right\rangle,\quad \vec{u}=\left\langle\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right\rangle$

6. Let $\vec{u}=\begin{pmatrix}2\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\4\end{pmatrix}$. Find $\left\|\frac{1}{\|2\vec{u}+\vec{v}\|}(2\vec{u}+\vec{v})\right\|$

7. Let $\vec{u}=\begin{pmatrix}2\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}3\\3\end{pmatrix}$. Find $\left\|\frac{1}{\|\vec{u}-\vec{v}\|}(\vec{u}-\vec{v})\right\|$

8. Let $\vec{u}=\begin{pmatrix}18\\6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-11\\12\end{pmatrix}$. Find $\left\|\frac{1}{\|5\vec{u}+3\vec{v}\|}(5\vec{u}+3\vec{v})\right\|$

9. Let $\vec{u}=\begin{pmatrix}2\\-1\\1\end{pmatrix},\ \vec{v}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}3\\2\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet\vec{v}$ *b)* $\vec{u}\bullet(\vec{v}+\vec{w})$ *c)* $(\vec{u}+2\vec{v})\bullet\vec{w}$ *d)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$

$(\mathbf{10-13})$ Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

10. $\vec{u}=(-6,\ -2),\quad \vec{v}=(5,\ -7)$

11. $\vec{u}=(6,\ 1,\ 4),\quad \vec{v}=(2,\ 0,\ -3)$

12. $\vec{u}=(1,\ -5,\ 4),\quad \vec{v}=(3,\ 3,\ 3)$

13. $\vec{u}=(-2,\ 2,\ 3),\quad \vec{v}=(1,\ 7,\ -4)$

$(\mathbf{14-16})$ Find the angles between the vectors:

14. $\vec{u} = 2\hat{i} + \hat{j}, \quad \vec{v} = \hat{i} + 2\hat{j} - \hat{k}$

15. $\vec{u} = \sqrt{3}\hat{i} - 7\hat{j}, \quad \vec{v} = \sqrt{3}\hat{i} + \hat{j} + \hat{k}$

16. $\vec{u} = \hat{i} + \sqrt{2}\hat{j} - \sqrt{2}\hat{k}, \quad \vec{v} = -\hat{i} + \hat{j} + \hat{k}$

$(\mathbf{17-20})$ Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

17. $\vec{u} = (6,\ 2), \quad \vec{a} = (3,\ -9)$

18. $\vec{u} = (3,\ 1,\ -7), \quad \vec{a} = (1,\ 0,\ 5)$

19. $\vec{u} = (1,\ 0,\ 0), \quad \vec{a} = (4,\ 3,\ 8)$

20. $\vec{u} = (1,\ 1,\ 1), \quad \vec{a} = (0,\ 2,\ -1)$

21. Consider $\vec{u} = -3\hat{j} + 4\hat{k}$, $\vec{v} = -4\hat{i} + \hat{j} + 5\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

22. Consider $\vec{u} = -\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{v} = 3\hat{i} + 6\hat{j} + 6\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

23. The direction angles α, β, and γ of a vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ are defined as follows:

α is the angle between ***v*** and the positive *x*-axis $(0 \le \alpha \le \pi)$

β is the angle between ***v*** and the positive *y*-axis $(0 \le \beta \le \pi)$

γ is the angle between ***v*** and the positive *z*-axis $(0 \le \gamma \le \pi)$

a) Show that $\cos\alpha = \frac{a}{|\vec{v}|}$, $\cos\beta = \frac{b}{|\vec{v}|}$, $\cos\gamma = \frac{c}{|\vec{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\vec{v}$.

b) Show that if $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ is a unit vector, then *a*, *b*, and *c* are the direction cosines of $\vec{v}$.

24. A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.

25. A gun with muzzle velocity of 1200 *ft/sec* is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

26. Suppose that a box is being towed up an inclined plane. Find the force ***w*** needed to make the component of the force parallel to the indicated plane equal to 2.5 *lb*.

27. Find the work done by a force $\vec{F} = 5\hat{i}$ (magnitude 5 *N*) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

28. How much work does it take to slide a crate 20 *m* along a loading dock by pulling on it with a 200 *N* force at an angle of 30° from the horizontal?

29. Use a dot product to find an equation of the line in the *xy*-plane passing through the point (x_0, y_0) perpendicular to the vector $\langle a, b\rangle$.

30. The wind passing over a boat's sail exerted a 100-*lb* magnitude force $\vec{F}$. How much work did the wind perform in moving the boat forward 1 *mile?* Answer in *foot-pounds*.

31. A 180-*lb* man stands on a hillside that makes an angle of 30° with the horizontal, producing a force of $W = \langle 0,\ -180 \rangle$ *lb*s.

a) Find the component of his weight in the downward direction perpendicular to the hillside and in the downward parallel to the hillside.

b) How much work is done when the man moves 10 *ft* up the hillside?

32. Suppose Ted weighs 180 *lbs*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.

b) Find the force acting to hold Ted against the slope.

Section 9.3 – Cross Products

9.3–1 Cross Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the ***cross product***.

- In 1773, ***Joseph Louis Lagrange*** introduced the component form of both the dot and cross products to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir ***William Rowan Hamilton*** introduced the quaternion product, and with it the terms "*vector*" and "*scalar*". Given two quaternions $[0, \vec{u}]$ and $[0, \vec{v}]$, where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, their quaternion product can be summarized as $\left[-\vec{u} \bullet \vec{v},\ \vec{u} \times \vec{v}\right]$.

  James Clerk Maxwell used Hamilton's quaternion tools to develop his famous ***electromagnetism*** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

We start with two nonzero vectors $\vec{u}$ and $\vec{v}$ in space. If $\vec{u}$ and $\vec{v}$ are no parallel, they determine a plane. We select a unit vector $\vec{n}$ perpendicular to the plane by the ***right-hand rule***. Then the cross product $\vec{u} \times \vec{v}$ ("$\vec{u}$ ***cross*** $\vec{v}$") is the vector defined as follows

9.3–2 *Definition*

$$\vec{u} \times \vec{v} = \left(|\vec{u}|\ |\vec{v}|\ \sin\theta\right) \cdot \vec{n}$$

9.3–3 Parallel Vectors

Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u}\times\vec{v}=\vec{0}$

9.3–4 Properties of the Cross Product

If $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ are any vectors and *r*, *s* are scalars, then:

Note:

- ✓ $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- ✓ $\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$
- ✓ $\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$

9.3–5 Determinant Formula for $\vec{u}\times\vec{v}$

Definition

The cross product of $\vec{u}=\left(u_1,\ u_2,\ u_3\right)$ and $\vec{v}=\left(v_1,\ v_2,\ v_3\right)$ is the vector:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\begin{vmatrix}u_2 & v_2\\ u_3 & v_3\end{vmatrix}\hat{i}-\begin{vmatrix}u_1 & v_1\\ u_3 & v_3\end{vmatrix}\hat{j}+\begin{vmatrix}u_1 & v_2\\ u_2 & v_1\end{vmatrix}\hat{k}$$

$$=\left(u_2v_3-u_3v_2\right)\hat{i}-\left(u_1v_3-u_3v_1\right)\hat{j}+\left(u_1v_2-u_2v_1\right)\hat{k}$$

$$=\left(u_2v_3-u_3v_2,\quad u_3v_1-u_1v_3,\quad u_1v_2-u_2v_1\right)$$

Example 1

Find $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ if $\vec{u}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{v}=-4\hat{i}+3\hat{j}+\hat{k}$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ -4 & 3 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix}\hat{i}-\begin{vmatrix}2 & 1\\ -4 & 1\end{vmatrix}\hat{j}+\begin{vmatrix}2 & 1\\ -4 & 3\end{vmatrix}\hat{k}$$

$$=-2\hat{i}-6\hat{j}+10\hat{k}$$

$$\vec{v}\times\vec{u}=-\vec{u}\times\vec{v}$$

$$=2\hat{i}+6\hat{j}-10\hat{k}$$

9.3–6 *Theorem:* Area of a Parallelogram

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

$$|\vec{u}\times\vec{v}|=|\vec{u}|\ |\vec{v}|\ |\sin\theta|$$

Example 2

Find the area of the triangle determined by the points $P_1(2,\ 2,\ 0)$, $P_2(-1,\ 0,\ 2)$, *and* $P_3(0,\ 4,\ 3)$

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\overrightarrow{P_1P_2}=(-1,\ 0,\ 2)-(2,\ 2,\ 0)$$

$$=(-3,\ -2,\ 2)$$

$$\overrightarrow{P_1P_3}=(0,\ 4,\ 3)-(2,\ 2,\ 0)$$

$$=(-2,\ 2,\ 3)$$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\left(\begin{vmatrix}-2 & 2\\ 2 & 3\end{vmatrix},\ -\begin{vmatrix}-3 & 2\\ -2 & 3\end{vmatrix},\ \begin{vmatrix}-3 & -2\\ -2 & 2\end{vmatrix}\right)$$

$$=(-10,\ 5,\ -10)$$

$$Area=\frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|$$

$$=\frac{1}{2}\sqrt{(-10)^2+5^2+(-10)^2}$$

$$=\frac{15}{2}\quad unit^2$$

Example 3

a) Find a vector perpendicular to the plane of $P(1,\ -1,\ 0)$, $Q(2,\ 1,\ -1)$, *and* $R(-1,\ 1,\ 2)$

b) Find the area of the triangle with vertices *P*, *Q*, and *R*.

c) Find a unit vector perpendicular to the *P*, *Q*, and *R*

Solution

a) The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane.

$$\overrightarrow{PQ}=(2-1)\hat{i}+(1+1)\hat{j}+(-1-0)\hat{k}$$

$$=\hat{i}+2\hat{j}-\hat{k}$$

$$\overrightarrow{PR}=-2\hat{i}+2\hat{j}+2\hat{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ -2 & 2 & 2\end{vmatrix}$$

$$=6\hat{i}+6\hat{k}$$

b) The area of the triangle is equal half the parallelogram determined by *P*, *Q*, and *R*.

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\sqrt{36+36}$$

$$=6\sqrt{2}$$

Area of the triangle:

$$A=\frac{1}{2}\left(6\sqrt{2}\right)$$

$$=3\sqrt{2}\ \ unit^2$$

c) Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\vec{n}$ is a unit vector $\perp$ to the plane

$$\vec{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$= \frac{6\hat{i} + 6\hat{k}}{6\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

9.3–7 Torque

When we turn a bolt by applying a force $\vec{F}$ to a wrench, we produce a torque that causes the bolt to rotate. The ***torque vector*** points in the direction of the axis of the bolt according to the right-hand rule (*ccw*).

$$\textbf{\textit{Magnitude of torque vector}} = \left|\vec{r}\right| \left|\vec{F}\right| \sin\theta$$

$$\textbf{\textit{Torque vector}} = \left(\left|\vec{r}\right| \cdot \left|\vec{F}\right| \cdot \sin\theta\right) \vec{n}$$

Example 4

Find the magnitude of the torque on generated by force $\vec{F}$ at the pivot point *P* on 20 lb. bar with 3 *feet* length.

Solution

$$\left|\overrightarrow{PQ} \times \vec{F}\right| = \left|\overrightarrow{PQ}\right| \left|\vec{F}\right| \sin 70°$$

$$\approx (3)(20)(0.94)$$

$$\approx 56.4 \ \ ft - lb$$

9.3–8 Triple Scalar or Box Product

The product $(\vec{u}\times\vec{v})\bullet\vec{w}$ is called the triple scalar product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ (in that order).

$$(\vec{u}\times\vec{v})\bullet\vec{w}=|\vec{u}\times\vec{v}|\ |\vec{w}|\ |\cos\theta|$$

Volume

The Volume of the Parallelepiped is:

$$\begin{aligned}V&=(area\ of\ base).(height)\\&=|\vec{u}\times\vec{v}|\ |\vec{w}|\ |\cos\theta|\\&=|\vec{u}\times\vec{v}|\frac{|(\vec{u}\times\vec{v})\bullet\vec{w}|}{|\vec{u}\times\vec{v}|}\\&=|(\vec{u}\times\vec{v})\bullet\vec{w}|\end{aligned}$$

$$\begin{aligned}Volume&=(\vec{u}\times\vec{v})\bullet\vec{w}\\&=\left|\det\begin{bmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{bmatrix}\right|\end{aligned}$$

Example 5

Find the volume of the box (parallelepiped) determined by $\vec{u}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{v}=-2\hat{i}+3\hat{k}$, and $\vec{w}=7\hat{j}-4\hat{k}$.

Solution

$$V=(\vec{u}\times\vec{v})\bullet\vec{w}$$

$$= \left| \det \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{bmatrix} \right|$$

$$= |-23|$$

$$= 23$$

The volume is 23 *units cubed*.

Example 6

Find the volume of the parallelepiped with sides $\vec{u} = (2,\ -6,\ 2)$, $\vec{v} = (0,\ 4,\ -2)$, and $\vec{w} = (2,\ 2,\ -4)$

Solution

$$V = \left| \det \begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} \right|$$

$$= 16 \quad unit^3$$

Exercises *Section* 9.3 – Cross Products

$(\mathbf{1-4})$ Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$

1. $\vec{u}=2\hat{i}-2\hat{j}-\hat{k},\quad \vec{v}=\hat{i}-\hat{k}$

2. $\vec{u}=\hat{i}+\hat{j}-\hat{k},\quad \vec{v}=0$

3. $\vec{u}=\hat{i}\times\hat{j},\quad \vec{v}=\hat{j}\times\hat{k}$

4. $\vec{u}=-8\hat{i}-2\hat{j}-4\hat{k},\quad \vec{v}=2\hat{i}+2\hat{j}+\hat{k}$

5. Given $\vec{u}=(3,\ 2,\ -1),\quad \vec{v}=(0,\ 2,\ -3),$ *and* $\vec{w}=(2,\ 6,\ 7)$ Compute the vectors:

a) $\vec{u}\times\vec{v}$

b) $\vec{v}\times\vec{w}$

c) $\vec{u}\times(\vec{v}\times\vec{w})$

d) $(\vec{u}\times\vec{v})\times\vec{w}$

e) $\vec{u}\times(\vec{v}-2\vec{w})$

6. Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u}=\hat{i}-\hat{k},\quad \vec{v}=\hat{j}$

7. Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u}=\hat{i}-\hat{k},\quad \vec{v}=\hat{j}+\hat{k}$

8. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(1,\ -1,\ 2),\quad Q(2,\ 0,\ -1),\quad$ *and* $R(0,\ 2,\ 1)$

9. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(1,\ 1,\ 1),\quad Q(2,\ 1,\ 3),\quad$ *and* $R(3,\ -1,\ 1)$

10. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(-2,\ 2,\ 0),\quad Q(0,\ 1,\ -1),\quad$ *and* $R(-1,\ 2,\ -2)$

11. Verify that $(\vec{u}\times\vec{v})\bullet\vec{w}=(\vec{v}\times\vec{w})\bullet\vec{u}=(\vec{w}\times\vec{u})\bullet\vec{v}$ and find the volume of the parallelepiped determined by $\vec{u}=2\hat{i},\quad \vec{v}=2\hat{j},\quad$ *and* $\vec{w}=2\hat{k}$

$(\mathbf{12-13})$ Find $|\vec{v}|,\ |\vec{u}|,\ \vec{v}\bullet\vec{u},\ \vec{u}\bullet\vec{v},\ \vec{v}\times\vec{u},\ \vec{u}\times\vec{v},\ |\vec{v}\times\vec{u}|$, the angle between $\vec{v}$ and $\vec{u}$, the scalar component of $\vec{u}$ in the direction of $\vec{v}$, and the vector $proj_{\vec{v}}\vec{u}$

12. $\vec{v}=\hat{i}+\hat{j}+2\hat{k},\quad \vec{u}=-\hat{i}-\hat{k}$

13. $\vec{v}=2\hat{i}+\hat{j}-\hat{k},\quad \vec{u}=\hat{i}+\hat{j}-5\hat{k}$

$(\mathbf{14-15})$ Find the area of the parallelogram determined by vectors $\vec{u}$ and $\vec{v}$, then the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

14. $\vec{u}=\hat{i}+\hat{j}-\hat{k},\quad \vec{v}=2\hat{i}+\hat{j}+\hat{k},\quad \vec{w}=-\hat{i}-2\hat{j}+3\hat{k}$

15. $\vec{u} = \hat{i} + \hat{j}, \quad \vec{v} = \hat{j}, \quad \vec{w} = \hat{i} + \hat{j} + \hat{k}$

$(\mathbf{16-17})$ Find the volume of the parallelepiped determined by:

16. $\vec{u} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \quad and \quad \vec{w} = -\hat{i} + 2\hat{j} - \hat{k}$

17. $\vec{u} = \hat{i} + \hat{j} - 2\hat{k}, \quad \vec{v} = -\hat{i} - \hat{k}, \quad and \quad \vec{w} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

18. Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\vec{F}\right| = 30\ lb.$

19. Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\vec{F}\right| = 30\ lb.$

20. Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 12\ in.$ and $\left|\vec{F}\right| = 40\ lb.$

$(\mathbf{21-30})$ Find the area of the parallelogram whose vertices are:

21. $A(1,\ 0), \quad B(0,\ 1), \quad C(-1,\ 0), \quad D(0,\ -1)$

22. $A(0,\ 0), \quad B(7,\ 3), \quad C(9,\ 8), \quad D(2,\ 5)$

23. $A(-1,\ 2), \quad B(2,\ 0), \quad C(7,\ 1), \quad D(4,\ 3)$

24. $A(0,\ 0,\ 0), \quad B(3,\ 2,\ 4), \quad C(5,\ 1,\ 4), \quad D(2,\ -1,\ 0)$

25. $A(1,\ 0,\ -1), \quad B(1,\ 7,\ 2), \quad C(2,\ 4,\ -1), \quad D(0,\ 3,\ 2)$

26. $(1,\ 2,\ 3)$, $(1,\ 0,\ 6)$, and $(4,\ 2,\ 4)$

27. $(1,\ 0,\ 3)$, $(5,\ 0,\ -1)$, and $(0,\ 2,\ -2)$

28. $\vec{u} = (1,\ -1,\ 2)$ *and* $\vec{v} = (0,\ 3,\ 1)$

29. $\vec{u} = (3,\ -1,\ 4)$ *and* $\vec{v} = (6,\ -2,\ 8)$

30. $\vec{u} = (2,\ 3,\ 0)$ *and* $\vec{v} = (-1,\ 2,\ -2)$

(31 – 34) Find the area of the triangle whose vertices are:

31. $A(0,\ 0)$, $B(-2,\ 3)$, $C(3,\ 1)$

32. $A(-1,\ -1)$, $B(3,\ 3)$, $C(2,\ 1)$

33. $A(1,\ 0,\ 0)$, $B(0,\ 0,\ 2)$, $C(0,\ 0,\ -1)$

34. $A(0,\ 0,\ 0)$, $B(-1,\ 1,\ -1)$, $C(3,\ 0,\ 3)$

35. Find the volume of the parallelepiped if four of its eight vertices are:

$$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$$

36. Let $\vec{u} = \langle 2,\ 4,\ -5 \rangle$ and $\vec{v} = \langle -6,\ 10,\ 2 \rangle$

a) Compute $\vec{u} - 3\vec{v}$

b) Compute $|\vec{u} + \vec{v}|$

c) Find the unit vector with the same direction as $\vec{u}$

d) Find a vector parallel to $\vec{v}$ with length 20.

e) Compute $\vec{u} \bullet \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

f) Compute $\vec{u} \times \vec{v}$, $\vec{v} \times \vec{u}$

g) Find the area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

37. Find a unit vector normal to the vectors $\langle 2,\ -6,\ 9 \rangle$ and $\langle -1,\ 0,\ 6 \rangle$

38. Find the angle between $\langle 2,\ 0,\ -2 \rangle$ and $\langle 2,\ 2,\ 0 \rangle$ using the dot product then the cross product.

39. You do leg lifts with 10-*kg* weight attached to your foot, so the resulting force is $mg \approx 98N$ directed vertically downward. If the distance from your knee to the weight is $0.4m$ and her lower leg makes an angle of θ to the vertical, find the magnitude of the torque about your knee as your leg is lifted (as a function of θ).

a) What is the minimum and maximum magnitude of the torque?
b) Does the direction of the torque change as your leg is lifted?

40. An automobile wheel has center at the origin and axle along the *y*-axis. One of the retaining nuts holding the wheel is at position $P_0(0,\ 0,\ 10)$. (Distances are measured in *cm*.)

A bent tire wrench with arm 25 *cm* long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction. If the horizontal force $\vec{F} = 500\hat{i}\ \ (N)$ is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?

41. An architect decides to cut a corner of the top floor of his design as shown below.

The sloping part of the roof is to be covered with sheet glass.
For heat transfer calculations, determine the area of the triangular cut *OAB*.
Where $A(0,\ 100,\ -50)$ & $B(-100,\ 100,\ 0)$ (*units in foot*)

Section 9.4 – Length of Lines and Curves in Space

9.4–1 Lines and Line Segments in Space

In space, a line has a distinct and unique form. An initial point on a line, $P_0\left(x_0, y_0, z_0\right)$, move toward a point $P(x, y, z)$ on a line ***L***. These two distinct points on the line ***L*** create a vector $\overrightarrow{P_0P}$. The vector $\vec{v} = \left(v_1, v_2, v_3\right)$ is the direction of the vector $\overrightarrow{P_0P}$ and has the magnitude. The point *P* will take ***t*** time to move away from P_0.

The expanded form of the equation $\overrightarrow{P_0P} = t\vec{v}$ is:

$$\left(x - x_0\right)\hat{i} + \left(y - y_0\right)\hat{j} + \left(z - z_0\right)\hat{k} = t\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

9.4–2 Vector Equation for a Line

A **vector equation for the line *L*** through $P_0\left(x_0, y_0, z_0\right)$ parallel to $\vec{v}$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

Where $\vec{r}(t)$ is the position vector of a point $P(x, y, z)$ on ***L*** and $\vec{r}_0$ is the position vector of $P_0\left(x_0, y_0, z_0\right)$.

9.4–3 Parametric Equations for a Line

A **standard parametrization** of the line through $P_0\left(x_0, y_0, z_0\right)$ parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is

$$\begin{cases} x = x_0 + t\,v_1 \\ y = y_0 + t\,v_2 \\ z = z_0 + t\,v_3 \end{cases} \qquad -\infty < t < \infty$$

Example 1

Find the parametric equations for the line through (–2, 0, 4) parallel to $\vec{v} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Solution

$$\begin{cases} x = -2 + 2t \\ y = 4t \\ z = 4 - 2t \end{cases} \qquad -\infty < t < \infty$$

Example 2

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution

$$\vec{v} = \overrightarrow{PQ}$$

$$= 4\,\hat{i} - 3\,\hat{j} + 7\,\hat{k}$$

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$$

The point $\left(x, y, z\right) = \left(-3 + 4t,\ 2 - 3t,\ -3 + 7t\right)$

On the line passes through P at $t = 0$ and Q at $t = 1$.

That implies the restriction $0 \le t \le 1$ to parameterize the segment:

$$\begin{cases} x = -3 + 4t \\ y = 2 - 3t \\ z = -3 + 7t \end{cases} \qquad 0 \le t \le 1$$

The position of a particle at time t is written:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$= \vec{r}_0 + t\,|\vec{v}|\,\frac{\vec{v}}{|\vec{v}|}$$

Initial position — Time — Speed — Direction

Example 3

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 *ft/sec*. What is the position of the helicopter after 10 *sec*.?

Solution

The unit vector: $= \dfrac{(1,\ 1,\ 1)}{\sqrt{1^2 + 1^2 + 1^2}}$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Therefore, the position of the helicopter at any time t is:

$$\vec{r}(t) = r_0 + t\,\vec{u}$$

$$= 0 + t(60)\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= 20\sqrt{3}\,t\left(\hat{i} + \hat{j} + \hat{k}\right)$$

The position after 10 *sec*:

$$\vec{r}(10) = 20\sqrt{3}\ (10)\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$= 200\sqrt{3}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

The distance is traveled:

$$\left|\vec{r}(10)\right| = 200\sqrt{3}\sqrt{1^2 + 1^2 + 1^2}$$

$$= 600\ \ ft$$

9.4–4 Distance from a Point *S* to a Line through *P* parallel to $\vec{v}$

$$d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{\left|\vec{v}\right|}$$

Example 4

Find the distance from the point S (1, 1, 5) to the line $L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t$

Solution

At $t = 0$, the equations for L passes through $P(1, 3, 0)$ parallel to $\vec{v} = \hat{i} - \hat{j} + 2\hat{k}$

$$\overrightarrow{PS} = (1-1)\hat{i} + (1-3)\hat{j} + (5-0)\hat{k}$$
$$= -2\hat{j} + 5\hat{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$
$$= \hat{i} + 5\hat{j} + 2\hat{k}$$

$$d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{\left|\vec{v}\right|}$$
$$= \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}}$$
$$= \frac{\sqrt{30}}{\sqrt{6}}$$
$$= \sqrt{5} \ \ unit$$

9.4–5 An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.

An initial point $P_0(x_0, y_0, z_0)$ move toward $P(x, y, z)$ on plane *M*, creating the vector $\overrightarrow{P_0P}$.

Let assume that $\vec{n}$ is a normal vector to the plane *M*. Then,

The dot product $\vec{n} \bullet \overrightarrow{P_0P} = 0$, since $\overrightarrow{P_0P}$ is orthogonal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$.

$$\vec{n} \bullet \overrightarrow{P_0P} = 0$$

$$\left(A\hat{i} + B\hat{j} + C\hat{k}\right) \cdot \left(\left(x - x_0\right)\hat{i} + \left(y - y_0\right)\hat{j} + \left(z - z_0\right)\hat{k}\right) = 0$$

$$A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$$

9.4–6 Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$ has:

Vector equation: $\vec{n} \bullet \overrightarrow{P_0P} = 0$

Component equation: $A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$

Component equation simplified: $Ax + By + Cz = D$ *where* $D = Ax_0 + By_0 + Cz_0$

Example 5

Find an equation for the plane through $P_0(-3,\ 0,\ 7)$ perpendicular to $\vec{n} = 5\hat{i} + 2\hat{j} - \hat{k}$

Solution

The component equation is

$$5\left(x - (-3)\right) + 2\left(y - 0\right) + (-1)\left(z - 7\right) = 0$$

$5(x+3)+2y-z+7=0$

$5x+15+2y-z+7=0$

$\underline{5x+2y-z=-22}$

$\vec{n}=5\hat{i}+2\hat{j}-\hat{k}$ substitute $\hat{i}$ *with x* $\hat{j}$ *with y* $\hat{k}$ *with z*

$5x+2y-z=5(-3)+2(0)-7$ *multiply* $5(-3)+2(0)-7$

$\underline{5x+2y-z=-22}$

Example 6

Find an equation for the plane through $A(0,\ 0,\ 1)$, $B(2,\ 0,\ 0)$, $C(0,\ 3,\ 0)$.

Solution

$\overrightarrow{AB}=2\hat{i}-\hat{k}$

$\overrightarrow{AC}=3\hat{j}-\hat{k}$

The cross product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -1\\ 0 & 3 & -1\end{vmatrix}$$

$$=3\hat{i}+2\hat{j}+6\hat{k}$$

Normal to the plane.

We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$3(x-0)+2(y-0)+6(z-1)=0$

$\underline{3x+2y+6z-6=0}$

or $\underline{3x+2y+6z=6}$

9.4–7 Lines of Intersection

Two distinct planes intersect in a line, otherwise if parallel, they have parallel normal vectors.

i. If the two planes are parallel, then the two normal vectors: $\vec{n}_1 = k\,\vec{n}_2 \quad (k \neq 0)$

ii. If the two planes intersect, then the cross-product vector of the two normal vectors $\vec{n}_1 \times \vec{n}_2$ lies on the intersection line (or parallel).

Example 7

Find equation of the line of intersection of the planes $3x - 6y - 2z = 15 \quad and \quad 2x + y - 2z = 5$

Solution

The line of intersection of two planes is perpendicular to both planes' normal vectors $\vec{n}_1$ and $\vec{n}_2$, and therefore parallel to $\vec{n}_1 \times \vec{n}_2$.

$$\vec{n}_1 = 3\,\hat{i} - 6\,\hat{j} - 2\hat{k}$$

$$\vec{n}_2 = 2\,\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 14\hat{i} + 2\,\hat{j} + 15\hat{k}$$

Let assume that $z = 0$, then:

$$\begin{cases} 3x - 6y = 15 \\ 2x + y = 5 \end{cases}$$

$$\begin{cases} x - 2y = 5 \\ 2x + y = 5 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} 5 & -2 \\ 5 & 1 \end{vmatrix} = 15 \quad \Delta_y = \begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} = -5$$

The point of intersection: $\underline{(x,\ y,\ z) = (3,\ -1,\ 0)}$

∴ The line of intersection is:

$$14x + 2y + 15z = 42 - 2$$

$$\underline{14x + 2y + 15z = 40}$$

Example 8

Find equation of the line of intersection of the planes $2x + y - z = 3$ *and* $2x + y - z = 4$

Solution

$$\vec{n}_1 = 2\,\hat{i} + \hat{j} - \hat{k}$$

$$\vec{n}_2 = 2\,\hat{i} + \hat{j} - \hat{k}$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\underline{= \vec{0}}$$

The two planes are parallel, therefore there is no line of intersection.

Other way, if you look at the given planes, $2x + y - z = 3 = 4$ are the same but equal to different constants.

Example 9

Find the point where the line $x = \frac{8}{3} + 2t,\quad y = -2t,\quad z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

Solution

The point: $\left(\frac{8}{3} + 2t,\ -2t,\ 1 + t\right)$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$\underline{t = -1}$$

The point of intersection is: $\left(\frac{8}{3} + 2t,\ -2t,\ 1 + t\right)\Big|_{t=-1} = \underline{\left(\frac{2}{3},\ 2,\ 0\right)}$

9.4–8 The distance from a Point to a Plane

To find the distance from a point S to a plane (short distance) is by projections to the plane.
Let P be any point on the plane with a normal vector $\vec{n}$.

Then the distance from S to the plane is the length (magnitude) of the projection of $\overrightarrow{PS}$ onto $\vec{n}$.

Let the plane is given by the equation:

$$Ax + By + Cz = D$$

The normal vector to the plane is:

$$\vec{n} = A\,\hat{i} + B\,\hat{j} + C\,\hat{k}$$

Let $S\left(x_s\ , y_s\ , z_s\right)$, then the distance from S to the plane is

$$distance = proj_{\vec{n}}\ \overrightarrow{PS}$$

$$= \left|\overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|}\right|$$

Example 10

Find the distance from $S(1,\ 1,\ 3)$ to the plane $3x + 2y + 6z = 6$

Solution

The coefficients in the equation
$3x + 2y + 6z = 6$ give $\vec{n} = 3\hat{i} + 2\,\hat{j} + 6\hat{k}$

To find a Point P on the plane from the plane equation by letting two components equal to zero and find the other component.

Assume that $x = y = 0 \quad \Rightarrow \quad z = 1$

$$P(0,\ 0,\ 1)$$

or $y = z = 0 \quad \Rightarrow \quad x = 2$

$$P(2,\ 0,\ 0)$$

Or $x = z = 0 \quad \Rightarrow \quad y = 3$

let consider: $P\,(0,\ 3,\ 0)$

$\overrightarrow{PS} = \hat{i} - 2\hat{j} + 3\hat{k}$

$|\vec{n}| = \sqrt{3^2 + 2^2 + 6^2}$

$= 7$

The distance from S to the plane is

$$d = \left| \overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|} \right|$$

$$= \left| \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \right|$$

$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right|$$

$$= \frac{17}{7} \quad unit$$

9.4–9 Angles Between Planes

To determine the angle between two planes, first find the normal vectors of each plane. Then, the angle between the two normal vectors is the same as the angle between two planes.

$$\theta = \cos^{-1} \frac{\vec{n}_1 \bullet \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$

Example 11

Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution

The vectors: $\vec{n}_1 = 3\hat{i}-6\hat{j}-2\hat{k}$, $\vec{n}_2 = 2\hat{i}+\hat{j}-2\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{6-6+4}{\sqrt{9+36+4}\sqrt{4+1+4}}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \qquad \approx 1.38 \ \ rad$$

9.4–10 Curves

The coordinates for a particle moving through space during a time interval *I*, are defined as function on *I*:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t)), \quad t \in I$, make up the curve in space that we call the particle's path.

$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

$\vec{r}(t) = (\sin 3t)(\cos t)\hat{i} + (\sin 3t)(\sin t)\hat{j} + t\hat{k}$

$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + (\sin 2t)\hat{k}$

$\vec{r}(t) = (4+\sin 20t)(\cos t)\hat{i} + (4+\sin 20t)(\sin t)\hat{j} + (\cos 20t)\hat{k}$

Example 12

Graph the vector function $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$

Solution

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

The curves traced by $\vec{r}(t)$ winds around a circular cylinder, satisfies the equation.

The curve rises as the $\hat{k}$ –components $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a helix (from an old Greek word for "spiral"). The equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$

9.4–11 Limits and Continuity

Definition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector function with domain D, and $\boldsymbol{L}$ a vector. We say that $\boldsymbol{r}$ has limit $\boldsymbol{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \vec{r}(t) = \boldsymbol{L}$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$\left|\vec{r}(t) - \boldsymbol{L}\right| < \varepsilon \quad \textit{whenever} \quad 0 < \left|t - t_0\right| < \delta$$

Example 13

Find the limit of $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ as t approaches $\frac{\pi}{4}$

Solution

$$\lim_{t\to\pi/4} \vec{r}(t) = \left(\lim_{t\to\pi/4} \cos t\right)\hat{i} + \left(\lim_{t\to\pi/4} \sin t\right)\hat{j} + \left(\lim_{t\to\pi/4} t\right)\hat{k}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} + \frac{\pi}{4}\hat{k}$$

9.4–12 *Definition*

A vector fumction $\vec{r}(t)$ is ***continuous at a point*** $t = t_0$ in its domain if $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$. The function is continuous if it is continuous at every point in its domain.

9.4–13 Derivative

Definition

The vector function $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ has a derivative (is differentiable) at *t* if *f*, *g*, and *h* have derivatives at *t*. The derivative is the vector function:

$$\vec{r}'(t) = \frac{d\vec{r}}{dt}$$

$$= \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

9.4–14 *Definitions*

If ***r*** is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

is the particle's ***velocity vector***, tangent to the curve. At any time *t*, the direction of $\vec{v}$ is the ***direction of motion***, the magnitude of $\vec{v}$ is the particle's ***speed***, and the derivative $\vec{a} = \frac{d\vec{v}}{dt}$, when it exists, is the particle's ***acceleration vector***. In summary,

1. Velocity is the derivative of position: $\vec{v}(t) = \frac{d\vec{r}}{dt}$

2. Speed is the magnitude of velocity: $Speed = |\vec{v}|$

3. Acceleration is the derivative of velocity: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

4. The unit vector $\frac{\vec{v}}{|\vec{v}|}$ is the direction of motion at time t.

Example 14

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\vec{r}(t) = 2\cos t\ \hat{i} + 2\sin t\ \hat{j} + 5\cos^2 t\ \hat{k}$. Sketch the velocity vector $\vec{v}\left(\frac{7\pi}{4}\right)$

Solution

The velocity vector at time t is:

$$\vec{v}(t) = -2\sin t\ \hat{i} + 2\cos t\ \hat{j} - 10\cos t\sin t\ \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= -2\sin t\ \hat{i} + 2\cos t\ \hat{j} - 5\sin 2t\ \hat{k}$$

The acceleration vector at time t is:

$$\vec{a}(t) = -2\cos t\ \hat{i} - 2\sin t\ \hat{j} - 10\cos 2t\ \hat{k} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

The speed is:

$$|\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 25\sin^2 2t}$$

$$= \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 25\sin^2 2t} \qquad \sin^2 t + \cos^2 t = 1$$

$$= \sqrt{4 + 25\sin^2 2t}$$

$$\vec{v}\left(\frac{7\pi}{4}\right) = -2\sin\left(\frac{7\pi}{4}\right)\hat{i} + 2\cos\left(\frac{7\pi}{4}\right)\hat{j} - 5\sin\left(\frac{7\pi}{2}\right)\hat{k}$$

$$= \sqrt{2}\hat{i} + \sqrt{2}\hat{j} + 5\hat{k}$$

$$\vec{a}\left(\frac{7\pi}{4}\right) = -2\cos\left(\frac{7\pi}{4}\right)\hat{i} - 2\sin\left(\frac{7\pi}{4}\right)\hat{j} - 10\cos\left(\frac{7\pi}{2}\right)\hat{k}$$

$$= -\sqrt{2}\hat{i} + \sqrt{2}\hat{j}$$

$$\left|\vec{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{4 + 25\sin^2\left(\frac{7\pi}{2}\right)}$$

$$= \sqrt{29}$$

9.4–15 *Differentiation Rules* for vector Functions

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of *t*, *C* a constant vector, *c* any scalar and *f* any differentiable scalar function.

1. *Constant Function Rule*: $\frac{d}{dt}\vec{C} = \vec{0}$

2. *Scalar Multiple Rules*: $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}'(t)$

   $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

3. *Sum Rule*: $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}'(t) + \vec{v}'(t)$

4. *Difference Rule*: $\frac{d}{dt}\left[\vec{u}(t) - \vec{v}(t)\right] = \vec{u}'(t) - \vec{v}'(t)$

5. *Dot Product Rule*: $\frac{d}{dt}\left[\vec{u}(t) \bullet \vec{v}(t)\right] = \vec{u}'(t) \bullet \vec{v}'(t) + \vec{u}(t) \bullet \vec{v}'(t)$

6. *Cross Product Rule*: $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

7. *Chain Rule*: $\frac{d}{dt}\left[\vec{u}\left(f(t)\right)\right] = f'(t)\vec{u}'\left(f(t)\right)$

9.4–16 Vector Functions of Constant Length

The position vector, of a particle that is moving on a sphere, has a constant length equal to the radius of the sphere. The velocity vector $\frac{d\vec{r}}{dt}$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\vec{r}(t)$. the vector and its first derivative are orthogonal.

$\vec{r}(t) \bullet \vec{r}(t) = c^2$ $\qquad$ $|\vec{r}(t)| = c$ *is constant*

$\frac{d}{dt}\left[\vec{r}(t) \bullet \vec{r}(t)\right] = 0$ $\qquad$ ***Differentiate both sides.***

$\vec{r}'(t) \bullet \vec{r}(t) + \vec{r}(t) \bullet \vec{r}'(t) = 0$

$2\,\vec{r}'(t) \bullet \vec{r}(t) = 0$

If $\vec{r}(t)$ is a differentiable vector function of t of constant length, then:

$$\vec{r}(t) \bullet \frac{d\vec{r}}{dt} = 0$$

9.4–17 Arc Length along a Space Curve

We assume that f has a continuous derivative at every point of $[a, b]$. Such function is called ***smooth***, and its graph is a ***smooth curve*** because it doesn't have any breaks, corners, or cusps.

9.4–18 *Definition*

If a curve C is defined parametrically by $x = f(t),\ y = g(t),\ and\ \ z = h(t)$ and $y = g(t)$, $a \le t \le b$, where f', g', and h' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Arc Length Formula $\qquad L = \int_a^b |\vec{v}|\ dt$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{\left[x'(\tau)\right]^2 + \left[y'(\tau)\right]^2 + \left[z'(\tau)\right]^2}\, d\tau$$

$$= \int_{t_0}^{t} \left|\vec{v}(\tau)\right|\, d\tau$$

Example 15

A glider is soaring upward along the helix $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

Solution

The path segment during this time corresponds to one full turn of the helix. The length of this portion of the curve is:

$$L = \int_{0}^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1^2}\, dt$$

$$= \int_{0}^{2\pi} \sqrt{2}\, dt$$

$$= \sqrt{2}\, t \Big|_{0}^{2\pi}$$

$$= 2\pi\sqrt{2} \quad unit$$

$\therefore$ This is $\sqrt{2}$ times the circumference of the circle in the *xy*-plane over which the helix stands.

Exercises *Section* 9.4 – Length of Lines and Curves in Space

$(\mathbf{1-8})$ Find the parametric equation for the line through:

1. The point $P(3,\ -4,\ -1)$ parallel to the vector $\hat{i}+\hat{j}+\hat{k}$

2. The points $P(1,\ 2,\ -1)$ and $Q(-1,\ 0,\ 1)$

3. The points $P(-2,\ 0,\ 3)$ and $Q(3,\ 5,\ -2)$

4. The origin parallel to the vector $2\hat{j}+\hat{k}$

5. The point $P(3,\ -2,\ 1)$ parallel to the line $x=1+2t,\quad y=2-t,\quad z=3t$

6. $(2,\ 4,\ 5)$ perpendicular to the plane $3x+7y-5z=21$

7. $(2,\ 3,\ 0)$ perpendicular to the vectors $\vec{u}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{v}=3\hat{i}+4\hat{j}+5\hat{k}$

8. Planes $x+2y+z=1$ *and* $x-y+2z=-8$ intersect.

9. Find the parameterization for the line segment joining the points $(0,\ 0,\ 0)$, $\left(1,\ 1,\ \frac{3}{2}\right)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.

10. Find the parameterization for the line segment joining the points $(1,\ 0,\ -1)$, $(0,\ 3,\ 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.

$(\mathbf{11-19})$ Find equation for the plane through:

11. $P_0(0,\ 2,\ -1)$ normal to $\vec{n}=3\hat{i}-2\hat{j}-\hat{k}$

12. $(1,\ -1,\ 3)$ parallel to the plane $3x+y+z=7$

13. $(1,\ 2,\ 3)$ parallel to the 2 vectors $\vec{u}=\langle 2,\ 3,\ 1\rangle$ & $\vec{v}=\langle 1,\ -1,\ 2\rangle$

14. $(1,\ 1,\ -1)$, $(2,\ 0,\ 2)$, and $(0,\ -2,\ 1)$

15. $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, & $(0,\ 0,\ 1)$

16. $P_0(2,\ 4,\ 5)$ perpendicular to the line $x=5+t,\quad y=1+3t,\quad z=4t$

17. $A(1,\ -2,\ 1)$ perpendicular to the vector from the origin to A.

18. Which perpendicular to the line from the origin to $(4,\ 5,\ 3)$ and passing through the point $(1,\ 3,\ 7)$

19. Which perpendicular to the line from the origin to $(2,\ -4,\ 3)$ and passing through the point $(3,\ -4,\ -5)$

20. Show that the planes $2x+y-z=-7$ *and* $4x+2y-2z=-9$ are parallel.

21. Show that the planes $2x+y+z=-7$ *and* $3x+4y-10z=4$ are perpendicular.

22. Find the point of intersection of the lines $x=2t+1,\quad y=3t+2,\quad z=4t+3$ and $x=s+2,\quad y=2s+4,\quad z=-4s-1$, and find the plane determined by these lines.

23. Find the plane determined by the intersecting lines:

$L_1:\ x=-1+t,\quad y=2+t,\quad z=1-t;\quad -\infty<t<\infty$

$L_2:\ x=1-4s,\quad y=1+2s,\quad z=2-2s;\quad -\infty<s<\infty$

24. Find a plane through $P_0(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x+y-z=3,\quad x+2y+z=2$

(25 – 34) Find the distance from the point to the plane.

25. $(0,\ 0,\ 12)$, $x=4t,\quad y=-2t,\quad z=2t$

26. $(2,\ 1,\ -1)$, $x=2t,\quad y=1+2t,\quad z=2t$

27. $(3,\ -1,\ 4)$, $x=4-t,\quad y=3+2t,\quad z=-5+3t$

28. $(2,\ -3,\ 4)$, $x+2y+2z=13$

29. $(0,\ 0,\ 0)$, $3x+2y+6z=6$

30. $(0,\ 1,\ 1)$, $4y+3z=-12$

31. $(6,\ 0,\ -6)$, $x-y=4$

32. $(3,\ 0,\ 10)$, $2x+3y+z=2$

33. $(2,\ 2,\ 3)$, $2x+3y+5z=0$

34. $(1,\ 4,\ 6)$; $O(0,\ 0,\ 0)$, $A(2,\ 0,\ -1)$, $B(2,\ -1,\ 0)$

(35 – 36) Find the distance from the point to the line.

35. $(2,\ 2,\ 0)$; $x=-t,\quad y=t,\quad z=-1+t$

36. $(0,\ 4,\ 1)$; $x=2+t,\quad y=2+t,\quad z=t$

$(\mathbf{37-40})$ Find the distance from the plane.

37. $x+2y+6z=1$ to the plane $x+2y+6z=10$

38. $x+y+z=-5$ *and* $2x+2y+2z=-7$

39. $x-y-z=-5$ *and* $2x-2y-2z=-7$

40. $6x+2y-3z=15$ *and* $6x+2y-3z=-25$

$(\mathbf{41-48})$ Find the angle between the planes.

41. $x+y=1$, $2x+y-2z=2$

42. $5x+y-z=10$, $x-2y+3z=-1$

43. $x=7$, $x+y+\sqrt{2}z=-3$

44. $x+y=1$, $y+z=1$

45. $2x-y+z=7$ & $x+y+2z=11$

46. $x+2y-z=12$ & $x-2y-2z=7$

47. $4x-3y+z=6$ & $2x+3y-5z=4$

48. $3x-z=-12$ & $x+3y=-17$

$(\mathbf{49-52})$ Find the point in which the line meets the plane.

49. $x=1-t$, $y=3t$, $z=1+t$; $2x-y+3z=6$

50. $x=2$, $y=3+2t$, $z=-2-2t$; $6x+3y-4z=-12$

51. $x=1+2t$, $y=3t$, $z=1+t$; meets the 3 coordinates planes.

52. The origin perpendicular to the plane $2x-y-z=4$ meets $3x-5y+2z=6$

$(\mathbf{53-55})$ Find an equation of the line through:

53. The point $(0,\ 1,\ 1)$ and parallel to the line $\vec{r}(t)=\langle 1+2t,\ 3-5t,\ 7+6t\rangle$

54. The point $(0,\ 1,\ 1)$ that is orthogonal to both $\langle 0,\ -1,\ 3\rangle$ and $\langle 2,\ -1,\ 2\rangle$

55. The point $(0,\ 1,\ 1)$ that is orthogonal to the vector $\langle -2,\ 1,\ 7\rangle$ and the *y-axis*

$(\mathbf{56-59})$ $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of the motion.

56. $\vec{r}(t)=2t\,\hat{i}+4t\,\hat{j}$

57. $\vec{r}(t)=4t^3\,\hat{i}-4t^2\,\hat{j}$

58. $\vec{r}(t)=r\cosh\omega t\,\hat{i}+r\sinh\omega t\,\hat{j}$

59. $\vec{r}(t)=e^t\,\hat{i}+e^{-t}\,\hat{j}$

(60 – 63) $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle? Then find the particle's velocity and acceleration vectors at the given value of *t*.

60. $\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}, \quad t=1$

61. $\vec{r}(t)=\frac{t}{t+1}\hat{i}+\frac{1}{t}\hat{j}, \quad t=-\frac{1}{2}$

62. $\vec{r}(t)=e^t\,\hat{i}+\frac{2}{9}e^{2t}\,\hat{j}, \quad t=\ln 3$

63. $\vec{r}(t)=(\cos 2t)\hat{i}+(3\sin 2t)\,\hat{j}, \quad t=0$

(64 – 72) $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

64. $\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}+2t\hat{k}, \quad t=1$

65. $\vec{r}(t)=(t+1)\hat{i}+\frac{t^2}{\sqrt{2}}\hat{j}+\frac{t^3}{3}\hat{k}, \quad t=1$

66. $\vec{r}(t)=(2\cos t)\hat{i}+(3\sin t)\,\hat{j}+4t\,\hat{k}, \quad t=\frac{\pi}{2}$

67. $\vec{r}(t)=(2\ln(t+1))\hat{i}+t^2\hat{j}+\frac{t^2}{2}\hat{k}, \quad t=1$

68. $\vec{r}(t)=\left(e^{-t}\right)\hat{i}+(2\cos 3t)\,\hat{j}+(2\sin 3t)\hat{k}, \quad t=0$

69. $\vec{r}(t)=(3\cos 2t)\hat{i}+(3\sin 2t)\,\hat{j}+5\hat{k}, \quad t=\frac{\pi}{4}$

70. $\vec{r}(t)=\left(e^{-t}\cos t\right)\hat{i}+\left(e^{-t}\sin t\right)\hat{j}+e^{-t}\,\hat{k}, \quad t=0$

71. $\vec{r}(t)=\left(e^{t}\sin t\right)\hat{i}+\left(e^{t}\cos t\right)\hat{j}+e^{t}\,\hat{k}, \quad t=0$

72. $\vec{r}(t)=\cosh 2t\,\hat{i}+\sinh 2t\,\hat{j}+t\,\hat{k}, \quad t=0$

(73 – 92) Find the length of the curve.

73. $\vec{r}(t)=\left\langle 2t^{9/2},\, t^3\right\rangle \quad for \quad 0\le t\le 2$

74. $\vec{r}(t)=\left(t^2-1\right)\hat{i}+2t\,\hat{j} \quad for \quad 0\le t\le 5$

75. $\vec{r}(t)=t^3\,\hat{i}+t^2\,\hat{j} \quad for \quad 0\le t\le 2$

76. $\vec{r}(t)=t^3\,\hat{i}+t^2\,\hat{j} \quad for \quad -2\le t\le 2$

77. $\vec{r}(t)=(4\cos t)\hat{i}+(4\sin t)\,\hat{j}+3t\,\hat{k}; \quad 0\le t\le\frac{\pi}{2}$

78. $\vec{r}(t)=\left(e^{t}\cos t\right)\hat{i}+\left(e^{t}\sin t\right)\hat{j}+e^{t}\,\hat{k}; \quad -\ln 4\le t\le 0$

79. $\vec{r}(t)=(1+2t)\hat{i}+(1+3t)\hat{j}+(6-6t)\hat{k}$; $-1\le t\le 0$

80. $\vec{r}(t)=\left\langle t^2,\ \frac{4\sqrt{2}}{3}t^{3/2},\ 2t\right\rangle$ *for* $1\le t\le 3$

81. $\vec{r}(t)=\langle t,\ \ln\sec t,\ \ln(\sec t+\tan t)\rangle$ *for* $0\le t\le \frac{\pi}{4}$

82. $\vec{r}(t)=(2\cos t)\hat{i}+(2\sin t)\hat{j}+t^2\,\hat{k}$; $0\le t\le \frac{\pi}{4}$

83. $\vec{r}(t)=(3\cos t)\hat{i}+(3\sin t)\hat{j}+2t^{3/2}\,\hat{k}$; $0\le t\le 3$

84. $\vec{r}(t)=(2\cos t)\hat{i}+(2\sin t)\hat{j}+\sqrt{5}\,t\,\hat{k}$; $0\le t\le \pi$

85. $\vec{r}(t)=\left(\cos^2 t\right)\hat{i}+\left(\sin^2 t\right)\hat{j}+\hat{k}$; $0\le t\le \frac{\pi}{4}$

86. $\vec{r}(t)=e^t\,\hat{i}+e^t\,\hat{j}+t\,\hat{k}$; $0\le t\le 1$

87. $\vec{r}(t)=e^t\cos t\,\hat{i}+e^t\sin t\,\hat{j}+e^t\,\hat{k}$; $0\le t\le 2\pi$

88. $\vec{r}(t)=t\,\hat{i}+\frac{2}{3}t^{3/2}\,\hat{k}$; $0\le t\le 8$

89. $\vec{r}(t)=(2+t)\hat{i}-(t+1)\hat{j}+t\,\hat{k}$; $0\le t\le 3$

90. $\vec{r}(t)=\left(\cos^3 t\right)\hat{i}+\left(\sin^3 t\right)\hat{k}$; $0\le t\le \frac{\pi}{2}$

91. $\vec{r}(t)=(t\sin t+\cos t)\hat{i}+(t\cos t-\sin t)\hat{j}$; $\sqrt{2}\le t\le 2$

92. $\vec{r}(t)=(t\cos t)\hat{i}+(t\sin t)\hat{j}+\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}$; $0\le t\le \pi$

(93 – 96) Find the formula for the distance traveled as a function of time starting at $t=0$

93. $x(t)=3t^2$ & $y(t)=4t^2$

94. $x(t)=a\cos^2 t$ & $y(t)=a\sin^2 t$

95. $x(t)=a\cos t$ & $y(t)=a\sin t$

96. $x(t)=5\cos 5t^2$ & $y(t)=5\sin 5t^2$

97. Suppose that $\vec{n}$ is normal to a plane and that $\vec{v}$ is parallel to the plane. Describe how you would find a vector $\vec{n}$ that is both perpendicular to $\vec{v}$ and parallel to the plane.

98. Given a point $(x_0,\ y_0,\ 0)$ and a vector $\vec{v}=\langle a,\ b,\ 0\rangle$ in $\mathbb{R}^3$, describe the set of points that satisfy the equation $\langle a,\ b,\ 0\rangle\times\langle x-x_0,\ y-y_0,\ 0\rangle=\vec{0}$. Use this result to determine an equation of a line in $\mathbb{R}^2$ passing through $(x_0,\ y_0)$ parallel to the vector $\langle a,\ b\rangle$.

99. The parallelogram has vertices at A (2, –1, 4), B (1, 0, –1), C (1, 2, 3) and D. Find

a) The coordinates of D,

b) The cosine of the interior angle of B

c) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,

d) The area of the parallelogram,

e) An equation for the plane of the parallelogram,

f) The areas of the orthogonal projection of the parallelogram on the three coordinate planes.

100. *a*) Find the distance from the point $P_0\left(x_0, y_0, z_0\right)$ to the plane P having equation

$$Ax + By + Cz = D$$

b) What is the distance from $(2,\ -1,\ 3)$ to the plane $2x - 2y - z = 9$?

(101 – 102) Given the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

101. Motion on the circle $x^2 + y^2 = 1$ $\quad \vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j}$, $\quad t = \frac{\pi}{4}$ *and* $\frac{\pi}{2}$

102. Motion on the cycloid $x = t - \sin t$, $y = 1 - \cos t$; $\quad \vec{r}(t) = (1 - \sin t)\hat{i} + (1 - \cos t)\hat{j}$; $\quad t = \pi$ & $\frac{3\pi}{2}$

103. Find all points on the ellipse $\vec{r}(t) = \langle 1,\ 8\sin t,\ \cos t\rangle$, for $0 \le t \le 2\pi$, at which $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

104. Find the point on the curve $\vec{r}(t) = (5\sin t)\hat{i} + (5\cos t)\hat{j} + 12t\,\hat{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

105. The acceleration of a wayward firework is given by $\vec{a}(t) = \sqrt{2}\,\hat{j} + 2t\,\hat{k}$ *for* $0 \le t \le 3$.

Suppose the initial velocity of the firework is $\vec{v}(0) = 1$.

a) Find the velocity of the firework, for $0 \le t \le 3$.

b) Find the length of the trajectory of the firework over the interval $0 \le t \le 3$

106. If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end *P* traces an involute of the circle. The circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at (1, 0). The unwound portion of the ***string*** is tangent to the circle at *Q*, and *t* is the radian measure of the angle from the position *x*-axis to segment *OQ*.

Derive the parametric equations $x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t > 0$

of the point $P(x, y)$ for the involute.

Section 9.5 – Calculus of Vector-Valued Functions

9.5–1 Unit Tangent Vector

The velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to the curve $\vec{r}(t)$ and that the vector:

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

A unit vector tangent to the (*smooth*) curve, called the ***unit tangent vector***.

Example 1

Find the unit tangent vector of the curve $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + t^2\,\hat{k}$ representing the path of the glider.

Solution

$$\vec{v}(t) = -(3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|}$$

$$= -\frac{3\sin t}{\sqrt{9+4t^2}}\hat{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\hat{j} + \frac{2t}{\sqrt{9+4t^2}}\hat{k}$$

9.5–2 Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\vec{T} = \frac{d\vec{r}}{ds}$ turns as the curve bends. Since $\vec{T}$ is a unit vector, its length remains constant and only its direction changes as particle moves along the curve. The rate at which $\vec{T}$ turns per unit of length along the curve is called the ***curvature***.

9.5–3 *Definition*

If $\vec{T}$ is the unit vector of a smooth curve, the ***curvature*** function of the curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

9.5–4 Formula for Calculating Curvature

If $\vec{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Where $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

Example 2

A straight line is parametrized by $\vec{r}(t) = \vec{C} + t\vec{v}$ for constant vectors $\vec{C}$ and $\vec{v}$. Thus $\vec{r}'(t) = \vec{v}$, and the unit tangent vector $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is a constant vector that always points in the same direction and has derivative $\vec{0}$.

It follows that, for any value of the parameter *t*, the curvature of the straight line is

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{|\vec{v}|}|\vec{0}|$$

$$= 0$$

Example 3

Find the curvature of a circle $\vec{r}(t) = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j}$ of radius a.

Solution

$$\vec{v}(t) = -(a\sin t)\hat{i} + (a\cos t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2}$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$= |a|\sqrt{\sin^2 + \cos^2 t}$$

$$= a \qquad \textit{Since } a > 0$$

$$\vec{T}(t) = -(\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -(\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t}$$

$$= 1 \qquad \checkmark$$

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{a}(1)$$

$$= \frac{1}{a} = \frac{1}{radius}$$

9.5–5 *Definition*

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

9.5–6 Formula for Calculating $\vec{N}$

If $\vec{r}(t)$ is a smooth curve, then the principal unit normal is

$$\vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

Where $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

Example 4

Find $\vec{T}$ and $\vec{N}$ for the circular motion $\vec{r}(t) = (\cos 2t)\hat{i} + (\sin 2t)\hat{j}$

Solution

$$\vec{v}(t) = -(2\sin 2t)\hat{i} + (2\cos 2t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t}$$
$$= 2$$

$$\vec{T}(t) = -(\sin 2t)\hat{i} + (\cos 2t)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -(2\cos 2t)\hat{i} - (2\sin 2t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t}$$
$$= 2$$

$$\vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{-(2\cos 2t)\hat{i} - (2\sin 2t)\hat{j}}{2}$$

$$= -(\cos 2t)\hat{i} - (\sin 2t)\hat{j}$$

9.5–7 Circle of Curvature for plane Curves

The ***circle of curvature*** or ***osculating circle*** at a point P on a plane where $\kappa \neq 0$ is the circle in the plane of the curve that:

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. lies toward the concave or inner side of the curve

The ***radius of curvature*** of the curve at P is the radius of the circle of curvature, which is

$$Radius\ of\ \ curvature = \rho = \frac{1}{\kappa}$$

To find ρ, we find κ and take the reciprocal. The ***center of curvature*** of the curve at P is the center of the circle of curvature.

Example 5

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution

Assume: $t = x$

$$\vec{r}(t) = x\hat{i} + y\hat{i}$$

$$= t\,\hat{i} + t^2\hat{j}$$

$\vec{v}(t) = \hat{i} + 2t\hat{j}$ $\qquad$ $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{1+4t^2}$

$\vec{T} = \frac{1}{\sqrt{1+4t^2}}\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\hat{j}$ $\qquad$ $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

$$\frac{d\vec{T}}{dt} = -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2\left(1+4t^2\right)^{1/2} - 8t^2\left(1+4t^2\right)^{-1/2}}{\left(1+4t^2\right)}\hat{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2\left(1+4t^2\right) - 8t^2}{\left(1+4t^2\right)^{3/2}}\hat{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2}{\left(1+4t^2\right)^{3/2}}\hat{j}$$

At the origin, $t = 0$, so the curvature is

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = 0\hat{i} + 2\hat{j}$$

$$= 2\hat{j}$$

$$\kappa(0) = \frac{1}{|\vec{v}(0)|}\left|\frac{d\vec{T}}{dt}(0)\right|$$

$$= \frac{1}{\sqrt{1}}\left|2\hat{j}\right|$$

$$= 2$$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{2}$

At the origin, $t = 0$:

$\vec{T} = \hat{i} \quad \vec{N} = \hat{j}$

The center of the circle is $\left(0, \frac{1}{2}\right)$

The equation of the osculating circle is: $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$

Example 6

Find the curvature and $\vec{N}$ for the helix $\vec{r}(t)=(a\cos t)\hat{i}+(a\sin t)\hat{j}+bt\,\hat{k},\quad a,b\geq 0,\quad a^2+b^2\neq 0$

Solution

$$\vec{v}=-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\,\hat{k} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}$$
$$=\sqrt{a^2+b^2}$$

$$\vec{T}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\,\hat{k}\right) \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\cos t)\hat{i}-(a\sin t)\hat{j}\right)$$
$$=\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)$$

$$\kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$
$$=\frac{1}{\sqrt{a^2+b^2}}\left|\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)\right|$$
$$=\frac{1}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}}\sqrt{\sin^2 t+\cos^2 t}$$
$$=\frac{a}{a^2+b^2}$$

$$\left|\frac{d\vec{T}}{dt}\right|=\frac{a}{\sqrt{a^2+b^2}}$$

$$\vec{N}=\frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$
$$=\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)\cdot\frac{\sqrt{a^2+b^2}}{a}$$
$$=-\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)$$
$$=-(\cos t)\hat{i}-(\sin t)\hat{j}$$

9.5–8 $\vec{T}\ \vec{N}\ \vec{B}$ Frame

The ***binormal vector*** of a curve in space $\vec{B} = \vec{T} \times \vec{N}$, a unit vector orthogonal to both $\vec{T}$ and $\vec{N}$. Together $\vec{T}$, $\vec{N}$, and $\vec{B}$ define a moving right-handed vector frame that play a significant role in calculating the paths of particles moving through space.

It is called the ***Frenet frame*** or $\vec{T}\ \vec{N}\ \vec{B}$ ***frame***.

9.5–9 Torsion

Definition

Let $\vec{B} = \vec{T} \times \vec{N}$. The torsion function of a smooth curve is:

$$\tau = -\frac{d\vec{B}}{ds} \bullet \vec{N}$$

$$= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} \quad \begin{matrix} \to \vec{v} : \textit{velocity} \\ \to \vec{a} : \textit{acceleration} \\ \to \vec{j} : \textit{Jerk} \end{matrix}$$

Where $s = rt$ is the arc length, and r is the radius.

$$\tau = -\frac{1}{r}\left(\frac{d\vec{B}}{dt} \bullet \vec{N}\right)$$

9.5–10 Tangential and Normal Components of Acceleration

When an object is accelerated by gravity, brakes, or a combination of rocket motors, how much of the acceleration acts in the direction of motion, in the tangential direction $\vec{T}$.

$$\vec{v} = \frac{d\vec{r}}{ds}\frac{ds}{dt} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \vec{T}\frac{ds}{dt}$$

$$\vec{a} = \frac{d}{dt}\left(\vec{T}\frac{ds}{dt}\right) \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\frac{d\vec{T}}{ds}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\kappa\vec{N}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$$

9.5–12 *Definition*

If the acceleration vector is written as

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

then

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}| \quad and \quad a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}|^2$$

are the ***tangential*** and ***normal*** scalar components of acceleration.

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

Example 7

Without finding $\vec{T}$ and $\vec{N}$, write the acceleration of the motion:

$$\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \quad t > 0$$

In the form $\vec{a} = a_T \vec{T} + a_N \vec{N}$

Solution

$$\vec{v}(t) = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\ \hat{i} + (t\sin t)\ \hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= |t| \qquad t > 0$$

$$= t$$

$$a_T = \frac{d}{dt}(t) \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$= 1$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j}$$

$$|\vec{a}|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$$

$$= \cos^2 t - 2t\cos t\ \sin t + t^2\sin^2 t + \sin^2 t + 2t\ \cos t\ \sin t + t^2\cos^2 t$$

$$= 1 + t^2\left(\sin^2 t + \cos^2 t\right)$$

$$= 1 + t^2$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

$$= \sqrt{1 + t^2 - 1}$$

$$= t$$

$$\vec{a} = \vec{T} + t\vec{N} \qquad \vec{a} = a_T \vec{T} + a_N \vec{N}$$

Example 8

For the given motion: $\vec{r}(t)=\frac{2}{3}(t-1)^{3/2}\,\hat{i}+\frac{2}{3}(t+1)^{3/2}\,\hat{j}+\sqrt{2}\,t\,\hat{k}\,,\quad t\geq 1$

a) Find the velocity, speed, and acceleration.
b) Find the tangent vector and the unit tangent vector.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right|=1$ *and* $\vec{T}\bullet\vec{N}=0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.
f) Find the curvature and the radius of the curvature.
g) Find the binormal vector and torsion.

Solution

a) $\vec{v}(t)=\sqrt{t-1}\,\hat{i}+\sqrt{t+1}\,\hat{j}+\sqrt{2}\,\hat{k}$ $\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right|=\sqrt{t-1+t+1+2}$$
$$=\sqrt{2}\,\sqrt{t+1}$$

$\vec{a}(t)=\frac{1}{2\sqrt{t-1}}\,\hat{i}+\frac{1}{2\sqrt{t+1}}\,\hat{j}$ $\qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$

b) $\vec{T}(t)=\frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}}\,\hat{i}+\frac{1}{\sqrt{2}}\,\hat{j}+\frac{1}{\sqrt{t+1}}\,\hat{k}$ $\qquad \vec{T}=\frac{\vec{v}}{\left|\vec{v}\right|}$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m=\frac{mn(ad-bc)\,x^{n-1}\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

$$\frac{d}{dt}\vec{T}(t)=\frac{1}{\sqrt{2}}\frac{1}{(t+1)^{3/2}\sqrt{t-1}}\,\hat{i}-\frac{1}{2}\frac{1}{(t+1)^{3/2}}\,\hat{k}$$

c) $$\left|\frac{d}{dt}\vec{T}(t)\right|=\sqrt{\frac{1}{2}\frac{1}{(t+1)^3\,(t-1)}+\frac{1}{4}\frac{1}{(t+1)^3}}$$
$$=\frac{1}{2}\sqrt{\frac{2+t-1}{(t+1)^3\,(t-1)}}$$
$$=\frac{1}{2}\,\frac{1}{(t+1)\sqrt{t-1}}$$

$$\vec{N}(t) = \frac{\sqrt{2}}{\sqrt{t+1}}\,\hat{i} - \sqrt{\frac{t-1}{t+1}}\,\hat{k}$$

$$\vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

d) $\left|\vec{N}\right| = \sqrt{\frac{2}{t+1} + \frac{t-1}{t+1}}$

$$= \sqrt{\frac{t+1}{t+1}}$$

$$= 1 \quad \checkmark$$

$$\vec{T} \bullet \vec{N} = \left(\frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}}\,\hat{i} + \frac{1}{\sqrt{2}}\,\hat{j} + \frac{1}{\sqrt{t+1}}\,\hat{k}\right) \bullet \left(\frac{\sqrt{2}}{\sqrt{t+1}}\,\hat{i} - \sqrt{\frac{t-1}{t+1}}\,\hat{k}\right)$$

$$= \frac{\sqrt{t-1}}{t+1} - \frac{\sqrt{t-1}}{t+1}$$

$$= 0 \quad \checkmark$$

e) At $t = 2$

$$\vec{r}(2) = \left\langle \frac{2}{3},\ \sqrt{3},\ 2\sqrt{2} \right\rangle$$

$$\vec{T}(2) = \frac{1}{\sqrt{6}}\,\hat{i} + \frac{1}{\sqrt{2}}\,\hat{j} + \frac{1}{\sqrt{3}}\,\hat{k}$$

$$\vec{N}(2) = \frac{\sqrt{2}}{\sqrt{3}}\,\hat{i} - \frac{1}{\sqrt{3}}\,\hat{k}$$

At $t = 3$

$$\vec{r}(3) = \left\langle \frac{4\sqrt{2}}{3},\ \frac{16}{3},\ 3\sqrt{2} \right\rangle$$

$$\vec{T}(3) = \frac{1}{2}\,\hat{i} + \frac{1}{\sqrt{2}}\,\hat{j} + \frac{1}{2}\,\hat{k}$$

$$\vec{N}(3) = \frac{\sqrt{2}}{2}\,\hat{i} - \frac{\sqrt{2}}{2}\,\hat{k}$$

f) $\kappa = \frac{1}{2}\,\frac{1}{(t+1)\sqrt{t-1}}\,\frac{1}{\sqrt{2}\sqrt{t+1}}$

$$\kappa = \frac{1}{\left|\vec{v}\right|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{2\sqrt{2}\,(t+1)^{3/2}\sqrt{t-1}}$$

curvatue radius $= 2\sqrt{2}\,(t+1)^{3/2}\sqrt{t-1}$

g) $\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{t+1}} \\ \frac{\sqrt{2}}{\sqrt{t+1}} & 0 & -\sqrt{\frac{t-1}{t+1}} \end{vmatrix}$ $\qquad \vec{B} = \vec{T} \times \vec{N}$

$$= \left\langle -\frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}},\ \frac{\sqrt{2}}{t+1} + \frac{1}{\sqrt{2}}\frac{t-1}{t+1},\ -\frac{1}{\sqrt{t+1}} \right\rangle$$

$$= -\frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}}\,\hat{j} + \frac{1}{\sqrt{2}}\,\hat{j} - \frac{1}{\sqrt{t+1}}\,\hat{k}$$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} \sqrt{t-1} & \sqrt{t+1} & \sqrt{2} \\ \frac{1}{2\sqrt{t-1}} & \frac{1}{2\sqrt{t+1}} & 0 \\ -\frac{1}{4(t-1)^{3/2}} & -\frac{1}{4(t+1)^{3/2}} & 0 \end{vmatrix}$$

$$= \sqrt{2}\left(-\frac{1}{8\sqrt{t-1}(t+1)^{3/2}} + \frac{1}{8\sqrt{t+1}(t-1)^{3/2}} \right)$$

$$= \frac{\sqrt{2}}{8}\,\frac{-t+1+t+1}{(t-1)^{3/2}(t+1)^{3/2}}$$

$$= \frac{\sqrt{2}}{4}\,\frac{1}{(t-1)^{3/2}(t+1)^{3/2}}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{t-1} & \sqrt{t+1} & \sqrt{2} \\ \frac{1}{2\sqrt{t-1}} & \frac{1}{2\sqrt{t+1}} & 0 \end{vmatrix}$$

$$= \left\langle -\frac{\sqrt{2}}{2\sqrt{t+1}},\ \frac{\sqrt{2}}{2\sqrt{t-1}},\ \frac{\sqrt{t-1}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t-1}} \right\rangle$$

$$= \left\langle -\frac{\sqrt{2}}{2\sqrt{t+1}},\ \frac{\sqrt{2}}{2\sqrt{t-1}},\ -\frac{1}{\sqrt{t^2-1}} \right\rangle$$

$$|\vec{v} \times \vec{a}|^2 = \frac{1}{2(t+1)} + \frac{1}{2(t-1)} + \frac{1}{t^2-1}$$

$$= \frac{t-1+t+1+2}{2\left(t^2-1\right)}$$

$$= \frac{1}{t-1}$$

$$\tau = \frac{\sqrt{2}}{4}\,\frac{(t-1)}{(t-1)^{3/2}(t+1)^{3/2}}$$

$$= -\frac{\sqrt{2}}{4}\,\frac{1}{\sqrt{t-1}\,(t+1)^{3/2}}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left|\vec{v}\times\vec{a}\right|^2}$$

Exercises *Section* 9.5 – Curvature and Normal Vectors

$(\mathbf{1-15})$ Find $\vec{T}$, $\vec{N}$, and κ for the plane curves:

1. $\vec{r}(t) = t\,\hat{i} + (\ln\cos t)\,\hat{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

2. $\vec{r}(t) = (\ln\sec t)\,\hat{i} + t\,\hat{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

3. $\vec{r}(t) = (2t+3)\,\hat{i} + \left(5-t^2\right)\hat{j}$

4. $\vec{r}(t) = \frac{t^3}{3}\,\hat{i} + \frac{t^2}{2}\,\hat{j}$, $t > 0$

5. $\vec{r}(t) = \left(\cos^3 t\right)i + \left(\sin^3 t\right)\hat{j}$, $0 < t < \frac{\pi}{2}$

6. $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\,\hat{j}$, $t > 0$

7. $\vec{r}(t) = (t\sin t + \cos t)\hat{i} + (t\cos t - \sin t)\,\hat{j}$; $\sqrt{2} \le t \le 2$

8. $\boldsymbol{r}(t) = (3\sin t)\hat{i} + (3\cos t)\,\hat{j} + 4t\,\hat{k}$

9. $\vec{r}(t) = \left(e^t\cos t\right)\hat{i} + \left(e^t\sin t\right)\hat{j} + 2\hat{k}$

10. $\vec{r}(t) = (\cosh t)\hat{i} - (\sinh t)\,\hat{j} + t\,\hat{k}$

11. $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\,\hat{j} + \sqrt{5}\,t\,\hat{k}$; $0 \le t \le \pi$

12. $\vec{r}(t) = t\,\hat{i} + \frac{2}{3}t^{3/2}\hat{k}$; $0 \le t \le 8$

13. $\vec{r}(t) = (2+t)\hat{i} - (t+1)\,\hat{j} + t\,\hat{k}$; $0 \le t \le 3$

14. $\vec{r}(t) = \left(\cos^3 t\right)\hat{i} + \left(\sin^3 t\right)\hat{k}$; $0 \le t \le \frac{\pi}{2}$

15. $\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\,\hat{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}$; $0 \le t \le \pi$

16. Find an equation for the circle of curvature of the curve $\vec{r}(t) = t\,\hat{i} + (\sin t)\,\hat{j}$, at the point $\left(\frac{\pi}{2}, 1\right)$. (The curve parameterizes the graph $y = \sin x$ in the xy-plane.)

$(\mathbf{17-26})$ Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$.

17. $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\,\hat{j} + bt\,\hat{k}$

18. $\vec{r}(t) = R(\omega t - \sin\omega t)\,\hat{i} + R(1-\cos\omega t)\,\hat{j}$

19. $\vec{r}(t) = (2t-1)\,\hat{i} + \left(4t^2-1\right)\hat{j}$

20. $\vec{r}(t)=\left(t^2+1\right)\hat{i}+2t\,\hat{j}$

21. $\vec{r}(t)=(1+3t)\,\hat{i}+(t-2)\,\hat{j}-3t\,\hat{k}$

22. $\vec{r}(t)=(t+1)\hat{i}+2t\hat{j}+t^2\hat{k},\quad t=1$

23. $\vec{r}(t)=(t\cos t)\hat{i}+(t\sin t)\,\hat{j}+t^2\hat{k},\quad t=0$

24. $\vec{r}(t)=\left(e^t\cos t\right)\hat{i}+\left(e^t\sin t\right)\hat{j}+\sqrt{2}e^t\hat{k},\quad t=0$

25. $\vec{r}(t)=\left(2+3t+3t^2\right)\hat{i}+\left(4t+4t^2\right)\hat{j}-(6\cos t)\hat{k}\quad t=0$

26. $\vec{r}(t)=(2+t)\hat{i}+\left(t+2t^2\right)\hat{j}+\left(1+t^2\right)\hat{k}\quad t=0$

(27 – 28) Graph the curves and sketch their velocity and acceleration vectors at the given values of *t*. Then write $\vec{a}$ of the motion $\vec{a}=a_T\vec{T}+a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$, and find the value of κ at the given values of *t*.

27. $\vec{r}(t)=(4\cos t)\hat{i}+\left(\sqrt{2}\sin t\right)\hat{j},\quad t=0\ and\ \frac{\pi}{4}$

28. $\vec{r}(t)=\left(\sqrt{3}\sec t\right)\hat{i}+\left(\sqrt{3}\tan t\right)\hat{j},\quad t=0$

(29 – 31) Find $\vec{T}$, $\vec{N}$, $\vec{B}$, τ, and κ at the given value of *t* for the plane curves

29. $\vec{r}(t)=\frac{4}{9}(1+t)^{3/2}\,\hat{i}+\frac{4}{9}(1-t)^{3/2}\,\hat{j}+\frac{1}{3}t\,\hat{k};\quad t=0$

30. $\vec{r}(t)=\left(e^t\sin 2t\right)\hat{i}+\left(e^t\cos 2t\right)\hat{j}+2e^t\,\hat{k};\quad t=0$

31. $\vec{r}(t)=t\,\hat{i}+\left(\frac{1}{2}e^{2t}\right)\hat{j};\quad t=\ln 2$

(32 – 33) Find $\vec{T}$, $\vec{N}$, $\vec{B}$, τ, and κ at the given value of *t*. Then find equations for the osculating, normal, and rectifying planes at that value of *t*.

32. $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\,\hat{j}-\hat{k},\quad t=\frac{\pi}{4}$

33. $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\,\hat{j}+t\hat{k},\quad t=0$

(34 – 36) Find $\vec{B}$ and τ for:

34. $\vec{r}(t)=(3\sin t)\hat{i}+(3\cos t)\,\hat{j}+4t\hat{k}$

35. $\vec{r}(t)=(\cos t+t\sin t)\vec{w}+(\sin t-t\cos t)\,\hat{j}+3\hat{k}$

36. $\vec{r}(t) = (6\sin 2t)\hat{i} + (6\cos 2t)\hat{j} + 5t\hat{k}$

37. The speedometer on your car reads a steady 35 *mph*, could you be accelerating? Explain.

38. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

39. Find $\vec{T}$, $\vec{N}$, $\vec{B}$, τ, and κ as functions of ***t*** for the plane curve:
$\vec{r}(t) = (\sin t)\hat{i} + \left(\sqrt{2}\cos t\right)\hat{j} + (\sin t)\hat{k}$, then write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$

40. Consider the ellipse $\vec{r}(t) = \langle 3\cos t,\ 4\sin t\rangle$ *for* $0 \le t \le 2\pi$.

a) Find the tangent vector $\frac{d\vec{r}}{dt}$, the unit vector $\vec{T}$, and the principal unit normal vector $\vec{N}$ at all points on the curve.

b) At what points does $\left|\frac{d\vec{r}}{dt}\right|$ have maximum and minimum values?

c) At what points does the curvature have maximum and minimum values? Interpret this result in light of part (*b*).

d) Find the points (if any) at which $\vec{r}$ and $\vec{N}$ are parallel.

41. Find the following for all values of *t* for which the given curve is defined by:
$\vec{r}(t) = \langle 6\cos t,\ 3\sin t\rangle$, $0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector.

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

42. Find the following for all values of *t* for which the given curve is defined by:
$\vec{r}(t) = \cos 2\pi t\ \hat{i} + \sin 2\pi t\ \hat{j}$, $0 \le t \le 1$

a) Find the tangent vector and the unit tangent vector.

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

43. Find the following for all values of t for which the given curve is defined by:

$\vec{r}(t) = \cos^2 t\ \hat{i} + \sin^2 t\ \hat{j}\ ,\quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector.
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

44. Find the following for all values of t for which the given curve is defined by:

$\vec{r}(t) = \cos^3 t\ \hat{i} + \sin^3 t\ \hat{j}\ ,\quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector.
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

45. Find the following for all values of t for which the given curve is defined by:

$\vec{r}(t) = (\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}\ ,\quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector.
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

46. Find the following for all values of t for which the given curve is defined by:

$\vec{r}(t) = t\hat{i} + (2\cos t)\hat{j} + (2\sin t)\hat{k}\ ,\quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector.
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

47. Find the following for all values of t for which the given curve is defined by:

$\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + \left(\sqrt{5}\sin t\right)\hat{k}\ ,\quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector
b) Find the curvature.
c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

48. Find equations for the osculating, normal and rectifying planes of the curve $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ at the point (1, 1, 1).

49. Consider the position vector $\vec{r}(t) = \left(t^2 + 1\right)\hat{i} + (2t)\,\hat{j}$, $t \geq 0$ of the moving objects.

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

50. Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\,\hat{j}$, $0 \leq t \leq 2\pi$ of the moving objects.

c) Find the normal and tangential components of the acceleration.

d) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

51. Consider the position vector $\vec{r}(t) = 3t\,\hat{i} + (4 - t)\,\hat{j} + t\,\hat{k}$, $t \geq 0$ of the moving objects. Find the normal and tangential components of the acceleration.

52. Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\,\hat{j} + (10t)\hat{k}$, $0 \leq t \leq 2\pi$ of the moving objects.

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

53. Compute the unit binormal vector $\vec{B}$ and the torsion of the curve $\vec{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle$, *at* $t = 1$

54. At point *P*, the velocity and acceleration of a particle moving in the plane are $\vec{v} = 3\hat{i} + 4\,\hat{j}$ and $\vec{a} = 5\hat{i} + 15\,\hat{j}$. Find the curvature of the particle's path at *P*.

55. Consider the curve $C:\ \vec{r}(t) = \langle 3\sin t,\ 4\sin t,\ 5\cos t \rangle$, *for* $0 \leq t \leq 2\pi$

a) Find $\vec{T}(t)$ at all points of *C*.

b) Find $\vec{N}(t)$ and the curvature at all points of *C*.

c) Sketch the curve and show $\vec{T}(t)$ and $\vec{N}(t)$ at the points of *C* corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (*a*) and (*b*) consistent with the graph?

e) Find $\vec{B}(t)$ at all points of *C*.

f) Describe three calculations that serve to check the accuracy of your results in part (*a*) – (*f*).

g) Compute the torsion at all points of *C*. Interpret this result.

56. Consider the curve $C: \vec{r}(t) = \langle 3\sin t,\ 3\cos t,\ 4t \rangle, \quad for\ 0 \le t \le 2\pi$

a) Find $\vec{T}(t)$ at all points of *C*.

b) Find $\vec{N}(t)$ and the curvature at all points of *C*.

c) Sketch the curve and show $\vec{T}(t)$ and $\vec{N}(t)$ at the points of *C* corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (*a*) and (*b*) consistent with the graph?

e) Find $\vec{B}(t)$ at all points of *C*.

f) Describe three calculations that serve to check the accuracy of your results in part (*a*) – (*f*).

g) Compute the torsion at all points of *C*. Interpret this result.

57. Suppose $\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$, where *f*, *g*, and *h* are the quadratic functions $f(t) = a_1 t^2 + b_1 t + c_1$, $g(t) = a_2 t^2 + b_2 t + c_2$, and $h(t) = a_3 t^2 + b_3 t + c_3$, and where at least one of the leading coefficients $a_1,\ a_2,\ or\ a_3$ is nonzero. Apart from a set of degenerate cases (for example $\vec{r}(t) = \langle t^2,\ t^2,\ t^2 \rangle$, whose graph is a line), it can be shown that the graph of $\vec{r}(t)$ is a parabola that lies in a plane.

a) Show by direct computation that $\vec{v} \times \vec{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?

b) Compute $\vec{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

58. Let *f* and *g* be continuous on an interval *I*. consider the curve.

$$C: \vec{r}(t) = \langle a_1 f(t) + a_2 g(t) + a_3,\ b_1 f(t) + b_2 g(t) + b_3,\ c_1 f(t) + c_2 g(t) + c_3 \rangle$$

For *t* in *I*, and where $a_i,\ b_i,\ and\ c_i$, for $i = 1,\ 2,\ and\ 3$, are real numbers.

a) Show that, in general, *C* lies in a plane.

b) Explain why the torsion is zero at all points of *C* for which the torsion is defined.

Section 9.6 – Motion in Space

9.6-1 Integrals of Vector Functions

A differentiable vector function $\vec{R}(t)$ is an antiderivative of a vector function $\vec{r}(t)$ on interval *I* if $\frac{d\vec{R}}{dt} = \vec{r}(t)$ at each point on *I*.

9.6-2 *Definition*

The indefinite integral of $\vec{r}(t)$ with respect to *t* is the set of all antiderivatives of $\vec{r}(t)$, denoted by $\int \vec{r}(t)\, dt$. If $\vec{R}$ is any antiderivative of $\vec{r}(t)$, then:

$$\int \vec{r}(t)dt = \vec{R}(t) + C$$

***Example* 1**

Integrate: $\int \left((\cos t)\hat{i} + \hat{j} - 2t\hat{k}\right) dt$

Solution

$$\int \left((\cos t)\hat{i} + \hat{j} - 2t\hat{k}\right) dt = \left(\int \cos t\, dt\right)\hat{i} + \left(\int dt\right)\hat{j} - \left(\int 2t dt\right)\hat{k}$$

$$= \left(\sin t + C_1\right)\hat{i} + \left(t + C_2\right)\hat{j} - \left(t^2 + C_3\right)\hat{k}$$

$$= \sin t\, \hat{i} + t\,\hat{j} - t^2\,\hat{k} + \vec{C} \qquad \vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$$

9.6-3 *Definition*

If the components of $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$ are integrable over [*a*, *b*], then so is $\vec{r}(t)$, and the ***definite integral*** of $\vec{r}(t)$ from *a* to *b* is

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{i} + \left(\int_a^b g(t)dt\right)\hat{j} + \left(\int_a^b h(t)dt\right)\hat{k}$$

Example 2

Evaluate the integral: $\int_0^{\pi} \left((\cos t)\hat{i} + \hat{j} - 2t\hat{k} \right) dt$

Solution

$$\int_0^{\pi} \left((\cos t)\hat{i} + \hat{j} - 2t\hat{k} \right) dt = (\sin t \Big|_0^{\pi} \hat{i} + (t \Big|_0^{\pi} \hat{j} - \left(\tfrac{1}{2}t^2 \Big|_0^{\pi} \hat{k}$$

$$= (0-0)\hat{i} + (\pi - 0)\hat{j} + \left(\pi^2 - 0\right)\hat{k}$$

$$= \pi\hat{j} - \pi^2\hat{k}$$

Example 3

Suppose the acceleration vector of the path of a hang glider is given by:

$$\vec{a}(t) = -(3\cos t)\hat{i} - (3\sin t)\hat{j} + 2\hat{k}.$$

At time $t = 0$, the glider departed from the point $(3,\ 0,\ 0)$ with velocity $\vec{v}(0) = 3\hat{j}$.

Find the glider's position as a function of t.

Solution

$$\vec{v}(t) = \int \left(-(3\cos t)\hat{i} - (3\sin t)\hat{j} + 2\hat{k} \right) dt \qquad \vec{v}(t) = \int \vec{a}(t)\,dt$$

$$= -(3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k} + \vec{C}_1$$

Given: $\vec{v}(0) = 3\hat{j}$

$$3\hat{j} = -(3\sin 0)\hat{i} + (3\cos 0)\hat{j} + 2(0)\hat{k} + \vec{C}_1$$

$$3\hat{j} = 3\hat{j} + \vec{C}_1$$

$$\vec{C}_1 = \vec{0}$$

$$\vec{v}(t) = -(3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k}$$

$$\vec{r}(t) = \int \left(-(3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k} \right) dt \qquad \vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= (3\cos t)\hat{i} + (3\sin t)\hat{j} + t^2\hat{k} + \vec{C}_2$$

Given: $\vec{r}(0) = 3\hat{i}$

$$3\hat{i} = (3\cos 0)\hat{i} + (3\sin 0)\hat{j} + (0)^2\hat{k} + \vec{C}_2$$

$$3\hat{i} = 3\hat{i} + \vec{C}_2$$

$$\vec{C}_2 = \vec{0}$$

$$\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + t^2\hat{k}$$

9.6-4 The vector and Parametric Equations for Ideal Projectile Motion

A projectile motion describes how an object fired at some angle from an initial position, and acted upon by only the force of gravity, moves in a vertical coordinate plane. We ignore the effects of any frictional drag on the object, which may vary with its speed and altitude, and the fact the force of gravity changes slightly with the projectile's changing height.

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down.

9.6-5 Ideal Projectile Motion Equation

$$\vec{r}(t) = \left(v_0 \cos\alpha\right)t\,\hat{i} + \left(\left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2\right)\hat{j}$$

This is the vector equation for ideal projectile motion. The angle α is the projectile's ***launch angle*** (***firing angle, angle of elevation***), and v_0 is the projectile's ***initial speed***.

The components of $\vec{r}(t)$ give the parametric equations:

$$x(t) = \left(v_0 \cos\alpha\right)t \quad \textit{and} \quad y(t) = \left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2$$

Where *x* is the distance downrange and *y* is the height of the projectile at time $t \geq 0$.

Example 4

A projectile is fired from the origin over horizontal ground at an initial speed of 500 *m/sec* and a launch angle of 60°. Where will the projectile be 10 *sec* later?

Solution

Given: $v_0 = 500, \quad \alpha = 60°, \quad g = 9.8, \quad t = 10$

$$\begin{aligned}\vec{r}(t) &= \left(v_0 \cos\alpha\right)t\,\hat{i} + \left(\left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2\right)\hat{j} \\ &= (500\cos 60°)(10)\hat{i} + \left((500\sin 60°)(10) - \tfrac{1}{2}9.8(10)^2\right)\hat{j} \\ &\simeq 2500\,\hat{i} + \left(2{,}500\sqrt{3} - 490\right)\hat{j} \\ &\approx 2500\,\hat{i} + 3840\,\hat{j}\end{aligned}$$

After 10 *sec*, the projectile is about 3840 *m* above the ground and 2500 *m* downrange from the origin.

9.6-6 Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum height: $y_{\max} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Maximum time: $t_{\max} = \dfrac{v_0 \sin\alpha}{g}$

Flight time: $t = \dfrac{2v_0 \sin\alpha}{g}$

Range: $R = \dfrac{v_0^2}{g}\sin 2\alpha$

If we fire a projectile from a point (x_0, y_0) instead of the origin, then the position vector for the path of motion is

$$\vec{r}(t) = \left(x_0 + (v_0 \cos\alpha)t\right)\hat{i} + \left(y_0 + (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2\right)\hat{j}$$

9.6-7 Projectile Motion with Wind Gusts

Example 5

A baseball is hit when it is 3 *feet* above the ground. It leaves the bat with initial speed of 152 *ft/sec*, making an angle of 20° with the horizontal. At the instant the ball hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\hat{i}$ $(ft/\sec)$ to the ball's initial velocity $(8.8\ ft/\sec = 6\ mph)$.

a) Find a vector equation (position vector) for the path of the baseball.
b) How high does the baseball go, and when does it reach maximum height?

c) Assuming that the ball is not caught, find its range and flight time.

Solution

a) The initial velocity of the baseball is:

$$\vec{v}(t) = \left(v_0 \cos\alpha\right)\hat{i} + \left(v_0 \sin\alpha\right)\hat{j} - 8.8\,\hat{i}$$

$$= \left(152\cos 20° - 8.8\right)\hat{i} + \left(152\sin 20°\right)\hat{j}$$

The initial position is $r_0 = 0\hat{i} + 3\hat{j}$.

$$\vec{r}(t) = -\tfrac{1}{2}gt^2\hat{j} + \vec{v}_0 t + \vec{r}_0$$

$$= -16t^2\hat{j} + \left(\left(152\cos 20° - 8.8\right)\hat{i} + \left(152\sin 20°\right)\hat{j}\right)t + 3\hat{j}$$

$$= \left(152\cos 20° - 8.8\right)t\,\hat{i} + \left(3 + \left(152\sin 20°\right)t - 16t^2\right)\hat{j}$$

$$\approx 134.033t\,\hat{i} + \left(3 + 51.987t - 16t^2\right)\hat{j}$$

b) The baseball reaches its highest point when the vertical component of velocity is zero:

$$\frac{dy}{dt} = \frac{d}{dt}\left(3 + \left(152\sin 20°\right)t - 16t^2\right)$$

$$= 152\sin 20° - 32t = 0$$

$$t = \frac{152\sin 20°}{32}$$

$$\approx 1.62 \text{ sec}$$

$$y_{Max} = 3 + \left(152\sin 20°\right)\left(1.62\right) - 16\left(1.62\right)^2$$

$$\approx 45.2 \text{ ft}$$

The maximum height of the baseball is about 45.2 *feet*, reached about 1.6 *sec* after leaving the bat.

c) The vertical component for $\vec{r}(t)$ equal to 0:

$$3 + 51.987t - 16t^2 = 0$$

$$-16t^2 + 51.987t + 3 = 0 \qquad \textit{Solve for t.}$$

$$\Rightarrow \quad t = 3.3 \text{ sec} \quad \textit{and} \quad t = -0.06 \text{ sec}$$

$$R = 134.033\left(3.3\right) \qquad\qquad R = \vec{r}_x = \left(152\cos 20° - 8.8\right)t$$

$$\approx 442 \text{ ft}$$

The horizontal range is about 442 *feet*, and the flight time is about 3.3 *sec*.

9.6-8 *Derivation* of a projectile with wind from a point

***Given*:**

The wind: $wind = w_x\ \hat{i} + w_y\ \hat{j}$

The initial position: $\vec{r}_0 = x_0\ \hat{i} + y_0\ \hat{j}$

The initial velocity: $\vec{v} = \left(v_0 \cos\alpha\right)\ \hat{i} + \left(v_0 \sin\alpha\right)\ \hat{j}$

Then, the velocity with wind is:

$$\vec{v}_0 = \left(v_0 \cos\alpha\right)\ \hat{i} + \left(v_0 \sin\alpha\right)\ \hat{j} + w_x\ \hat{i} + w_y\ \hat{j}$$

$$= \left(v_0 \cos\alpha + w_x\right)\ \hat{i} + \left(v_0 \sin\alpha + w_y\right)\ \hat{j}$$

$$\vec{a}(t) = -g\ \hat{j}$$

$$\vec{v}(t) = \int \left(-g\ \hat{j}\right) dt \qquad \vec{v}(t) = \int \vec{a}(t)\, dt$$

$$= \left(-g\ t\right)\ \hat{j} + \vec{v}_0$$

$$= \left(v_0 \cos\alpha + w_x\right)\ \hat{i} + \left(-g\ t + \left(v_0 \sin\alpha + w_y\right)\right)\ \hat{j}$$

$$\vec{r}(t) = \int \left(\left(v_0 \cos\alpha + w_x\right)\ \hat{i} + \left(-g\ t + \left(v_0 \sin\alpha + w_y\right)\right)\ \hat{j}\right) dt \qquad \vec{r}(t) = \int \vec{v}(t)\, dt$$

$$= \left(v_0 \cos\alpha + w_x\right) t\ \hat{i} + \left(-\frac{1}{2} g\ t^2 + \left(v_0 \sin\alpha + w_y\right) t\right)\ \hat{j} + \vec{r}_0$$

$$= -\frac{1}{2} g t^2\ \hat{j} + \left(\left(v_0 \cos\alpha + w_x\right)\ \hat{i} + \left(v_0 \sin\alpha + w_y\right)\ \hat{j}\right) t + x_0\ \hat{i} + y_0\ \hat{j} \qquad r(t) = -\frac{1}{2} g t^2 + v_0 t + r_0$$

$$= \underbrace{\left(x_0 + \left(v_0 \cos\alpha + w_x\right) t\right)}_{x(t)}\ \hat{i} + \underbrace{\left(-\frac{1}{2} g t^2 + \left(v_0 \sin\alpha + w_y\right) t + y_0\right)}_{y(t)}\ \hat{j}$$

$$y(t) = -\frac{1}{2} g t^2 + \left(v_0 \sin\alpha + w_y\right) t + y_0$$

Maximum time when the object is at maximum height.

$$y' = -gt + v_0 \sin\alpha + w_y = 0$$

$$t_{Max} = \frac{v_0 \sin\alpha + w_y}{g}$$

$$y(t) = -\frac{1}{2} g \left(\frac{v_0 \sin\alpha + w_y}{g} \right)^2 + \left(v_0 \sin\alpha + w_y \right) \left(\frac{v_0 \sin\alpha + w_y}{g} \right) + y_0$$

$$= -\frac{1}{2} \frac{\left(v_0 \sin\alpha + w_y \right)^2}{g} + \frac{\left(v_0 \sin\alpha + w_y \right)^2}{g} + y_0$$

$$= \frac{\left(v_0 \sin\alpha + w_y \right)^2}{2g} + y_0$$

Maximum Height

$$y(t) = -\frac{1}{2} g t^2 + \left(v_0 \sin\alpha + w_y \right) t + y_0 = 0$$

$$t_{1,\,2} = \frac{-\left(v_0 \sin\alpha + w_y \right) \pm \sqrt{\left(v_0 \sin\alpha + w_y \right)^2 - 4\left(-\frac{1}{2} g \right) y_0}}{2\left(-\frac{1}{2} g \right)}$$

$$= \frac{v_0 \sin\alpha + w_y \mp \sqrt{\left(v_0 \sin\alpha + w_y \right)^2 + 2 g y_0}}{g}$$

Range:

$$Range = x_0 + \left(v_0 \cos\alpha + w_x \right) t$$

$$= x_0 + \left(v_0 \cos\alpha + w_x \right) \frac{v_0 \sin\alpha + w_y + \sqrt{\left(v_0 \sin\alpha + w_y \right)^2 + 2 g y_0}}{g}$$

If $y_0 = 0$

$$Range = x_0 + \left(v_0 \cos\alpha + w_x \right) \frac{v_0 \sin\alpha + w_y + \sqrt{\left(v_0 \sin\alpha + w_y \right)^2}}{g}$$

$$= x_0 + \frac{2\left(v_0 \sin\alpha + w_y \right)\left(v_0 \cos\alpha + w_x \right)}{g}$$

$$= x_0 + \frac{2}{g}\left(v_0^2 \sin\alpha\cos\alpha + v_0 w_x \sin\alpha + v_0 w_y \cos\alpha + w_x w_y\right)$$

$$= x_0 + \frac{v_0^2 \sin 2\alpha}{g} + \frac{2v_0\left(w_x \sin\alpha + w_y \cos\alpha\right)}{g} + \frac{2w_x w_y}{g}$$

Exercises *Section* 9.6 – Motion in Space

$(\mathbf{1-8})$ Evaluate the integral

1. $\displaystyle\int_0^1 \left(t^3\,\hat{i} + 7\,\hat{j} + (t+1)\,\hat{k}\right) dt$

2. $\displaystyle\int_1^2 \left((6-6t)\,\hat{i} + 3\sqrt{t}\,\hat{j} + \frac{4}{t^2}\,\hat{k}\right) dt$

3. $\displaystyle\int_{-\pi/4}^{\pi/4} \left((\sin t)\,\hat{i} + (1+\cos t)\,\hat{j} + \left(\sec^2 t\right)\hat{k}\right) dt$

4. $\displaystyle\int_0^{\pi/3} \left((\sec t \tan t)\hat{i} + (\tan t)\,\hat{j} + (2\sin t \cos t)\hat{k}\right) dt$

5. $\displaystyle\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\,\hat{i} + \frac{\sqrt{3}}{1+t^2}\,\hat{k}\right) dt$

6. $\displaystyle\int_1^{\ln 3} \left(te^t\,\hat{i} + e^t\,\hat{j} + (\ln t)\,\hat{k}\right) dt$

7. $\displaystyle\int_0^{\pi/2} \left(\cos t\,\hat{i} - \sin 2t\,\hat{j} + \sin^2 t\,\hat{k}\right) dt$

8. $\displaystyle\int_0^1 \left(\frac{1}{t+1}\,\hat{i} + \frac{t}{t^2+1}\,\hat{j} + \frac{1}{t^2+1}\,\hat{k}\right) dt$

9. Solve the initial value problem for $\vec{r}(t)$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{r}}{dt} = -t\hat{i} - t\hat{j} - t\hat{k} \\ \textit{Initial condition}: & \vec{r}(0) = \hat{i} + 2\hat{j} + 3\hat{k} \end{cases}$$

10. Solve the initial value problem for $\vec{r}(t)$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{r}}{dt} = (180t)\hat{i} + \left(180t - 16t^2\right)\hat{j} \\ \textit{Initial condition}: & \vec{r}(0) = 100\,\hat{j} \end{cases}$$

11. Solve the initial value problem for $\vec{r}(t)$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\,\hat{i} + e^{-t}\,\hat{j} + \dfrac{1}{t+1}\hat{k} \\ \textit{Initial condition}: & \vec{r}(0) = \hat{k} \end{cases}$$

12. Solve the initial value problem for $\vec{r}(t)$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\vec{r}}{dt^2} = -32\hat{k}$$

$$\textit{Initial condition}: \quad \vec{r}(0) = 100\hat{k};$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\hat{i} + 8\,\hat{j}$$

13. Solve the initial value problem for $\vec{r}(t)$ as a vector function of *t*.

$$\textit{Differential equation}: \quad \frac{d^2\vec{r}}{dt^2} = -\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$\textit{Initial condition}: \quad \vec{r}(0) = 10\hat{i} + 10\hat{j} + 10\hat{k};$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{0}$$

14. Consider $\vec{r}(t) = \left\langle t+1,\ t^2 - 3 \right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\int \vec{r}(t)\,dt$

15. Consider $\vec{r}(t) = \left\langle \frac{1}{2t+1},\ \frac{t}{t+1} \right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\int \vec{r}(t)\,dt$

16. Consider $\vec{r}(t) = \left\langle e^{-2t},\ te^{-t},\ \tan^{-1} t \right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\int \vec{r}(t)\,dt$

17. Consider $\vec{r}(t) = \langle \sin 2t,\ 3\cos 4t,\ t \rangle$

a) Evaluate $\lim\limits_{t \to 0} \vec{r}(t)$ and $\lim\limits_{t \to \infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\int \vec{r}(t)\,dt$

18. At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$. It travels in a straight line to the point $(4,\ 1,\ 4)$, has speed 2 at $(1,\ 2,\ 3)$ and constant acceleration $3\hat{i} - \hat{j} + \hat{k}$. Find an equation for the position vector $\vec{r}(t)$ of the particle at time *t*.

19. A projectile is fired at a speed of 840 *m/sec* at an angle of 60°. How long will it take to get 21 *km* downrange?

20. Find the muzzle speed of a gun whose maximum range is 24.5 *km*.

21. A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 *m* away.

a) What was the ball's initial speed?
b) For the same initial speed, find the two firing angles that make the range 6 *m*.

22. An electron in a TV tube is beamed horizontally at a speed of $5 \times 10^6\ m/\sec$ toward the face of the tube 40 *cm* away. About how far will the electron drop before it hits?

23. A golf ball is hit with an initial speed of 116 *ft/sec* at an angle of elevation of 45° from the tee to a green that is elevated 45 *feet* above the tee. Assuming that the pin, 369 *feet* downrange, does not get in the way, where will the ball land in relation to the pin?

24. A toddler on level ground throws a baseball into the air at an angle of 30° with the ground from a height of 2 *feet*. If the ball lands 10 *feet* from the child, determine the initial speed of the ball.

25. An ideal projectile is launched straight down an inclined plane.

a) Show that the greatest downhill range is achieved when the initial velocity vector bisects angle *AOR*.

b) If the projectile were fired uphill instead of down, what launch angle would maximize its range?

26. A volleyball is hit when it is 4 *feet* above the ground and 12 *feet* from a 6-*foot*-high net. It leaves the point of impact with an initial velocity of 35 *ft/sec* at an angle of 27° and slips by the opposing team untouched.

a) Find a vector equation for the path of the volleyball.

b) How high does the volleyball go, and when does it reach maximum height?

c) Find its range and flight time.

d) When is the volleyball 7 *feet* above the ground? How far (ground distance) is the volleyball from where it will land?

e) Suppose that the net is raised to 8 *feet*. Does this changes thing? Explain.

27. A basketball player tosses a basketball into the air at an angle 45° with the ground from a height of 6 *feet* above the ground. If the ball goes through the basket 15 *feet* away and 10 *feet* above the ground, determine the initial velocity of the ball.

28. A circular wheel with radius 1 *foot* and center *C* rolls to the right along the *x*-axis at a half-run per second. At time *t* seconds, the position vector of the point *P* on the wheel's circumference is

$$\vec{r}(t) = (\pi t - \sin \pi t)\hat{i} + (1 - \cos \pi t)\hat{j}$$

a) Sketch the curve traced by *P* during the interval $0 \le t \le 3$

b) Find $\vec{v}$ and $\vec{a}$ at $t = 0, 1, 2,$ and 3 and add these vectors to your sketch

c) At any given time, what is the forward speed of the topmost point of the wheel? Of *C*?

29. The position of a particle in the plane at time t is $\vec{r}(t) = \frac{1}{\sqrt{1+t^2}}\hat{i} + \frac{t}{\sqrt{1+t^2}}\hat{j}$.

Find the particle's highest speed.

30. A particle traveling in a straight line located at the point $(1,\ -1,\ 2)$ and has speed 2 at time t = 0. The particle moves toward the point $(3,\ 0,\ 3)$ with constant acceleration $2\hat{i} + \hat{j} + \hat{k}$. Find the position vector $\vec{r}(t)$ at time t.

31. A shot leaves the thrower's hand 6.5 *feet* above the ground at a 45° angle at 44 *ft/sec*. Where is it 3 *sec* later?

32. An object is fired with an initial velocity v_0 and at angle θ.

a) Find an expression that represents the magnitude of the velocity at any time t of its flight?

b) Show that part (a) can be simplified as $\sqrt{v_0^2 - 64y}$

33. A ball is hit by a bat at a point (***h*** *feet)* above the ground and the ball is projected with an initial velocity v_0 and at angle θ.

a) Find the range d, where the ball hit the ground?

b) Show that the distance can be written as $\frac{1}{64}v_0^2\left(\sin 2\theta + \sqrt{\sin^2 2\theta + \frac{256h\cos^2\theta}{v_0^2}}\right)$

c) Find the angle θ for which the maximum range is obtained?

d) Can you give a geometrical argument to show that the value of θ should be less than $45°$?

e) Show that the result reduces to $45°$ when $h = 0$

f) Show that the maximum range for all possible values of θ is $\frac{1}{32}v_0^2$ $\tan\theta$ or $\cot\theta$?

34. A projectile is fired with an initial velocity v_0 and at angle α. However, the ground slope upward at an angle β.

a) Find the range of the projectile (distance OP)

b) Find the maximum range by varying α

c) Find the maximum range up a hillside which has a slope of $\frac{1}{20}$ with initial velocity of $v_0 = 600\ ft/\sec$

35. A projectile is launched from the origin, which is a point 50 *feet* from a 30-*foot* vertical cliff. It is launched at a speed of $50\sqrt{2}$ *ft/s* at an angle of 45° to the horizontal. Assume the ground is horizontal on top of the cliff and only the gravitational force affects the motion of the object

a) Give the coordinates of the landing spot of the projectile on the top of the cliff
b) What is the maximum height reached by the projectile?
c) What is the time of flight?
d) Write the integral that gives the length of the trajectory.
e) Approximate the length of the trajectory.
f) What is the range of launch angles needed to clear the edge of the cliff?

36. A jet is flying at a constant speed $\left|\vec{v}_j\right|$ at altitude ***h***, parallel to the ground (horizontally). At time zero, the pilot fires a missile at with a velocity of $\vec{v}_{m0}$ and of an angle α. Later on, another type of missile fires with a velocity of $\vec{v}_{m1}$ and of an angle β. At the end, the pilot drops a bomb with no initial velocity.

For each of the three-firing positions,

a) Write a parametric equation of the motion for first missile.
b) Calculate the tangential and normal acceleration at any point of the path.
c) Find the tangent vector.

d) Find the principal unit normal vector.
e) Find the time take to hit the target?
f) Find the distance travel, if it hits the target?
g) Find the range.
h) Determine the angle that will hit the target.

37. You propelled an object upward from the origin at an angle $\theta < 90°$, with initial value v_0 *ft / sec* . The distance of the object travels in the air given by $r(\theta)$, and the distance from the point of launch to landing, is given by $d(t)$.

The object will land on a ramp with an α, see the figure below. (*Neglect the air resistance*)

a) Determine the equations $x(t)$ *and* $y(t)$ of the projectile in function of time.
b) Find a formula for the distance $d(t)$ in function v_0, α *and* θ.
c) Find a formula of the path $y(x)$ and $\vec{r}(t)$ in function v_0 *and* θ.
d) Find a formula for the distance $r(\theta)$ in function v_0, θ, *and* α.
e) Find the time of the object to approach its maximum height.
f) Find the maximum height of the object and how far from you.
g) How long it will take for the object to land on the ramp.
h) Assume $\alpha = 45°$, $\theta = 60°$, and $v_0 = 16$ *ft / sec*, find $d(t)$, $\vec{r}(t)$, and $y(x)$
i) Find the total distance that the object travels $\vec{r}(t)$, $y(x)$, and $r(\theta)$.
j) What should be the angle θ to maximize the distance.
k) From the θ_{max}, determine the time will take, the distance will land from you and the distance travels.
l) Find the $\vec{r}(t)$, velocity, speed, and acceleration parametric equation at its maximum angle.
m) Find the position point *P* (landing point), velocity, and speed at its maximum impact.

Summary

Distance Formula: $\left|P_1P_2\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Standard Equation for the Sphere $\left(x - x_0\right)^2 + \left(y - y_0\right)^2 + \left(z - z_0\right)^2 = a^2$

Magnitude: $\left|\vec{v}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Direction / Unit vector: $\frac{\vec{v}}{\left|\vec{v}\right|}$

Angle between Vectors: $\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{\left|\vec{u}\right|\left|\vec{v}\right|}\right)$

Dot Product: $\vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Vector Projection: $proj_{\vec{v}}\vec{u} = \left(\left|\vec{u}\right| \cos\theta\right) \frac{\vec{u}}{\left|\vec{v}\right|}$

$$= \left(\frac{\vec{u} \bullet \vec{v}}{\left|\vec{v}\right|^2}\right)\vec{v}$$

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is the scalar:

$$\left|\vec{u}\right|\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\left|\vec{v}\right|} \quad = \vec{u} \bullet \frac{\vec{v}}{\left|\vec{v}\right|}$$

Work: $W = \vec{F} \bullet \vec{D}$

$$= \left|\vec{F}\right| \left|\vec{D}\right| \cos\theta$$

Cross Product: $\vec{u} \times \vec{v} = \left(\left|\vec{u}\right| \left|\vec{v}\right| \sin\theta\right)\vec{n}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Magnitude of torque vector: $\left|\vec{r}\right| \left|\vec{F}\right| \sin\theta$

Torque vector: $\left(\left|\vec{r}\right| \left|\vec{F}\right| \sin\theta\right)\vec{n}$

Triple scalar product: $\left(\vec{u}\times\vec{v}\right)\bullet\vec{w}=\left|\vec{u}\times\vec{v}\right|\,\left|\vec{w}\right|\,\left|\cos\theta\right|$

Volume: $V=\left(\vec{u}\times\vec{v}\right)\bullet\vec{w}\;=\left|\det\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{bmatrix}\right|$

Vector equation for the line ***L***: $\vec{r}\left(t\right)=r_0+t\,\vec{v},\quad -\infty<t<\infty$

The distance from a Point to a line: $d=\dfrac{\left|\overrightarrow{PS}\times\vec{v}\right|}{\left|\vec{v}\right|}$

The distance from a Point to a Plane: $d=\left|\overrightarrow{PS}\bullet\dfrac{\vec{n}}{\left|\vec{n}\right|}\right|$

Angle between the planes: $\theta=\cos^{-1}\left(\dfrac{\vec{n}_1\bullet\vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)$

Arc Length: $s\left(t\right)=\displaystyle\int_{t_0}^{t}\sqrt{\left[x'\left(\tau\right)\right]^2+\left[y'\left(\tau\right)\right]^2+\left[z'\left(\tau\right)\right]^2}\;d\tau\;=\int_{t_0}^{t}\left|\vec{v}\left(\tau\right)\right|\,d\tau$

Maximum height: $y_{\max}=\dfrac{\left(v_0\sin\alpha\right)^2}{2g}$

Maximum time: $t=\dfrac{v_0\sin\alpha}{g}$

Flight time: $t=\dfrac{2v_0\sin\alpha}{g}$

Range: $R=\dfrac{v_0^2}{g}\sin 2\alpha$

Unit tangent vector: $\vec{T}=\dfrac{\vec{v}}{\left|\vec{v}\right|}$

Principal unit normal vector: $\vec{N}=\dfrac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$

$=\dfrac{1}{\kappa}\dfrac{d\vec{T}}{ds}$

Binormal vector: $\vec{B} = \vec{T} \times \vec{N}$

$$= \frac{1}{\vec{v}} \left| \frac{d\vec{T}}{dt} \right|$$

Curvature: $\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{\left| \vec{v} \times \vec{a} \right|}{\left| \vec{v} \right|^3}$

$$= \frac{1}{\left| \vec{v} \right|} \left| \frac{d\vec{T}}{dt} \right|$$

Torsion: $\tau = -\frac{d\vec{B}}{ds} \bullet \vec{N}$

$$= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left| \vec{v} \times \vec{a} \right|^2}$$

Tangential and normal scalar.

Components of acceleration: $\vec{a} = a_T \vec{T} + a_N \vec{N}$

$$a_T = \frac{d}{dt} \left| \vec{v} \right|$$

$$a_N = \kappa \left| \vec{v} \right|^2$$

$$= \sqrt{\left| \vec{a} \right|^2 - a_T^2}$$

$$= \left| \vec{v} \right| \left| \frac{dT}{dt} \right|$$

Exercises *Lecture* 9 – Review

$(\mathbf{1}-\mathbf{5})$ Find the parametric equation for the line

1. Through the point $(1,\ 2,\ 3)$ parallel to $\vec{v}=\hat{i}-2\hat{j}+3\hat{k}$

2. Through the point $(-1,\ -5,\ 3)$ parallel to $\vec{v}=\hat{i}+2\hat{j}-4\hat{k}$

3. Through the point $(2,\ 4,\ -9)$ perpendicular to the plane $2x-3y+z=5$

4. Through the point $(2,\ 4,\ -9)$ parallel to the line $x=2t-1,\quad y=5,\quad z=-3t+5$

5. Passes through the points $P_0(1,\ 1,\ 1)$ and $P_1(2,\ 2,\ 2)$

$(\mathbf{6}-\mathbf{10})$ Find the parametric equation for the plane.

6. Through the point $P(2,\ 4,\ 6)$ perpendicular to the vector from the origin to P.

7. Through the point $P_0(1,\ 3,\ 2)$ parallel to the plane $2x-y+2z=6$.

8. Through the point $P_0(-2,\ 1,\ 3)$ normal to the plane $2x-y+2z=6$.

9. Through the point $P_0(3,\ 2,\ 1)$ perpendicular to the line $x=3t+1,\quad y=t-5,\quad z=2t+3$.

10. Through the point $P_0(6,\ -3,\ 2)$ perpendicular to the line of intersection of the two planes $x+2y-z=1,\quad 2x-y-3z=3$.

$(\mathbf{11}-\mathbf{12})$ Find the distance from the point to the plane.

11. $(1,\ 1,\ -1),\quad 2x-5y+7z=2$

12. $(1,\ 3,\ 1),\quad x=2t-1,\quad y=2t+1,\quad z=t+3$

$(\mathbf{13}-\mathbf{14})$ Find the distance from the point to the line.

13. $(2,\ 3,\ 0);\quad x=-t+2,\quad y=t-8,\quad z=7t+9$

14. $(-1,\ 3,\ 4);\quad x=2,\quad y=3t+5,\quad z=2t-2$

15. Find the distance from the plane $x-y+2z=3$ to the plane $2x+y-z=5$

16. Find the intersection, if exists, between the line $x=1+t,\quad y=4-2t,\quad z=5-t$ and the line $x=t,\quad y=2t,\quad z=-t$.

17. Find the parametric line equation that intersect the 2 planes $x-y+z=0$ and $x+2y+3z=6$

18. Find the point, if exits, that intersects the line $x = 1 + 3t$, $y = -1 + t$, $z = 2 - 2t$ and the plane $2x - 3y + z = 6$

(19 – 21) $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's ***velocity*** and ***acceleration*** vectors. Then find the particle's ***speed*** and ***direction*** of motion at the given value of *t*.

19. $\vec{r}(t) = \sqrt{t+1}\,\hat{i} + \left(t^2 + 1\right)\hat{j} + 2t\,\hat{k}$, $t = 0$

20. $\vec{r}(t) = \sin 3t\,\hat{i} + \cos 3t\,\hat{j} + t\,\hat{k}$, $t = \frac{\pi}{4}$

21. $\vec{r}(t) = e^{2t}\,\hat{i} + e^{2t}\,\hat{j} + t\,\hat{k}$, $t = 0$

(22 – 25) Find the length of the curve:

22. $\vec{r}(t) = e^t\,\hat{i} + e^t\,\hat{j} - 2t\,\hat{k}$ *for* $0 \le t \le 1$

23. $\vec{r}(t) = \sin(5t)\,\hat{i} + \cos(5t)\,\hat{j} - t\,\hat{k}$ *for* $0 \le t \le \frac{\pi}{2}$

24. $\vec{r}(t) = 2t\,\hat{i} + t^2\,\hat{j} + \ln(t)\,\hat{k}$ *for* $1 \le t \le e$

25. $\vec{r}(t) = (\sin t - t\cos t)\,\hat{i} + (\cos t + t\sin t)\,\hat{j} + t^2\,\hat{k}$ *for* $0 \le t \le \frac{\pi}{2}$

(26 – 28) Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at $t = 0$ ***without*** finding $\vec{T}$ and $\vec{N}$.

26. $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$

27. $\vec{r}(t) = e^t\cos t\,\hat{i} + e^t\sin t\,\hat{j} + e^t\,\hat{k}$

28. $\vec{r}(t) = \left(t^3 + 1\right)\hat{i} + \left(t^2 - 1\right)\hat{j} + t\,\hat{k}$

29. Find the following for all values of *t* for which the given curve is defined by:

$$\vec{r}(t) = (\sin 2t)\,\hat{i} + (\cos 2t)\,\hat{j} + 3t\,\hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector.

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

30. Consider the weight suspended by two wires. Find the magnitudes and components of vectors $\vec{F}_1$, $\vec{F}_2$ and angles α and β.

31. Consider the weight $w\ N$ suspended by two wires where $\vec{F}_1$ and $\vec{F}_2$ are force vectors directed along the wires.

a) Find the vectors $\vec{F}_1$ and $\vec{F}_2$, and show that their magnitudes are

$$\left|\vec{F}_1\right| = \frac{w\cos\beta}{\sin(\alpha+\beta)} \quad \textit{and} \quad \left|\vec{F}_2\right| = \frac{w\cos\alpha}{\sin(\alpha+\beta)}$$

b) For a fixed β determine the value of α which minimizes the magnitude $\left|\vec{F}_1\right|$

c) For a fixed α determine the value of β which minimizes the magnitude $\left|\vec{F}_2\right|$

32. Consider the weight of 50 *lbs*. suspended by two wires. Find the magnitudes and components of vectors $\vec{F}_1$, $\vec{F}_2$.

33. Consider the weight of 100 *lbs*. suspended by two wires. Find the magnitudes and components of vectors $\vec{F}_1$, $\vec{F}_2$.

$(\mathbf{34-35})$ Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P

34. $\left|\overrightarrow{PQ}\right| = 2\ ft\ ,\quad \left|\vec{F}\right| = 25\ lb.\ ,\quad \alpha = 45°$

35. $\left|\overrightarrow{PQ}\right| = 6\ in\ ,\quad \left|\vec{F}\right| = 50\ lb.\ ,\quad \alpha = 120°$

36. The diameter of a Ferry wheel is $100\ ft$ and above the ground $12\ ft$. It will take $10\ munites$ to circle one revolution. Describe the motion of the wheel at a parametric equation by finding the maximum speed at which you move directly downward.

37. You threw a ball from the ground with initial velocity v_0 and with angle θ. Find a formula for the angle that it will hit a target at point $P(d,\ h)$

38. A ball is thrown with a speed $128\ ft/sec$ from a height of $12\ ft$ at an angle of $30°$ from horizontal.

a) How long will it take the ball to hit the ground.
b) What the range and the speed of the ball when hit the ground?
c) What is the maximum height of the ball will reach?

39. Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. Ship ***A*** is located at (4, 0, 0), whereas ship ***B*** is located at (0, 5, 0). All coordinates are given in thousands of feet. Ship ***A*** locates the submarine in the direction of the vector $2\,\hat{i} + 3\,\hat{j} - \frac{1}{3}\hat{k}$, and ship ***B*** locates the submarine in the direction of the vector $18\,\hat{i} - 6\,\hat{j} - \hat{k}$. Four minutes ago, the submarine was located at $\left(2,\ -1,\ -\frac{1}{3}\right)$. The aircraft is due in 20 *min*. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface direct the aircraft?

40. Two helicopters, H_1 and H_2, are traveling together. At time $t = 0$, they separate and follow different straight-line paths given by

$$H_1: \quad x = 6 + 40t, \quad y = -3 + 10t, \quad z = -3 + 2t$$

$$H_2: \quad x = 6 + 110t, \quad y = -3 + 4t, \quad z = -3 + t$$

Time t is measured in hours and all coordinates are measured in miles. Due to system malfunctions, H_2 stops its flight at (446, 13, 1) and, in a negligible amount of time, lands at (446, 13, 0). Two hours later, H_1 is advised of this fact and heads toward H_2 at 150 *mph*. How long will it take H_1 to reach H_2

41. The capacity of a single-sided, single-layer digital versatile disc (*DVD*) is approximately 4.7 billion bytes–enough to store a two-hour movie. (Newer Blu-ray discs are in the range of 50 gigabytes.) A *DVD* consists of a single "groove" that spirals outward from the inner edge to the outer edge of the storage region.

a) First consider the spiral given by the polar coordinates by $r = \frac{t\theta}{2\pi}$, where $0 \le \theta \le 2\pi N$ and successive loops of the spiral are t units apart. Explain why this spiral has N loops and why the entire spiral has a radius of $R = Nt$ units. Sketch three loops of the spiral.

b) Find the length L of the spiral with N loops.

c) The integral in part (*b*) can be evaluated exactly, but a good approximation can also be made. Assuming N is large, explain why $\theta^2 + 1 \approx \theta^2$. Use this approximation to simplify the integral in part (*b*) and show that $L \approx t\pi N^2 = \frac{\pi R^2}{t}$.

d) Now consider a *DVD* with an inner radius of $r = 2.5$ *cm* and an outer radius of $R = 5.9$ *cm*. Model the groove by a spiral with a thickness of $t = 1.5\ microns = 1.5 \times 10^{-6} m$. Because of the hole in the *DVD*, the lower limit in the arc length integral is not $\theta = 0$. What are the limits of integration?

e) Use the approximation in part (*c*) to find the length of the *DVD* groove. Express your answer in *centimeters* and *miles*.

Practice *Exam*

1. Consider a 100-*N* weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

2. Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at *P* if $\left|\overrightarrow{PQ}\right| = 1$ *foot.* and $\left|\vec{F}\right| = 50$ *lb.*

3. $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's ***velocity*** and ***acceleration*** vectors. Then find the particle's ***speed*** and ***direction*** of motion at the given value of *t*.

$$\vec{r}(t) = (t+1)\hat{i} + \left(t^2 - 1\right)\hat{j} + 2t\hat{k}, \quad t = 1$$

4. Find the length of the curve $\vec{r}(t) = \left\langle t^2,\ \frac{4\sqrt{2}}{3}t^{3/2},\ 2t \right\rangle$ *for* $1 \le t \le 3$

5. Find the parametric equations for the line through (–3, 2, 0) parallel to $\vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

6. Find equation for the plane through $A(1,\ -2,\ 1)$ perpendicular to the vector from the origin to *A*.

7. Find the distance from the point to the plane $(0,\ 1,\ 1)$, $4y + 3z = -12$

8. Find the distance from the point to the line $(2,\ 2,\ 0)$; $x = -t$, $y = t$, $z = -1 + t$

9. Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at $t = 0$ ***without*** finding $\vec{T}$ and $\vec{N}$.

$$\vec{r}(t) = (2+t)\hat{i} + \left(t + 2t^2\right)\hat{j} + \left(1 + t^2\right)\hat{k}$$

10. Find the following for all values of t for which the given curve is defined by:

$$\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector.

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

Lecture Ten

Functions of Several Variables

Section 10.1 – Graphs, Limits, and Continuity

10.1−1 Domains

Suppose D is a set of n-tuples of real numbers. A ***real-valued function*** f on D is a rule that assigns a unique (single) real number

$$w = f\left(x_1, x_2, \ldots, x_n\right)$$

To each element in D. $\left(x_1, x_2, \ldots, x_n\right)$

The set D is the function's ***domain***.

The set of w-values taken on by f is the function's ***range***.

The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***independent variables*** x_1 to x_n. We also call the x_j's, the function's ***input variables*** and call w the function's ***output variable***.

10.1−2 *Examples* of Domains and Ranges

Functions of two variables

Function	*Domain*	*Range*
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	*Entire plane*	$[-1, 1]$

Functions of three variables

Function	*Domain*	*Range*
$w=\sqrt{x^2+y^2+z^2}$	*Entire plane*	$[0,\ \infty)$
$w=\dfrac{1}{x^2+y^2+z^2}$	$(x,\ y,\ z)\neq(0,\ 0,\ 0)$	$(0,\ \infty)$
$w=xy\ln z$	*Half – space* $z>0$	$(-\infty,\ \infty)$

10.1–3 Functions of Two Variables – *Definitions*

A point $(x_0,\ y_0)$ in a region (set) *R* in the *xy*-plane is an ***interior point*** of *R* if it is the center of a disk of positive radius that lies entirely in *R*. A point $(x_0,\ y_0)$ is a ***boundary point*** of *R* if every disk centered at $(x_0,\ y_0)$ contains points that lie outside of *R* as well as points that lie in *R*. (The boundary point itself need not belong to *R*.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its ***boundary***.

A region is ***open*** if it consists entirely of interior points.

A region is ***closed*** if it contains all its boundary points.

Interior point

$x^2+y^2<1$

Open unit disk

Boundary point

$x^2+y^2=1$

Boundary of unit disk

Closed Boundary

$x^2+y^2\leq 1$

Closed unit disk

10.1–4 Functions of Two Variables – *Definitions*

A point $(x_0,\ y_0,\ z_0)$ in a region ***R*** in space is an ***interior point*** of ***R*** if it is the center of a solid ball that lies entirely in ***R***. A point $(x_0,\ y_0,\ z_0)$ is a ***boundary point*** of ***R*** if every solid ball centered at .

(x_0, y_0, z_0). contains points that lie outside of ***R*** as well as that lie inside ***R***. The ***interior*** of R is the set of interior points of ***R***. The ***boundary*** of ***R*** is the set of boundary points of ***R***.

A region is ***open*** if it consists entirely of interior points.
A region is ***closed*** if it contains its entire boundary.
A region is ***unbounded*** if it is not bounded.

Interior point ***Closed Boundary*** ***Unbounded***

10.1−5 Graphs, Level Curves, and contours of Functions of two Variables

The set of points in the plane where a function $f(x, y) = c$ is called a ***level curve*** of *f*.

The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the ***graph*** of *f*.

The graph of *f* is also called the ***surface*** $z = f(x, y)$

Example 1

Graph $f(x,y) = 100 - x^2 - y^2$ and plot the level curves

$f(x,y) = 0, \quad f(x,y) = 51, \quad and \quad f(x,y) = 75$ in the domain of *f* in the plane.

Solution

The domain of *f* is the entire *xy*-plane, and the range of *f* is the set of real numbers less than or equal to 100.

The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in the picture.

At $f(x,y) = 0$

$$\Rightarrow \quad x^2 + y^2 = 100$$

Which is the circle of radius 10 centered at the origin (level curve).

At $f(x,y)=51$

$\Rightarrow \quad x^2+y^2=49$

Which is the circle of radius 7 centered at the origin.

At $f(x,y)=75$

$\Rightarrow \quad x^2+y^2=25$

Which is the circle of radius 5 centered at the origin.

If $x^2+y^2>100$, then the values of $f(x,y)$ are negative.

10.1–6 Functions of Three Variables

Definition

The set of points $(x,\ y,\ z)$ in space where a function of three independent variables has a constant value $f(x,\ y,\ z)=c$ is called a ***level surface*** of *f*.

Example 2

Describe the level surfaces of the function $f(x,\ y,\ z)=\sqrt{x^2+y^2+z^2}$

Solution

The value of *f* is the distance from the origin to the point $(x,\ y,\ z)$.

Each surface $\sqrt{x^2+y^2+z^2}=c\ (>0)$, is a sphere of radius *c* centered at the origin.

Example 3

The temperature *w* beneath the Earth's surface is a function of the depth *x* beneath the surface and the time *t* of the year. If we measure *x* in feet and *t* as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos\left(1.7\times10^{-2}t - 0.2x\right)e^{-0.2x}$$

The temperature at 9 *feet* is scaled to vary from +1 to –1, so that the variation at *x feet*. can be interpreted as a fraction of the variation at the surface.

10.1–7 Limit – *Definition*

We say that a function $f(x, y)$ approaches the limit *L*, as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of *f*.

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

For a function with three variables, then we say that a function $f(x, y, z)$ approaches the limit *L*, as (x, y, z) approaches (x_0, y_0, z_0), and write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y, z) in the domain of *f*.

$$\left| f(x, y, z) - L \right| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

10.1–8 Theorem

The following rules hold if *L*, *M*, *K* are real numbers and

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \quad \textit{and} \quad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M$$

Sum Rule: $$\lim_{(x, y) \to (x_0, y_0)} \left(f(x, y) + g(x, y) \right) = L + M$$

Difference Rule: $$\lim_{(x, y) \to (x_0, y_0)} \left(f(x, y) - g(x, y) \right) = L - M$$

Constant Multiple Rule: $$\lim_{(x, y) \to (x_0, y_0)} kf(x, y) = kL$$

Product Rule: $$\lim_{(x, y) \to (x_0, y_0)} \left(f(x, y) \cdot g(x, y) \right) = L \cdot M$$

Quotient Rule: $$\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$$

Power Rule: $$\lim_{(x, y) \to (x_0, y_0)} \left[f(x, y) \right]^n = L^n \quad (\textbf{n} \textit{ a positive integer})$$

Root Rule: $$\lim_{(x, y) \to (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} \quad (\textbf{n} \textit{ a positive integer})$$

Example 4

Find $$\lim_{(x, y) \to (0, 1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3}$$

Solution

$$\lim_{(x, y) \to (0, 1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2 (1) + 5(0)(1) - (1)^3}$$

$$= -3$$

Example 5

Find $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

Solution

$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2}$$

$$= 5$$

Example 6

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{0}{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y}$$

$$= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x}+\sqrt{y}\right)$$

$$= 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x}+\sqrt{y}\right)$$

$$= 0$$

Example 7

Find $\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2 \sin x}{x}$

Solution

$$\lim_{(x,y)\to(0,1)} \frac{y^2 \sin x}{x} = \frac{0}{0}$$

$$= \lim_{y\to 1}\left(y^2\right) \times \lim_{x\to 0} \frac{\sin x}{x} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= 1 \cdot 1$$

$$= 1$$

Example 8

Find $\displaystyle\lim_{(x,y,z)\to\left(\pi, \frac{\pi}{2}, 2\pi\right)} \left(e^{x+\cos y} + z\right)$

Solution

$$\lim_{(x,y,z)\to\left(\pi, \frac{\pi}{2}, 2\pi\right)} \left(e^{x+\cos y} + z\right) = e^{\pi+\cos\frac{\pi}{2}} + 2\pi$$

$$= e^{\pi} + 2\pi$$

Example 9

Find $\displaystyle\lim_{(x,y,z)\to(1,-2,3)} \left(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz + 4\right)$

Solution

$$\lim_{(x,y,z)\to(1,-2,3)} \left(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz + 4\right) = 1 + 4 + 9 + 4 - 6 + 12 + 4$$

$$= 28$$

Example 10

Find $\displaystyle\lim_{(x,y,z)\to(1,3,-4)} \frac{x^2 y - 2z}{2x + 3y - z}$

Solution

$$\lim_{(x,y,z)\to(1,3,-4)} \frac{x^2 y - 2z}{2x + 3y - z} = \frac{3+8}{2+9+4}$$

$$= \frac{11}{15}$$

10.1–9 Continuity – *Definition*

A function $f(x, y)$ is ***continuous at the point*** (x_0, y_0), if:

1. f is defined at (x_0, y_0)
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

A function is ***continuous*** if it is continuous at every point of its domain.

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the ***composite*** function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (x_0, y_0).

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x+y+z) \quad \textit{and} \quad \frac{y\sin z}{x-1}$$

10.1–10 Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Example 11

Show that the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as (x, y) approaches $(0, 0)$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{0}{0}$$

We examine the curve:

$$y = kx^2, \quad x \neq 0$$

$$\left.\frac{2x^2y}{x^4+y^2}\right|_{y=kx^2} = \frac{2x^2\left(kx^2\right)}{x^4+\left(kx^2\right)^2}$$

$$= \frac{2kx^4}{x^4+k^2x^4}$$

$$= \frac{2kx^4}{x^4\left(1+k^2\right)}$$

$$= \frac{2k}{1+k^2}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ along\ y=kx^2}} f(x,y) = \frac{2k}{1+k^2}$$

This limit varies with the path of approach. If (x, y) approaches (0, 0) along the parabola $y = x^2$.

Exercises *Section* 10.1 – Graphs, Limits, and Continuity

1. Find the specific values for $f(x,y,z)=\dfrac{x-y}{y^2+z^2}$

a) $f(3,-1,2)$ *b)* $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)$ *c)* $f\left(0,\ -\frac{1}{3},\ 0\right)$ *d)* $f(2,\ 2,\ 100)$

2. Find the specific values for $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

a) $f(0,\ 0,\ 0)$ *b)* $f(2,\ -3,\ 6)$ *c)* $f(-1,\ 2,\ 3)$ *d)* $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)$

$(\mathbf{3-17})$ Find and sketch the domain for each function

3. $f(x,y)=\sqrt{y-x-2}$

4. $f(x,y)=\ln\left(x^2+y^2-4\right)$

5. $f(x,y)=\dfrac{\sin(xy)}{x^2+y^2-25}$

6. $f(x,y)=\ln(xy+x-y-1)$

7. $f(x,y)=\sqrt{\left(x^2-4\right)\left(y^2-9\right)}$

8. $f(x,\ y)=\dfrac{1}{x^2+y^2}$

9. $f(x,\ y)=\ln xy$

10. $f(x,\ y)=\sqrt{x-y^2}$

11. $f(x,\ y)=\tan(x+y)$

12. $f(x,\ y)=\sqrt{x^2-y}$

13. $f(x,\ y)=e^{x+y}$

14. $f(x,\ y)=x^2+4y^2$

15. $f(x,\ y,\ z)=x^2+y^2-z$

16. $f(x,\ y,\ z)=\sqrt{x^2+y^2-z}$

17. $f(x,\ y,\ z)=\dfrac{2}{x^2+y^2+z^2}$

$(\mathbf{18-19})$ Find and sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for the given values of *c*, we refer to these level curves as a contour map.

18. $f(x,y)=x+y-1,\quad c=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$

19. $f(x,y)=x^2+y^2,\quad c=0,\ 1,\ 4,\ 9,\ 16,\ 25$

20. For the function: $f(x,y)=4x^2+9y^2$:

a) Find the function's domain.
b) Find the function's range.
c) Find the function's level curves.
d) Find the boundary of the function's domain.
e) Determine if the domain is an open region, a closed region, or neither.
f) Decide if the domain is bounded or unbounded.

21. For the function: $f(x,y)=xy$:

a) Find the function's domain.
b) Find the function's range.
c) Find the function's level curves.
d) Find the boundary of the function's domain.
e) Determine if the domain is an open region, a closed region, or neither.
f) Decide if the domain is bounded or unbounded.

22. For the function: $f(x,y)=e^{-\left(x^2+y^2\right)}$:

a) Find the function's domain.
b) Find the function's range.
c) Find the function's level curves.
d) Find the boundary of the function's domain.
e) Determine if the domain is an open region, a closed region, or neither.
f) Decide if the domain is bounded or unbounded.

23. For the function: $f(x,y)=\ln\left(9-x^2-y^2\right)$:

a) Find the function's domain.
b) Find the function's range.
c) Find the function's level curves.
d) Find the boundary of the function's domain.
e) Determine if the domain is an open region, a closed region, or neither.
f) Decide if the domain is bounded or unbounded.

24. Find an equation for $f(x,y)=16-x^2-y^2$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $\left(2\sqrt{2},\ \sqrt{2}\right)$

25. Find an equation for $f(x,y)=\dfrac{2y-x}{x+y+1}$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $(-1,\ 1)$

$(\mathbf{26}-\mathbf{29})$ Sketch a typical level surface for the function:

26. $f(x,y,z)=x^2+y^2+z^2$

27. $f(x,y,z)=\ln\left(x^2+y^2+z^2\right)$

28. $f(x,y,z)=y^2+z^2$

29. $f(x,y,z)=z-x^2-y^2$

$(\mathbf{30}-\mathbf{80})$ Find the limit

30. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$

31. $\displaystyle\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

32. $\displaystyle\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$

33. $\displaystyle\lim_{(x,y)\to(0,0)} \cos\frac{x^2+y^3}{x+y+1}$

34. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y\sin x}{x}$

35. $\displaystyle\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x}$

36. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$

37. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y}$

38. $\displaystyle\lim_{\substack{(x,y)\to(2,-4)\\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y-xy+4x^2-4x}$

39. $\displaystyle\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

40. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

41. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

42. $\displaystyle\lim_{(x,y)\to(4,-2)} (10x-5y+6xy)$

43. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x+y}$

44. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{xy}$

45. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2}$

46. $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2}$

47. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2y}{x^4+2y^2}$

48. $\displaystyle\lim_{(x,y)\to(\ln 2,\pi)} e^x\cos y$

49. $\displaystyle\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} e^y\sin x$

50. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2}{x+\cos y}$

51. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$

52. $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{x+y+1}$

53. $\displaystyle\lim_{(x,y)\to(4,-2)} \sqrt[3]{\frac{x^2-y}{x^2+y+1}}$

54. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{2x^2-xy}{4x^2-y^2}$

55. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x+\sin y}{x+y}$

56. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ y\neq x^2}} \frac{y}{x^2-y^2}$

57. $\displaystyle\lim_{(x,y)\to\left(0,\frac{\pi}{2}\right)} \frac{1-\cos xy}{4x^2y^3}$

58. $\lim\limits_{(x,y)\to(0,\ 0)} \dfrac{x^2-2xy+y^2}{x-y}$

59. $\lim\limits_{(x,y)\to(0,\ 0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}\ -1}$

60. $\lim\limits_{(x,y)\to(2,\ 1)} \dfrac{x-y-1}{\sqrt{x-y}\ -1}$

61. $\lim\limits_{(x,y)\to(0,\ 0)} \dfrac{x^4-9y^4}{x^2+3y^2}$

62. $\lim\limits_{(x,y)\to(0,\ 0)} \dfrac{x^3-y^3}{x-y}$

63. $\lim\limits_{(x,y)\to(0,\ 0)} \ln\left(x^2+y^2\right)$

64. $\lim\limits_{(x,y)\to(-1,\ 1)} \ln\left(x^2+y^2\right)$

65. $\lim\limits_{(x,y)\to\left(0,\ \frac{\pi}{4}\right)} \dfrac{\sec x+1}{2x-\tan y}$

66. $\lim\limits_{P\to(1,\ -1,\ 1)} \cos^{-1}(x+y+z)$

67. $\lim\limits_{P\to(1,\ 2,\ 3)} \dfrac{x^2-y^2-z^2}{xyz-2}$

68. $\lim\limits_{P\to(0,\ 0,\ 0)} \dfrac{x^2-y^2-z^2}{x^2+y^2-z^2}$

69. $\lim\limits_{P\to(\frac{\pi}{3},\ e,\ \ln 2)} \ln|y|\ e^z\ \cos 2x$

70. $\lim\limits_{P\to(4,\ -1,\ 3)} \sqrt{13-x^2-2y^2+z^2}$

71. $\lim\limits_{P\to(e,\ 1,\ -1)} \ln|x+y+z|$

72. $\lim\limits_{P\to(\ln 2,\ 1,\ -1)} e^{x+y+z}$

73. $\lim\limits_{P\to(0,\ 1,\ -1)} e^{x^2+y^2+z^2}$

74. $\lim\limits_{P\to(2,\ 1,\ -1)} \dfrac{xy^2-3z}{3x+4y-2z}$

75. $\lim\limits_{P\to(\pi,\ 0,\ 2)} z\,e^{-2y}\cos 2x$

76. $\lim\limits_{P\to(2,\ -3,\ 6)} \ln\sqrt{x^2+y^2+z^2}$

77. $\lim\limits_{P\to(1,\ 3,\ 4)} \left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$

78. $\lim\limits_{P\to(1,\ -1,\ -1)} \dfrac{2xy+yz}{x^2+z^2}$

79. $\lim\limits_{(x,\ y,\ z)\to\left(\frac{\pi}{2},\ 0,\ \frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz}$

80. $\lim\limits_{(x,\ y,\ z)\to(5,\ 2,\ -3)} \tan^{-1}\left(\dfrac{x+y^2}{z^2}\right)$

(81 – 86) At what points (x, y, z) in space are the functions continuous

81. $f(x,y,z)=x^2+y^2-2z^2$

82. $f(x,y,z)=\sqrt{x^2+y^2-1}$

83. $f(x,y,z)=\ln(xyz)$

84. $f(x,y,z)=e^{x+y}\cos z$

85. $h(x,y,z)=\dfrac{1}{|y|+|z|}$

86. $h(x,y,z)=\dfrac{1}{z-\sqrt{x^2+y^2}}$

Section 10.2 – Partial Derivatives

10.2−1 Partial Derivatives of a Function of Two Variables

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ ***rather*** than the **d** symbol.

10.2−2 *Definition*

The ***partial derivative*** of $f(x, y)$ with ***respect to x*** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exists.

The partial derivative with respect to y is denoted:

$$\frac{\partial f}{\partial y}(x_0, y_0), \quad f_y(x_0, y_0), \quad \frac{\partial f}{\partial y}, \quad f_y$$

Example 1

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,\ -5)$ if $f(x,\ y) = x^2 + 3xy + y - 1$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right)$$

$$= 2x + 3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,\ -5)} = 2(4) + 3(-5)$$

$$= -7$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right)$$

$$= 3x + 1 \Big|_{(4,\ -5)}$$

$$= 13$$

Example 2

Find $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ as a function if $f(x,\ y) = y \sin xy$

Solution

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy)$$

$$= \sin xy \frac{\partial}{\partial y}(y) + y \frac{\partial}{\partial y}(\sin xy)$$

$$= \sin xy + (y \cos xy)\frac{\partial}{\partial y}(xy)$$

$$= \sin xy + xy \cos xy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y \sin xy)$$

$$= y^2 \cos xy$$

Example 3

Find f_x and f_y as a function if $f(x, y) = \dfrac{2y}{y+\cos x}$

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)(2) - (2y)(1)}{(y+\cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

OR

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{\cos x + y}\right) \qquad \left(\frac{\alpha + b}{\beta + d}\right)' = \frac{\alpha' d - \beta' b}{(\beta + d)^2} \qquad \begin{vmatrix} 0 & 2y \\ (\cos x)' & y \end{vmatrix}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) \qquad \left(\frac{ay+b}{cy+d}\right)' = \frac{ad - bc}{(cy+d)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

Example 4

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exist.

Find $\frac{\partial y}{\partial x}$, $\frac{\partial z}{\partial y}$

Solution

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\left(\frac{yz-1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{z}{yz-1}}$$

$$\frac{\partial}{\partial x}(yz - \ln z = x + y)$$

$$z\frac{\partial y}{\partial x} = 1 + \frac{\partial y}{\partial x}$$

$$(z-1)\frac{\partial y}{\partial x} = 1$$

$$\boxed{\frac{\partial y}{\partial x} = \frac{1}{z-1}}$$

$$\frac{\partial}{\partial y}(yz - \ln z = x + y)$$

$$z + y\frac{\partial z}{\partial y} - \frac{1}{z}\frac{\partial z}{\partial y} = 1$$

$$\left(\frac{yz-1}{z}\right)\frac{\partial z}{\partial y} = 1 - z$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{z - z^2}{yz-1}}$$

Example 5

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,\ 2,\ 5)$.

Solution

The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1,\ 2)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\ 2)} = \left.\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right|_{(1,\ 2)}$$

$$= 2y \Big|_{(1,\ 2)}$$

$$= 4$$

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola.

$\Rightarrow\ z = 1 + y^2$

$$\left.\frac{\partial z}{\partial y}\right|_{y=2} = \left.\frac{\partial}{\partial y}\left(1 + y^2\right)\right|_{y=2}$$

$$= 2y \Big|_{y=2}$$

$$= 4$$

10.2−3 Functions of More than Two Variables

The partial derivatives of more than two variables are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example 6

If *x*, *y*, and *z* are independent variables and $f(x, y, z) = x\sin(y+3z)$.

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\sin(y+3z)\right)$$
$$= \sin(y+3z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x\sin(y+3z)\right)$$
$$= x\cos(y+3z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x\sin(y+3z)\right)$$
$$= 3x\cos(y+3z)$$

Example 7

If resistors of R_1, R_2, *and* R_3 ohms are connected in parallel to make an *R*-ohm resistor, the value of *R* can be found from the equation.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30\ \Omega$, $R_2 = 45\ \Omega$, *and* $R_3 = 90\ \Omega$

Solution

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = \frac{\partial}{\partial R_2}\left(\frac{1}{R_2}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = -\frac{1}{R_2^2}$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \left(\frac{R}{R_2}\right)^2$$

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{6}{90} = \frac{1}{15}$$

$$R = 15$$

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2$$

$$= \frac{1}{9}$$

A small change in the resistance R_2 leads to a change in R about $\frac{1}{9}th$ as large.

10.2−4 Partial Derivatives and Continuity

Example 8

Let $\quad f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

a) Find the limit of f as (x, y) approaches (0, 0) along the line $y = x$.
b) Prove that f is not continuous at the origin.
c) Show that both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin.

Solution

a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin)

$$\lim_{(x,y)\to(0,0)} f(x,y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0$$

$$= 0$$

b) Since $f(0,\ 0)=1$, and the limit proves that *f* is ***not*** continuous at $(0,\ 0)$.

c) $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = \dfrac{\partial}{\partial x}1\Big|_{(0,0)}$

$$= 0$$

It is the slope of the line at any *x*.

∴The slope of the line at any *y*:

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

10.2−5 Second-Order Partial Derivatives

The second-order derivatives are denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx},\quad \frac{\partial^2 f}{\partial y\partial x} \text{ or } f_{xy},\quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},\quad \text{and}\quad \frac{\partial^2 f}{\partial x\partial y} \text{ or } f_{yx}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),\qquad \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\dfrac{\partial^2 f}{\partial y\partial x}$ *or* $f_{xy} = \left(f_x\right)_y$ Differentiate first with respect to *x*, then with respect to *y*.

Example 9

If $f(x,\ y) = x\cos y + ye^x$. Find the second derivatives $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y\partial x}$, $\dfrac{\partial^2 f}{\partial y^2}$, *and* $\dfrac{\partial^2 f}{\partial x\partial y}$

Solution

$$\frac{\partial f}{\partial x} = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos y + ye^x\right)$$

$$= ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\cos y + ye^x\right)$$

$$= -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(-x\sin y + e^x\right)$$

$$= -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right)$$

$$= -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^x$$

10.2–6 *Theorem* – The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, *and* f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example 10

Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$, then find $\frac{\partial^2 w}{\partial y \partial x}$

Solution

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(xy + \frac{e^y}{y^2+1}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(x + \frac{e^y\left(y^2+1\right) - 2ye^y}{\left(y^2+1\right)^2} \right)$$

$$= 1$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1} \right) \right]$$

$$= \frac{\partial}{\partial y}(y)$$

$$= 1$$

Example 11

Find second partial derivatives: $f(x,y,z) = x^2y + 2xz^2 - 3y^2z$

Solution

$f_x = 2xy + 2z^2$ $\quad$ $f_y = x^2 - 6yz$ $\quad$ $f_z = 4xz - 3y^2$

$f_{xx} = 2y$ $\quad$ $f_{yy} = -6z$ $\quad$ $f_{zz} = 4x$

$f_{xy} = f_{yx} = 2x$

$f_{xz} = f_{zx} = 4z$

$f_{zy} = f_{yz} = -6y$

Example 11

Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$

Solution

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

Exercises *Section* 10.2 – Partial Derivatives

(1 – 39) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

1. $f(x,y)=2x^2-3y-4$

2. $f(x,y)=x^2-xy+y^2$

3. $f(x,y)=x^2+2y^2-x^2y^2$

4. $f(x,y)=x^3-2x^2y^2+y^3$

5. $f(x,y)=5xy-7x^2-y^2+3x-6y+2$

6. $f(x,y)=(xy-1)^2$

7. $f(x,y)=\left(x^3+\frac{y}{2}\right)^{2/3}$

8. $f(x,y)=\frac{1}{x+y}$

9. $f(x,y)=\frac{x}{x^2+y^2}$

10. $f(x,y)=\tan^{-1}\frac{y}{x}$

11. $f(x,y)=e^{-x}\sin(x+y)$

12. $f(x,y)=e^{xy}\ln y$

13. $f(x,y)=\sin^2(x-3y)$

14. $f(x,y)=\cos^2\left(3x-y^2\right)$

15. $f(x,y)=x^y$

16. $f(x,y)=3x^2y^5$

17. $f(x,y)=x\cos y-y\sin x$

18. $f(x,y)=\cos\left(x^2y-3xy+2\right)$

19. $f(x,y)=\tan\left(x^3-3x^2y^2+2y^3\right)$

20. $f(x,y)=\frac{x^2}{x^2+y^2}$

21. $f(x,y)=xy\,e^{xy}$

22. $f(x,y)=\ln\sqrt{x^2+y^2}$

23. $f(x,y)=\frac{1}{2}\ln\left(x^2+y^2\right)+\tan^{-1}\frac{y}{x}$

24. $f(x,y)=\sqrt{4-x^2-y^2}$

25. $f(x,y)=\sin\left(x^2y-2x\right)+\cos\left(x^2y+xy\right)$

26. $f(x,y)=\frac{xy}{x+y}$

27. $f(x,y)=\sec\left(x^2-y^2\right)$

28. $f(x,y)=\tanh(2x+3y)$

29. $f(x,y)=\cosh\left(\frac{x}{y}\right)$

30. $f(x,y)=\sinh\left(\frac{x}{x^2+y^2}\right)$

31. $f(x,y)=\operatorname{sech}\left(x^2+y^2\right)$

32. $f(x,y)=\tan^{-1}\left(x^2y\right)-\tan^{-1}\left(xy^2\right)$

33. $f(x,y)=e^{x^2y}$

34. $f(x,y)=e^{2x^2-y}$

35. $f(x,y)=\frac{x+y}{x^2+y^2}$

36. $f(x,y)=\frac{x-y}{x+y}$

37. $f(x,y)=\frac{2xy}{x^2+y^2}$

38. $f(x,y)=\frac{y}{x^2}+\frac{x}{y^2}$

39. $f(x,y)=\frac{y^2}{x}-\frac{x^2}{y}$

$(\mathbf{40-59})$ Find f_x, f_y, *and* f_z :

40. $f(x,y,z)=1+xy^2-2z^2$

41. $f(x,y,z)=xy+yz+xz$

42. $f(x,y,z)=x-\sqrt{y^2+z^2}$

43. $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

44. $f(x,y,z)=\sec^{-1}(x+yz)$

45. $f(x,y,z)=\ln(x+2y+3z)$

46. $f(x,y,z)=e^{-\left(x^2+y^2+z^2\right)}$

47. $f(x,y,z)=\tanh(x+2y+3z)$

48. $f(x,y,z)=\sinh\left(xy-z^2\right)$

49. $f(x,\ y,\ z)=4xyz^2-\dfrac{3x}{y}$

50. $f(x,\ y,\ z)=\dfrac{xyz}{x+y}$

51. $f(x,\ y,\ z)=e^{x+2y+3z}$

52. $f(x,\ y,\ z)=x^2\sqrt{y+z}$

53. $f(x,\ y,\ z)=x^2-3xyz+y^2+yz^2$

54. $f(x,\ y,\ z)=\dfrac{x^2y-4xyz+y^2}{x+4yz}$

55. $f(x,y,z)=\sin\left(x^2y+z\right)+\cos\left(x^2y-yz\right)$

56. $f(x,y,z)=xe^{-2yz}+\cos(xz-yz)$

57. $f(x,y,z)=\ln\left(\sqrt{x^2+y^2+z^2}\right)$

58. $f(x,y,z)=\sec(xyz)$

59. $f(x,y,z)=x^2z-y^2z+xz^2$

$(\mathbf{60-69})$ Find partial derivatives of the function with respect to each variable:

60. $g(r,\theta)=r\cos\theta+r\sin\theta$

61. $f(x,y)=\frac{1}{2}\ln\left(x^2+y^2\right)+\tan^{-1}\dfrac{y}{x}$

62. $h(x,y,z)=\sin(2\pi x+y-3z)$

63. $f(r,l,T,w)=\dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi w}}$

64. $f(x,y,t)=\pi^2\sin(2\pi x)\sin(2\pi y)\cos(10\pi t)$

65. $u(x,y,t)=2\sin\left(\frac{1}{3}x\right)\sin\left(\frac{1}{4}y\right)e^{-25t/16}$

66. $f(x,y,\theta)=x^2+y^2-2xy\cos\theta$

67. $f(x,y,z)=\cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

68. $P(\rho,\varphi,\theta)=\dfrac{1}{\rho^2}P_0\cos^2\theta\sin^4\varphi$

69. $P(\rho,\varphi,\theta)=\rho^2\cos\theta\sin\varphi+2\rho\sec\theta$

(70 – 95) Find all the *second*-order partial derivatives of:

70. $f(x, y) = x + y + xy$

71. $f(x, y) = \sin xy$

72. $g(x, y) = x^2 y + \cos y + y \sin x$

73. $r(x, y) = \ln(x + y)$

74. $w = x^2 \tan(xy)$

75. $w = y e^{x^2 - y}$

76. $g(x, y) = y + \dfrac{x}{y}$

77. $g(x, y) = e^x + y \sin x$

78. $f(x, y) = y^2 - 3xy + \cos y + 7e^y$

79. $f(x, y) = y^2 e^{xy} + \dfrac{y}{x}$

80. $f(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$

81. $f(x, y) = x^2 + 2y^3 - xy^2$

82. $f(x, y) = \cos\left(x^2 y - 2xy - 3\right)$

83. $f(x, y) = \tan\left(x^3 - 2x^2 y^2 + 3y^4\right)$

84. $f(x, y) = x^2 - 2xy + 2y^2 + 3x + 25$

85. $f(x, y) = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

86. $f(x, y, z) = xyz$

87. $f(x, y, z) = xy + yz + xz$

88. $f(x, y, z) = x^2 + y^2 + z^2 - 9$

89. $f(x, y, z) = x^3 yz + 2x^2 y^2 + 3xz - 5yz^2 + 2$

90. $f(x, y, z) = x^2 yz^3 - 3xy^2 z + 3x^2 z - 4$

91. $f(x, y, z) = \sin\left(x^2 y - yz\right) - \cos\left(x^2 y + z\right)$

92. $f(x, y, z) = e^{-3x} \sin\left(yz^2\right)$

93. $f(x, y, z) = (\ln x) \sec\left(y^2 + z\right)$

94. $f(x, y, z) = \sinh(xyz)$

95. $f(x, y, z) = \tan(x + y + z)$

(96 – 105) Verify that the function satisfies Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

96. $u(x, y) = x^2 - y^2$

97. $u(x, y) = y\left(3x^2 - y^2\right)$

98. $u(x, y) = \ln\left(x^2 + y^2\right)$

99. $u(x, y) = e^x \sin y$

100. $u(x, y) = e^{-2x} \cosh y$

101. $u(x, y) = \sinh x \cos y$

102. $u(x, y) = e^{x+y}$

103. $u(x, y) = \tan\left(\dfrac{y}{x}\right)$

104. $u(x, y) = \sin^{-1}\left(\dfrac{x}{y}\right)$

105. $u(x, y) = e^{\ln(x+y)}$

$(\mathbf{106-110})$ The three-dimensional ***Laplace equation*** $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$

Is satisfied by steady-state temperature distribution $T=f(x,y,z)$ in space, by gravitational potentials, and by electrostatic potentials. Show that the function satisfies a Laplace equation.

106. $f(x,\ y,\ z)=x^2+y^2+z^2$

107. $f(x,\ y,\ z)=2z^3-3\left(x^2+y^2\right)z$

108. $f(x,\ y,\ z)=e^{-2y}\cos 2x$

109. $f(x,\ y,\ z)=\ln\sqrt{x^2+y^2}$

110. $f(x,\ y,\ z)=\tan^{-1}\dfrac{x}{y}$

111. Let $f(x,y)=2x+3y-4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the

a) Plane $x=2$
b) Plane $y=-1$.

112. Let $w=f(x,y,z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\dfrac{\partial f}{\partial y}$ at $\left(x_0,y_0,z_0\right)$. Use this definition to find $\dfrac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x,y,z)=-2xy^2+yz^2$.

113. Find the value of $\dfrac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation $xz+y\ln x-x^2+4=0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

114. Express A implicitly as a function of a, b, and c and calculate $\dfrac{\partial A}{\partial a}$ and $\dfrac{\partial A}{\partial b}$.

115. An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t)=\sin(\alpha x)\cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

Section 10.3 – Chain Rule

10.3–1 *Theorem* – Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ is differentiable and if $x = x(t),\ y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \bullet x'(t) + f_y(x(t), y(t)) \bullet y'(t)$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t,\quad y = \sin t$.

What is the derivative's value at $t = \frac{\pi}{2}$?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= y(-\sin t) + x(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t
\end{aligned}$$

$$\begin{aligned}
w &= xy \\
&= \cos t \sin t \\
&= \frac{1}{2}\sin 2t
\end{aligned}$$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{1}{2}(2\cos 2t) \\
&= \cos 2t
\end{aligned}$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos 2\left(\frac{\pi}{2}\right)$$
$$= \cos(\pi)$$
$$= -1$$

10.3–2 *Theorem* – Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and if $x, y, \text{ and } z$ are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 2

Find $\frac{dw}{dt}$ if $w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t$

In this example the values of $w(t)$ are changing along the path of a helix as t changes.

What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$
$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$
$$= -\sin^2 t + \cos^2 t + 1$$
$$= \cos 2t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos(0) + 1$$
$$= 2$$

10.3–3 *Theorem* – Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example 3

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)\\ &= \frac{1}{s} + 4r + (4r)(2)\\ &= \frac{1}{s} + 12r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\ &= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)\\ &= -\frac{r}{s^2} + \frac{2}{s}\end{aligned}$$

- If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Example 4

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2, \quad x = r - s, \quad y = r + s$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r-s) + 2(r+s) \\ &= 2r - 2s + 2r + 2s \\ &= 4r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r-s) + 2(r+s) \\ &= -2r + 2s + 2r + 2s \\ &= 4s\end{aligned}$$

- If $w = f(x),\ x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$

10.3–4 *Theorem* – A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z} \qquad \frac{dz}{dy} = -\frac{F_y}{F_z}$$

Example 5

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$

Solution

$$F(x, y) = y^2 - x^2 - \sin xy$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}$$

Example 6

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at $(0,\ 0,\ 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$

Solution

$$F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$$

$$F_x = 3x^2 + yze^{xz}$$

$$F_y = e^{xz} - z\sin y$$

$$F_z = 2z + xye^{xz} + \cos y$$

$$F(0,\ 0,\ 0) = 0 \quad F_z = 1 \neq 0$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z}$$

$$= -\frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y}\Bigg|_{(0,\ 0,\ 0)}$$

$$= -\frac{0}{1}$$

$$= 0$$

$$\frac{dz}{dy} = -\frac{F_y}{F_z}$$

$$= -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}\Bigg|_{(0,\ 0,\ 0)}$$

$$= -1$$

Exercises *Section* 10.3 – Chain Rule

(1 – 15) Express $\frac{dz}{dt}$ as a function of t (in two different ways):

1. $z = x^2 - y^2, \quad x = \sin t, \quad y = \cos t$

2. $z = x^2 + y^2, \quad x = 1 - 2\sin 2\pi t, \quad y = 1 + \cos 2\pi t$

3. $z = x^2 + y^2, \quad x = t^2, \quad y = \frac{1}{t}$

4. $z = 2xy + y^2, \quad x = -3t^2, \quad y = 1 + t^3$

5. $z = 4x + xy^2, \quad x = e^{3t}, \quad y = e^{2t}$

6. $z = x^2 y, \quad x = \tan 3t, \quad y = \sec 3t$

7. $z = \sqrt{x^2 + y^2}, \quad x = \cos at, \quad y = \sin at$

8. $z = x^2 + 4xy + 4y^2, \quad x = \cos t, \quad y = \sin t$

9. $z = e^x \sin y, \quad x = ts^2, \quad y = t + 2s$

10. $z = x\sin y - y\cos x, \quad x = t, \quad y = t^2$

11. $z = x^2 - 2xy + y^2, \quad x = e^t, \quad y = \ln t$

12. $z = x^3 + x^2 y^2 + y^3, \quad x = -\sqrt{t}, \quad y = t$

13. $z = x^3 - y^3, \quad x = \sqrt{t+1}, \quad y = \sqrt{t}$

14. $z = \ln x + \ln y, \quad x = e^{t+1}, \quad y = e^t$

15. $z = x^2 - y^2, \quad x = 2\cosh t, \quad y = 2\sinh t$

(16 – 30) Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

16. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

17. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

18. $w = x^4 - y^4, \quad x = 4\sinh(2t), \quad y = 4\cosh(2t), \quad t = 0$

19. $w = xy^3 - x^3 y, \quad x = e^t, \quad y = \ln t, \quad t = 1$

20. $w = x^2 y^2, \quad x = 3t^2, \quad y = 2t^3, \quad t = 1$

21. $w = x^2 + xy + y^2, \quad x = 3t, \quad y = -2t, \quad t = 1$

22. $w = \ln\left(x^2 + y^2\right), \quad x = e^{-t}, \quad y = e^t, \quad t = 1$

23. $w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

24. $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$, $t = 1$

25. $w = \sin(xy + \pi)$, $x = e^t$, $y = \ln(t+1)$ $t = 0$

26. $w = xe^y + y\sin z - \cos z$, $x = 2\sqrt{t}$, $y = t - 1 + \ln t$, $z = \pi t$, $t = 1$

27. $w = xy + yz + xz$, $x = e^t$, $y = e^{-t}$, $z = e^{-t} + e^t$, $t = 1$

28. $w = \ln\left(x + 2y - z^2\right)$, $x = 2t$, $y = \frac{1}{t}$, $z = 4\sqrt{t}$, $t = 1$

29. $w = \sin xyz$, $x = 1 - 3t$, $y = 4t$, $z = e^{t-1}$, $t = 1$

30. $w = x\,e^{yz}$, $x = \tan 3t$, $y = \cos t$, $z = \sin t$, $t = 1$

(31 – 32) Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point

31. $z = 4e^x \ln y$, $x = \ln(u\cos v)$, $y = u\sin v$ at $(u,v) = \left(2,\ \frac{\pi}{4}\right)$.

32. $z = 4x - y^2$, $x = uv^2$, $y = u^2 v$ at $(u,v) = (1,\ 1)$.

(33 – 34) Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point

33. $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$ at $(u,v) = \left(\frac{1}{2},\ 1\right)$.

34. $w = 4x + y^2 + z^2$, $x = \frac{u}{v}$, $y = uv$, $z = u + v$ at $(u,v) = (0,\ 1)$.

35. Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of x, y and z if

$u = e^{qr}\sin^{-1} p$, $p = \sin x$, $q = z^2 \ln y$, $r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point

$(x,y,z) = \left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)$.

(36 – 38) Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

36. $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,\ 1,\ 1)$

37. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point $(\pi,\ \pi,\ \pi)$

38. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1,\ \ln 2,\ \ln 3)$

$(\mathbf{39-40})$ Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ if

39. $w=(x+y+z)^2,\quad x=r-s,\quad y=\cos(r+s),\quad z=\sin(r+s)$ when $r=1,\ s=-1$

40. $w=\sin(2x-y),\quad x=r+\sin s,\quad y=rs$ when $r=\pi,\ s=0$

$(\mathbf{41-43})$ Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when:

41. $z=\sin xy+x\sin y,\quad x=u^2+v^2,\quad y=uv$ $\quad u=0,\ v=1$

42. $z=5\tan^{-1}x,\quad x=e^u+\ln v,\quad u=\ln 2,\ v=1$

43. $z=\ln q,\quad q=\sqrt{v+3}\ \tan^{-1}u,\quad u=1,\ v=-2$

44. Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

$(\mathbf{45-48})$ Evaluate the derivatives:

45. $w'(t)$, where $w=xy\sin z,\quad x=t^2,\quad y=4t^3,\quad and\ \ z=t+1$

46. $w'(t)$, where $w=\sqrt{x^2+y^2+z^2},\quad x=\sin t,\quad y=\cos t,\quad and\ \ z=\cos t$

47. w_s and w_t, where $w=xyz,\quad x=2st,\quad y=st^2,\quad and\ \ z=s^2t$

48. w_r, w_s, and w_t, where $w=\ln\left(xy^2\right),\quad x=rst,\quad and\ \ y=r+s$

49. The voltage V in a circuit that satisfies the law $V=IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation:

$$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$$

To find how the current is changing at the instant when $R=600\ \Omega$, $I=0.04A$, $\frac{dR}{dt}=0.5\ ohm/\sec$, and $\frac{dV}{dt}=-0.01\ volt/\sec$.

50. The lengths *a*, *b*, and *c* of the edges of a rectangular box are changing with time. At the instant in question, $a = 1\ m$, $b = 2\ m$, $c = 3\ m$, $\frac{da}{dt} = \frac{db}{dt} = 1\ m/\sec$, and $\frac{dc}{dt} = -3\ m/\sec$. At what rates the box's volume *V* and surface area *S* changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

51. Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of *T* on the circle.

(52 – 63) Evaluate $\frac{dy}{dx}$:

52. $x^2 - 2y^2 - 1 = 0$

53. $x^3 + 3xy^2 - y^5 = 0$

54. $2\sin xy = 1$

55. $ye^{xy} - 2 = 0$

56. $\sqrt{x^2 + 2xy + y^4} = 3$

57. $y \ln\left(x^2 + y^2 + 4\right) = 3$

58. $y \ln\left(x^2 + y^2\right) = 4$

59. $2x^2 + 3xy - 3y^4 = 2$

60. $y = \sin(x + y) + \cos(x - y)$

61. $x = \sec(x - y) + \tan(x + y)$

62. $x \cos y = y + \sin x$

63. $x^3 - 2xy + y^3 = 0$

(63 – 70) Evaluate $\frac{dy}{dx}$, $\frac{dy}{dz}$, $\frac{dz}{dx}$:

64. $\ln(x + y) = y^2 + 2z$

65. $x \tan y = y \sec z$

66. $z^2 + \tan x = \sec y$

67. $yz\, e^{xy} - 2\ln(xy) = z^2$

68. $\ln\left(x^2 + y^2\right) = y\, e^z$

69. $z = \sin(x + y) + \cos(x - y)$

70. $\sqrt{x^2 + y^2} = 3z + 2xyx$

(71 – 74) Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point.

71. $z^3 - xy + yz + y^3 - 2 = 0$; $(1,\ 1,\ 1)$

72. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0$; $(2,\ 3,\ 6)$

73. $\sin(x+y)+\sin(y+z)+\sin(x+z)=0$; $(\pi,\ \pi,\ \pi)$

74. $xe^{y}+ye^{z}+2\ln x-2-3\ln 2=0$; $(1,\ \ln 2,\ \ln 3)$

75. Consider the surface and parameterized curves C in the xy-plane.

$$z=4x^2+y^2-2;\quad C:\ x=\cos t,\quad y=\sin t,\quad \textit{for}\ \ 0\le t\le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of $\boldsymbol{t}$ for which you are walking uphill.

76. Consider the surface and parameterized curves C in the xy-plane.

$$z=4x^2-2y^2+4;\quad C:\ x=2\cos t,\quad y=2\sin t,\quad \textit{for}\ \ 0\le t\le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of $\boldsymbol{t}$ for which you are walking uphill.

77. A demand of two name different brands items is modeled by the function $D(x,y)$ *gal / month*

$$D(x,y)=210-12x^2+18y;\quad x=4+.18t,\quad y=5+.3\sqrt{t},\quad \textit{for}\ \ t\ge 0$$

Where x, y are the prices of the first and second brand.

At what rate will the demand D will be changing with respect to time 9 *months* from now?

78. Find the value of the derivative of $f(x,y,z)=xy+yz+xz$ with respect to t on the curve $x=\cos t,\quad y=\sin t,\quad z=\cos 2t$ at $t=1$

79. Define y as a differentiable function of x for $2xy+e^{x+y}-2=0$, find the values of $\frac{dy}{dx}$ at point $P(0,\ \ln 2)$

80. A point P is moving along an intersection curve of a circle $x^2+y^2=5$ and the hyperbolic $z=\frac{x^2}{16}-\frac{y^2}{9}$. If x is increasing at the rate of 0.2 *in/min*, how fast is z changing when $x=2$.

81. The altitude of a right circular cone is 16 *inches* and is increasing $0.2 \ \ in/min$, the radius is 8 *inches* and is increasing $0.3 \ \ in/min$. How fast is the volume changing?

82. A radius r of a right circular cylinder is increasing at the rate of $2 \ \ in/min$ and the height h is decreasing at the rate of $3 \ \ in/min$.

a) At what rate is the volume of the cylinder changing when the radius is 8 *inches*, and the height is 6 *inches*.

b) At what rate is the volume of the cylinder changing when the radius is 6 *inches*, and the height is 8 *inches*.

c) At what rate is the volume of the cylinder changing when the radius is 8 *inches*, and the height is 4 *inches*.

d) Assume that the radius and height of the cylinder is given in function of time, $r = \frac{1}{2}t \ \ and \ \ h = \frac{1}{3}t$, both are increasing with time. How fast is the volume increasing when $r = 8 \ ft. \ \ and \ \ h = 6 \ ft.$

83. A police cruiser, approaching a right-angle intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. At that time, the cruiser is 0.4 *mi* north of the intersection and the car is 0.6 *mi* to the east. while the cruiser is moving at speed 65 *mph* the instant of measurement and the speed of the car is at 70 *mph*.

What is the rate of the distance between police and the car.

84. The volume of a frustum of a cone is given by: $V = \frac{\pi}{3}\left(r_1^2 + r_2^2 + r_1 r_2\right)h$ (see the picture below).

Find the rate of change of the volume when $r_1 = 8\ in.,\quad r_2 = 12\ in.,\quad and\quad h = 16\ in.$ and the rate of the change of the radius are equal and given $a\ \ in/sec$, while the height remains constant.

85. Water runs into a conical tank; the tank stands point down and has a height of 10 *feet* and is decreasing at rate of $3\ \ ft^3/min$ and a base radius of 4 *feet* decreasing at rate of $2\ \ ft^3/min$. What rate is the volume of the conical tank?

86. An electrostatic potential $\left(U_E = k\frac{q_1 q_2}{r}\right)$ along an inner circle with a region:

$D_1 = \left\{(x, y) \mid \ (x-2)^2 + y^2 \le 4\right\}$ is kept at 50 *volts*. While the electrostatic potential along an outer circle with a region: $D_2 = \left\{(x, y) \mid \ (x-1)^2 + y^2 \le 1\right\}$ is kept at 100 *volts*.

The electrostatic potential at any point (x, y) is given by:

$$U(x, y) = 150 - \frac{200x}{x^2 + y^2}$$

Find $U_x\ \ and\ \ U_y$

Section 10.4 – Directional Derivatives and The Gradient

10.4–1 Directional Derivatives in the Plane

The rate at which *f* changes with respect to *t* along a differentiable curve $x = g(t),\ y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that the function $f(x, y)$ is defined throughout a region *R* in the *xy*-plane, that $P_0(x_0, y_0)$ is a point in *R*, and that $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector. Then the equations

10.4–2 *Definition*

The derivative of *f* at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\vec{u},\, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

The directional derivative is also noted by: $\left(D_{\vec{u}} f\right)_{P_0}$

Example 1

Find the derivative of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction of the unit vector

$\vec{u} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Solution

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s\to 0} \frac{f\left(x_0+su_1,\, y_0+su_2\right)-f\left(x_0,\, y_0\right)}{s}$$

$$= \lim_{s\to 0} \frac{f\left(1+s\frac{1}{\sqrt{2}},\ 2+s\frac{1}{\sqrt{2}}\right)-f\left(1,\ 2\right)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1+\frac{s}{\sqrt{2}}\right)^2+\left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right)-\left(1^2+1\bullet 2\right)}{s}$$

$$= \lim_{s\to 0} \frac{1+\frac{2}{\sqrt{2}}s+\frac{1}{2}s^2+2+\frac{3}{\sqrt{2}}s+\frac{1}{2}s^2-3}{s}$$

$$= \lim_{s\to 0} \frac{s^2+\frac{5}{\sqrt{2}}s}{s}$$

$$= \lim_{s\to 0} \left(s+\frac{5}{\sqrt{2}}\right)$$

$$= \frac{5}{\sqrt{2}}$$

The rate of change of $f\left(x,\, y\right)=x^2+xy$ at $P_0\left(1,\ 2\right)$ in the direction $\vec{u}$ is $\frac{5}{\sqrt{2}}$

10.4–3 Interpretation of the Directional Derivative

The equation $z=f\left(x,y\right)$ represents a surface *S* in space. If $z_0=f\left(x_0,y_0\right)$, then the point $P_0\left(x_0,\, y_0,\, z_0\right)$ lies on *S*. The vertical plane that passes through *P* and $P_0\left(x_0,\, y_0,\, z_0\right)$ parallel to $\vec{u}$ intersects *S* in a curve *C*.

When $\vec{u} = \hat{i}$, the directional derivative at P_0 is $\frac{\partial f}{\partial x}$ evaluated at $\left(x_0, y_0\right)$.

When $\vec{u} = \hat{j}$, the directional derivative at P_0 is $\frac{\partial f}{\partial y}$ evaluated at $\left(x_0, y_0\right)$.

The directional derivative generalizes the two partial derivatives.

10.4−4 Calculation and Gradients

$$\begin{aligned}\left(\frac{df}{ds}\right)\Bigg|_{\vec{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} \\ &= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 \\ &= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j}\right]}_{\textit{Gradient of at } P_0} \bullet \underbrace{\left(u_1\hat{i} + u_2\hat{j}\right)}_{\textit{Directional } \vec{u}}\end{aligned}$$

10.4−5 *Definition*

Gradient is an increase/decrease in the magnitude (change in the value) of an object such temperature or pressure with change in position in a given variable to another.

The ***gradient vector*** (*gradient*) of $f(x, y)$ at a point $P_0\left(x_0, y_0\right)$ is the vector:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

obtained by evaluating the partial derivatives of *f* at P_0.

∴ ∇ pronounces "***del***"

10.4−6 *Theorem* – The directional Derivative is a Dot Product

If $f(x, y)$ is differentiable in an open region containing $P_0\left(x_0, y_0\right)$, then:

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \left(\nabla f\right)_{P_0} \bullet \vec{u}$$

The dot product of the gradient ∇f at P_0 and $\vec{u}$.

Example 2

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\vec{v} = 3\hat{i} - 4\hat{j}$

Solution

$$\vec{u} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

The partial derivatives of *f* are continuous and at (2, 0)

$$f_x(2,0) = \left(e^y - y\sin(xy)\right)\Big|_{(2,0)}$$

$$= e^0 - 0 = 1$$

$$= 1$$

$$f_y(2,0) = \left(xe^y - x\sin(xy)\right)\Big|_{(2,0)}$$

$$= 2e^0 - 0$$

$$= 2$$

$$\nabla f = f_x(2,0)\hat{i} + f_y(2,0)\hat{j}$$

$$= \hat{i} + 2\hat{j}$$

Therefore, the derivative of *f* at (2, 0) in the direction of $\vec{v}$ is

$$\left(D_{\vec{u}} f\right)\Big|_{(2,0)} = (\nabla f)_{(2,0)} \bullet \vec{u}$$

$$= \left(\hat{i} + 2\hat{j}\right) \bullet \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= \frac{3}{5} - \frac{8}{5}$$

$$= -1$$

10.4–7 Properties of the Directional Derivative $D_{\vec{u}} f = \nabla f \bullet \vec{u} = |\nabla f| \cos\theta$

1. The function *f* increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\vec{u}$ is the direction of ∇f. That is, at each point *P* in its domain, *f* increases most rapidly in the direction of the gradient vector ∇f at *P*.

The derivative in this direction is:

$$D_{\vec{u}} f = |\nabla f| \cos 0$$
$$= |\nabla f|$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$.

The derivative in this direction is

$$D_{\vec{u}} f = |\nabla f| \cos(\pi)$$
$$= -|\nabla f|$$

3. Any direction $\vec{u}$ orthogonal to a gradient ∇f is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and

$$D_{\vec{u}} f = |\nabla f| \cos\left(\frac{\pi}{2}\right)$$
$$= |\nabla f| \cdot (0)$$
$$= 0$$

Example 3

Find the directions in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) Increases most rapidly at the point (1, 1).
b) Decreases most rapidly at the point (1, 1).
c) What are the directions of zero change in f at (1, 1).

Solution

a) The function increases most rapidly at the point (1, 1).

The gradient is

$$\nabla f = f_x \hat{i} + f_y \hat{j}$$
$$= x\hat{i} + y\hat{j}$$

$$(\nabla f)\Big|_{(1,\,1)} = \left(x\hat{i} + y\hat{j}\right)\Big|_{(1,\,1)}$$
$$= \hat{i} + \hat{j}$$

Its direction is:

$$\vec{u} = \frac{\hat{i} + \hat{j}}{\left|\hat{i} + \hat{j}\right|}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

b) The function decreases most rapidly at the point (1, 1).

The gradient is: $-(\nabla f)_{(1,\,1)} = -\hat{i} - \hat{j}$

Its direction is: $-\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

c) The directions of zero change at (1, 1) are the direction orthogonal to ∇f :

$$\vec{n} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \quad \textit{and} \quad -\vec{n} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

10.4–8 Gradients and Tangents to Level Curves

If a differentiable function $f(x,y)$ has a constant value c along a smooth curve $r = g(t)\hat{i} + h(t)\hat{j}$, then $f(g(t),h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f(g(t),h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}\right)}_{\nabla f}\bullet\underbrace{\left(\frac{dg}{dt}\hat{i}+\frac{dh}{dt}\hat{j}\right)}_{\frac{d\vec{r}}{dt}}=0$$

- At every point $\left(x_0, y_0\right)$ in the domain of a differentiable function $f\left(x, y\right)$, the gradient of f is normal to the level curve through $\left(x_0, y_0\right)$.

Example 4

Find an equation for the tangent to the ellipse $\frac{x^2}{4}+y^2=2$ at the point $\left(-2, 1\right)$

Solution

$$f\left(x,y\right)=\frac{x^2}{4}+y^2$$

The gradient of f at $\left(-2, 1\right)$ is

$$\nabla f = f_x\hat{i}+f_y\hat{j}$$

$$=\frac{1}{2}x\hat{i}+2y\hat{j}$$

$$\nabla f\Big|_{(-2,\,1)}=\left(\frac{x}{2}\hat{i}+2y\hat{j}\right)\Big|_{(-2,\,1)}$$

$$=-\hat{i}+2\hat{j}$$

The equation of the line is given by:

$$f_x\left(x-x_0\right)+f_y\left(y-y_0\right)=0$$

$$\left(-1\right)\left(x-\left(-2\right)\right)+\left(2\right)\left(y-1\right)=0$$

$$-x-2+2y-2=0$$

$$-x+2y=4$$

$$x-2y=-4$$

10.4–9 Algebra Rules for Gradients

Sum Rule: $\nabla(f+g)=\nabla f+\nabla g$

Difference Rule: $\nabla(f-g)=\nabla f-\nabla g$

Constant Multiple Rule: $\nabla(kf)=k\nabla f \qquad \forall k$

Product Rule: $\nabla(fg)=f\nabla g+g\nabla f$

Quotient Rule: $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$

10.4–10 Functions of Three Variables

For a differentiable function $f(x,y,z)$ and a unit $\vec{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ in space, we have

$$\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$

$$\begin{aligned}D_{\vec{u}}f&=\nabla f\bullet\vec{u}\\&=\frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2+\frac{\partial f}{\partial z}u_3\end{aligned}$$

The directional derivative can be written in the form

$$\begin{aligned}D_{\vec{u}}f&=\nabla f\bullet\vec{u}\\&=|\nabla f||\vec{u}|\cos\theta\\&=|\nabla f|\cos\theta\end{aligned}$$

Example 5

a) Find the derivative of $f(x,y,z)=x^3-xy^2-z$ at $P_0(1,\ 1,\ 0)$ in the direction of $\vec{v}=2\hat{i}-3\hat{j}+6\hat{k}$

b) In what directions does *f* change most rapidly at P_0, and what are the rates of change in these directions?

Solution

a) The direction of ***v*** is obtained by dividing ***v*** by its length:

$$\begin{aligned}|\vec{v}|&=\sqrt{2^2+(-3)^2+6^2}\\&=7\end{aligned}$$

$\vec{u} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$ $\qquad\qquad$ $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$

The partial derivatives of f at P_0 are:

$$f_x \Big|_{(1,1,0)} = \left(3x^2 - y^2\right)\Big|_{(1,1,0)}$$
$$= 2$$

$$f_y \Big|_{(1,1,0)} = -2y\Big|_{(1,1,0)}$$
$$= -2$$

$$f_z \Big|_{(1,1,0)} = -1$$

The gradient of f at P_0 is:

$$\nabla f \Big|_{(1,\ 1,\ 0)} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\Big|_{(1,\ 1,\ 0)}$$
$$= 2\hat{i} - 2\hat{j} - \hat{k}$$

Therefore, the derivative of f at P_0 in the direction of $\vec{v}$ is:

$$D_{\vec{u}} f \Big|_{(1,\ 1,\ 0)} = \nabla f \Big|_{(1,\ 1,\ 0)} \bullet \vec{u}$$
$$= \left(2\hat{i} - 2\hat{j} - \hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$
$$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right)$$
$$= \frac{4}{7}$$

b) The function increases most rapidly in the direction of $\nabla f = 2\hat{i} - 2\hat{j} - \hat{k}$ and decreases most rapidly in the direction of $-\nabla f$.

The rates of change in the directions are:

$$|\nabla f| = \sqrt{4+4+1} = 3 \quad \textit{and} \quad -|\nabla f| = -3$$

Exercises *Section* 10.4 – Directional Derivatives and The Gradient

(1 – 3) Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x,y) = y - x$, $(2,\ 1)$

2. $f(x,y) = \ln\left(x^2 + y^2\right)$, $(1,\ 1)$

3. $f(x,y) = \sqrt{2x + 3y}$, $(-1,\ 2)$

(4 – 8) Use the formula $\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ to find the directional derivatives of the given functions at the given point.

4. $z = x^2 + xy + y^2$, $(1,\ 1)$, $\theta = \frac{\pi}{6}$

5. $z = x^2 - 4y^2$, $(5,\ 2)$, $\theta = \frac{\pi}{4}$

6. $z = x^2 + 3y^2$, $(1,\ 2)$, $\theta = \frac{\pi}{3}$

7. $z = x + y\cos xy$, $(0,\ 0)$, $\theta = \frac{\pi}{3}$

8. $z = 2x^2 - 3xy + 2y^2$, $(1,\ -1)$, $\theta = \frac{\pi}{2}$

(9 – 22) Find ∇f at the given point.

9. $f(x,y) = x^2 + y^2$, $(1,\ 1)$

10. $f(x,y) = \sqrt{2x^2 + 3y^2}$, $(3,\ 2)$

11. $f(x,y) = e^{-x}\cos y$, $\left(0,\ \frac{\pi}{4}\right)$

12. $f(x,y) = x^2 + 2xy + y^2$, $(1,\ 2)$

13. $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x$, $(1,\ 1,\ 1)$

14. $f(x,y,z) = 2x^3 - 3\left(x^2 + y^2\right)z + \tan^{-1} xz$, $(1,\ 1,\ 1)$

15. $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1} x$, $\left(0,\ 0,\ \frac{\pi}{6}\right)$

16. $f(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2}$, $(1,\ 1,\ 1)$

17. $f(x,y,z) = \ln\left(x^2 + y^2 + z^2\right)$, $(1,\ 1,\ 1)$

18. $f(x,y,z)=e^{x^2+y^2+z^2}$, $(1,\ 0,\ 1)$

19. $f(x,y,z)=e^{-x-y-2z}$, $(\ln 8,\ \ln 4,\ \ln 2)$

20. $f(\alpha,\ \beta,\ \theta)=\cos(\alpha-\beta)+\sin(\alpha+\beta)+e^{\cos\theta}$, $\left(\frac{\pi}{3},\ \frac{\pi}{6},\ \frac{\pi}{4}\right)$

21. $f(\rho,\ \varphi,\ \theta)=\rho^2\sin\varphi\cos\theta$, $\left(2,\ \frac{\pi}{4},\ \frac{\pi}{2}\right)$

22. $f(x,\ y,\ z)=\sin\pi x\cos y\sin 2z$, $\left(1,\ \frac{\pi}{4},\ \frac{\pi}{3}\right)$

(23 – 37) Find the directional derivative of the function.

23. $f(x,y)=2xy-3y^2$ at $P_0(5,\ 5)$ in the direction of $\vec{v}=4\hat{i}+3\hat{j}$

24. $f(x,y)=\dfrac{x-y}{xy+2}$ at $P_0(1,\ -1)$ in the direction of $\vec{v}=12\hat{i}+5\hat{j}$

25. $h(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)+\sqrt{3}\sin^{-1}\left(\dfrac{xy}{2}\right)$ at $P_0(1,\ 1)$ in the direction of $\vec{v}=3\hat{i}-2\hat{j}$

26. $g(x,y)=e^x\cos 2y$ at $P_0\left(0,\ \frac{\pi}{4}\right)$ in the direction of $\vec{v}=2\hat{i}+3\hat{j}$

27. $f(x,y)=x^3-x^2y^2+xy+y^3$ at $P_0(1,\ -1)$ in the direction of $\vec{v}=\hat{i}-2\hat{j}$

28. $f(x,y)=x^3-y^3$ at $P_0(2,\ -1)$ in the direction of $\vec{v}=\dfrac{1}{\sqrt{2}}\hat{i}+\dfrac{1}{\sqrt{2}}\hat{j}$

29. $f(x,y,\ z)=x^2y^3z^4$ at $P_0(-1,\ 3,\ 2)$ in the direction of $\vec{v}=\hat{i}+\hat{j}+\hat{k}$

30. $f(x,y,\ z)=\sqrt{xyz}$ at $P_0(2,\ 2,\ 4)$ in the direction of $\vec{v}=2\hat{i}+4\hat{j}-4\hat{k}$

31. $f(x,y,z)=xy+yz+zx$ at $P_0(1,\ -1,\ 2)$ in the direction of $\vec{v}=3\hat{i}+6\hat{j}-2\hat{k}$

32. $g(x,y,z)=3e^x\cos yz$ at $P_0(0,\ 0,\ 0)$ in the direction of $\vec{v}=2\hat{i}+\hat{j}-2\hat{k}$

33. $h(x,y,z)=\cos xy+e^{yz}+\ln zx$ at $P\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\vec{v}=\hat{i}+2\hat{j}+2\hat{k}$

34. $f(x,y,\ z)=x\tan^{-1}\left(\dfrac{y}{z}\right)$ at $P_0(3,\ -2,\ 2)$ in the direction of $\vec{v}=\hat{i}+2\hat{j}-\hat{k}$

35. $f(x,y,\ z)=(\cos y+2\sin z)e^x$ at $P_0\left(1,\ \frac{\pi}{3},\ \frac{\pi}{3}\right)$ in the direction of $\vec{v}=\hat{i}-\hat{j}+2\hat{k}$

36. $f(x,y,\ z)=x^2y\cos 2z$ at $P_0\left(-1,\ 2,\ \frac{\pi}{2}\right)$ in the direction of $\vec{v}=\hat{i}+\hat{j}-\hat{k}$

37. $f(x,y,z)=xy+2xz-y^2+z^2$ at $P(1,\ -2,\ 1)$ along the curve $x=t$, $y=t-3$, $z=t^2$ in the direction of increasing z.

$(\mathbf{38-41})$ Find the directions in which the function, then find the derivatives of the function in these directions.

38. $f(x,y)=x^2+xy+y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$.

39. $f(x,y)=x^2y+e^{xy}\sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$.

40. $g(x,y,z)=xe^y+z^2$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$.

41. $h(x,y,z)=\ln\left(x^2+y^2-1\right)+y+6z$ increase and decrease most rapidly at $P_0(1,\ 1,\ 0)$

$(\mathbf{42-44})$ Sketch the given curve, then write an equation for the tangent line

42. $x^2+y^2=4;\quad \left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ \sqrt{2}\right)$.

43. $x^2-y=1;\quad \left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$.

44. $x^2-xy+y^2=7;\quad \left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $(-1,\ 2)$.

45. In what direction is the derivative of $f(x,y)=xy+y^2$ at $P(3,\ 2)$ equal to zero?

$(\mathbf{46-53})$ Compute the gradient of the function, evaluate it at the given point *P*, and evaluate the directional derivative at that point in the given direction.

46. $f(x,\ y)=x^2;\quad P=(1,\ 2);\quad \vec{u}=\left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle$

47. $f(x,\ y)=x^2y^3;\quad P=(-1,\ 1);\quad \vec{u}=\left\langle \frac{5}{13},\ \frac{12}{13}\right\rangle$

48. $f(x,\ y)=\frac{x}{y^2};\quad P=(0,\ 3);\quad \vec{u}=\left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle$

49. $f(x,\ y)=\sqrt{2+x^2+2y^2};\quad P=(2,\ 1);\quad \vec{u}=\left\langle \frac{3}{5},\ \frac{4}{5}\right\rangle$

50. $f(x,\ y,\ z)=xy+yz+xz+4;\quad P=(2,\ -2,\ 1);\quad \vec{u}=\left\langle 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle$

51. $f(x,\ y,\ z)=1+\sin(x+2y-z);\quad P=\left(\frac{\pi}{6},\ \frac{\pi}{6},\ -\frac{\pi}{6}\right);\quad \vec{u}=\left\langle \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right\rangle$

52. $f(x,\ y,\ z)=\ln\sqrt{x^2+y^2+z^2};\quad P=(1,\ 1,\ -1);\quad \vec{u}=\left\langle \frac{1}{6},\ -\frac{2}{6},\ \frac{1}{6}\right\rangle$

53. $f(x,\ y,\ z)=x^2y^2+y^2z^2+x^2z^2;\quad P=(1,\ 1,\ 1);\quad \vec{u}=\left\langle \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right\rangle$

(54 – 57) Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\vec{v}$.

54. $f(x,y)=\cos x\cos y,\quad P_0\left(\frac{\pi}{4},\ \frac{\pi}{4}\right),\quad \vec{v}=3\hat{i}+4\hat{j}$

55. $f(x,y)=x^2e^{-2y},\quad P_0(1,\ 0),\quad \vec{v}=\hat{i}+\hat{j}$

56. $f(x,y,z)=\ln(2x+3y+6z),\quad P_0(-1,\ -1,\ 1),\quad \vec{v}=2\hat{i}+3\hat{j}+6\hat{k}$

57. $f(x,y,z)=x^2+3xy-z^2+2y+z+4,\quad P_0(0,\ 0,\ 0),\quad \vec{v}=\hat{i}+\hat{j}+\hat{k}$

58. Let $f(x,\ y)=\ln(1+xy);\quad P=(2,\ 3)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.
b) Find a unit vector that points in ta direction of no change.

59. Let $f(x,\ y)=\sqrt{4-x^2-y^2};\quad P=(-1,\ 1)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.
b) Find a unit vector that points in ta direction of no change.

(60 – 61) Let $f(x,\ y)=8-2x^2-y^2$, for the level curves $f(x,\ y)=C$ and points $(a,\ b)$, compute the slope of the line tangent to the level curve at $(a,\ b)$ and verify that the tangent line is orthogonal to the gradient at that point.

60. $f(x,\ y)=5;\quad (a,\ b)=(1,\ 1)$

61. $f(x,\ y)=0;\quad (a,\ b)=(2,\ 0)$

62. Find the direction in which the function $f(x,\ y)=4x^2-y^2$ has zero change at the point $(1,\ 1,\ 3)$. Express the directions in terms of unit vectors.

63. An infinitely long charged cylinder of radius *R* with its axis along *z*-axis has an electric potential $V=k\ln\left(\frac{R}{r}\right)$, where *r* is the distance between a variable point $P(x,\ y)$ and the axis of the cylinder $\left(r^2=x^2+y^2\right)$ and *k* is a physical constant. The electric field at a point $(x,\ y)$ in the *xy*-plane is given by $\boldsymbol{E}=-\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point $(x,\ y)$ with $r>R$.

Section 10.5 – Tangent Planes and Linear Approximation

10.5–1 Tangent Planes and Normal Lines

If $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function *f*, then $f(g(t), h(t), k(t)) = c$.

Differentiating both sides of this equation with respect to *t* leads to

$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \bullet \underbrace{\left(\frac{dg}{dt}\hat{i} + \frac{dh}{dt}\hat{j} + \frac{dk}{dt}\hat{k}\right)}_{\frac{d\vec{r}}{dt}} = 0$$

10.5–2 *Definition*

The ***tangent plane*** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function *f* is the plane through P_0 normal to $\nabla f \big|_{P_0}$.

The ***normal line*** of the surface at P_0 is the line through P_0 parallel to $\nabla f \big|_{P_0}$.

Normal Line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$

Tangent Plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Example 1

Find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $P_0(1,\ 2,\ 4)$.

Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0.

The gradient is:

$$\nabla f\Big|_{P_0}=\left(2x\hat{i}+2y\hat{j}+\hat{k}\right)\Big|_{(1,\ 2,\ 4)}$$

$$=2\hat{i}+4\hat{j}+\hat{k}$$

The tangent plane is the plane:

$$2(x-1)+4(y-2)+(z-4)=0$$

$$2x+4y+z=14$$

The line normal to the surface at P_0 is:

$$x=1+2t,\quad y=2+4t,\quad z=4+t$$

10.5–3 Plane Tangent to a Surface $z=f(x,y)$ at $\left(x_0,\ y_0,\ f(x_0,y_0)\right)$

The plane tangent to the surface $z=f(x,y)$ of a differentiable function f at the point $P_0(x_0,\ y_0,\ z_0)=\left(x_0,\ y_0,\ f(x_0,y_0)\right)$ is:

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

Example 2

Find the plane tangent and the normal line to the surface $z=x\cos y-ye^x$ at (0, 0, 0)

Solution

$$f(x,y,z)=x\cos y-ye^x-z$$

$$\nabla f=\left(\cos y-ye^x\right)\hat{i}+\left(-x\sin y-e^x\right)\hat{j}-\hat{k}\Big|_{(0,\ 0)}$$

$$=\hat{i}-\hat{j}-\hat{k}$$

Therefore, the tangent plane is:

$$1(x-0)-(y-0)-(z-0)=0$$

$$\underline{x-y-z=0}$$

The normal line:

$$\begin{cases} x=t \\ y=-t \\ z=-t \end{cases}$$

Example 3

The surfaces $f(x,y,z)=x^2+y^2-2=0$ and $g(x,y,z)=x+z-4=0$ meet in an ellipse E. Find the parametric equations for the line tangent to E at the point $P_0(1,\ 1,\ 3)$

Solution

The tangent line is orthogonal to both ∇f and ∇g at P_0 and therefore parallel to $\vec{v}=\nabla f\times\nabla g$.

The components of $\vec{v}$ and the coordinates of P_0 give us equations for the line.

$$\nabla f\Big|_{(1,\ 1,\ 3)}=\left(2x\,\hat{i}+2y\,\hat{j}\right)\Big|_{(1,\ 1,\ 3)}$$

$$\underline{=2\hat{i}+2\hat{j}}$$

$$\nabla g\Big|_{(1,\ 1,\ 3)}=\left(\hat{i}+\hat{k}\right)\Big|_{(1,\ 1,\ 3)}$$

$$\underline{=\hat{i}+\hat{k}}$$

$$\vec{v}=\left(2\hat{i}+2\hat{j}\right)\times\left(\hat{i}+\hat{k}\right)$$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\underline{=2\hat{i}-2\hat{j}-2\hat{k}}$$

The tangent line is:

$$x=1+2t,\quad y=1-2t,\quad z=3-2t$$

10.5–4 Estimating Change in a Specific Direction

How much the value of a function *f* changes if we move a small distance *ds* from a point P_0 to another point neary.

$$df = f'\left(P_0\right)ds \qquad (\textit{single variable})$$

$$df = \left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)ds \qquad (\textit{two or more variables})$$

$\vec{u}$ is the direction of the motion away from P_0.

10.5–5 Estimating the Change in *f* in a Direction $\vec{u}$

To estimate the change in the value of a differentiable function *f* when we move a small distance *ds* from a point P_0 in a paricular direction $\vec{u}$ is given by

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)}_{\substack{\textit{Directional}\\ \textit{derivative}}} \cdot \underset{\textit{Distance}}{ds}$$

Example 4

Estimate how much the value of $f\left(x,y,z\right) = y\sin x + 2yz$ will change if the point $P\left(x,y,z\right)$ moves 0.1 unit from $P_0\left(0,\ 1,\ 0\right)$ straight toward $P_1\left(2,\ 2,\ -2\right)$

Solution

$$\overrightarrow{P_0P_1} = 2\hat{i} + \hat{j} - 2\hat{k}$$

The direction of the vector is:

$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{\left|\overrightarrow{P_0P_1}\right|}$$

$$= \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

$$\nabla f\Big|_{(0,1,0)} = \left(\left(y\cos x\right)\hat{i} + \left(\sin x + 2z\right)\hat{j} + 2y\hat{k}\right)\Big|_{(0,1,0)}$$

$$= \hat{i} + 2\hat{k}$$

$$\nabla f \Big|_{P_0} \bullet \vec{u} = \left(\hat{i} + 2\hat{k}\right) \bullet \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3} - \frac{4}{3}$$

$$= -\frac{2}{3}$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\vec{u}$ is:

$$df = \left(\nabla f \Big|_{P_0} \bullet \vec{u}\right)(ds)$$

$$= \left(-\frac{2}{3}\right)(0.1)$$

$$\approx -0.067 \ \ unit$$

10.5–6 *Definition*

The ***linearization*** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x \Big|_{(x_0, y_0)} (x - x_0) + f_y \Big|_{(x_0, y_0)} (y - y_0)$$

The approximation $\qquad f(x, y) \approx L(x, y)$

is the ***standard linear*** approximation of f at (x_0, y_0)

Example 5

Find the linearization of $f(x,y)=x^2-xy+\frac{1}{2}y^2+3$ at the point (3, 2)

Solution

$$f(3,\ 2)=3^2-(3)(2)+\frac{1}{2}2^2+3$$
$$=8$$

$$f_x(3,\ 2)=\frac{\partial}{\partial x}\left(x^2-xy+\frac{1}{2}y^2+3\right)\Big|_{(3,\,2)}$$
$$=2x-y\Big|_{(3,\,2)}$$
$$=2(3)-2$$
$$=4$$

$$f_y(3,\ 2)=-x+y\Big|_{(3,\,2)}$$
$$=-3+2$$
$$=-1$$

$$L(x,y)=8+4(x-3)-1(y-2)$$
$$=4x-y-2$$

10.5–7 The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at $(x_0,\ y_0)$ and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x,y)$ incurred in replacing $F(x,y)$ on R by its linearization.

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Satisfies the inequality:

$$|E(x,y)| \le \tfrac{1}{2}M\left(|x-x_0|+|y-y_0|\right)^2$$

$$R: \quad |x-x_0| \le h, \quad |y-y_0| \le k$$

Example 6

Find an upper bound for the error in the approximation $f(x,y) \approx L(x,y)$ of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ over the rectangle:

$$R: \quad |x-3| \le 0.1, \quad |y-2| \le 0.1$$

Express the upper bound as a percentage of $f(3,2)$, the value of f at the center of the rectangle.

Solution

$$f_{xx} = \frac{\partial}{dx}(2x-y)$$

$$= 2$$

$$\boxed{|f_{xx}| = 2}$$

$$f_{yy} = \frac{\partial}{dy}(-x+y)$$

$$= 1$$

$$\boxed{|f_{yy}| = 1}$$

$$f_{xy} = \frac{\partial}{dy}(2x-y)$$

$$= -1$$

$$\boxed{|f_{xy}| = 1}$$

The largest of these is 2, so let $M = 2$.

$$|E(x,y)| \le \tfrac{1}{2}M\left(|x-x_0|+|y-y_0|\right)^2$$

$$= \tfrac{1}{2}(2)\left(|x-3|+|y-2|\right)^2$$

$$= \left(|x-3|+|y-2|\right)^2$$

Since $|x-3| \le 0.1, \quad |y-2| \le 0.1$

$$|E(x,y)| \le (0.1+0.1)^2$$

$$= 0.04$$

As a percentage of $f(3,2)=8$, the error is no greater than

$$\frac{0.04}{8} \times 100 = 0.5\%$$

10.5–8 Differentials – *Definition*

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In the linearization of *f* is called the ***total differential of f***.

10.5–9 *Theorem* – The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region *R* containing a point (x_0, y_0) and that f_x *and* f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

In the value of *f* that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

10.5–10 *Definition*

A function $z = f(x, y)$ is ***differentiable at*** (x_0, y_0) If $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x\left(x_0, y_0\right)\Delta x + f_y\left(x_0, y_0\right)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1,\ \varepsilon_2 \to 0$ as both $\Delta x,\ \Delta y \to 0$.

We call *f* ***differentiable*** if it is differentiable at every point in its domain and say that its graph is a ***smooth surface***.

Example 7

Suppose that a cylindrical can is designed to have a radius of 1 *in*. and a height of 5 *in*., but that the radius and height are off the amounts $dr = +0.03$ and $dh = -0.1$.
Estimate the resulting absolute change in the volume of the can.

Solution

To estimate the absolute change in $V = \pi r^2 h$,

$$\Delta V \approx dV = V_r\left(r_0,\ h_0\right)dr + V_h\left(r_0,\ h_0\right)dh$$

$$dV = \left(2\pi r_0 h_0\right)(0.03) + \left(\pi r_0^2\right)(-0.1)$$

$$= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)$$

$$= 0.2\pi$$

$$\approx 0.63\ \ in^3$$

Example 8

Your company manufactures right circular cylindrical molasses storage tanks that are 25 *ft* with a radius of 5 *ft*. How sensitive are the tanks' volumes to small variations in height and radius?

Solution

$$V = \pi r^2 h$$

$$dV = V_r\left(r_0,\ h_0\right)dr + V_h\left(r_0,\ h_0\right)dh$$

$$= V_r(5,\ 25)dr + V_h(5,\ 25)dh$$

$$= (2\pi rh)_{(5,\ 25)}dr + \left(\pi r^2\right)_{(5,\ 25)}dh$$

$$= 250\pi dr + 25\pi dh$$

A 1-*unit* change in *r* will change *V* about 250π units.

A 1-*unit* change in *h* will change *V* about 25π units.

The tanks' volume is 10 times more sensitive to a small change in *r* than it is to a small change of equal size in *h*.

$$dV = (2\pi rh)_{(25,\ 5)}\, dr + \left(\pi r^2\right)_{(25,\ 5)} dh$$

$$= 250\pi dr + 625\pi dh$$

Now the volume is more sensitive to changes in *h* than to changes in *r*.

The general rule is that functions are most sensitive to small changes in the variables that generated the largest partial derivatives.

Example 9

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of *r* and *h*. Suppose that *r* is measured with an error of no more than 2% and *h* with an error of no more than 0.5%.
Estimate the resulting possible percentage error in the calculation of *V*.

Solution

$$\left|\frac{dr}{r}\times 100\right| \le 2 \qquad \left|\frac{dh}{h}\times 100\right| \le 0.5$$

$$\frac{dV}{V} = \frac{2\pi rhdr + \pi r^2 dh}{\pi r^2 h}$$

$$= \frac{2dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| = \left|\frac{2dr}{r} + \frac{dh}{h}\right|$$

$$\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$

$$\le 2(0.02) + 0.005$$

$$= 0.045$$

The error in the volume is at the most 4.5%.

10.5–11 Functions of More Than Two Variables

- The linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ is:

$$L(x,y,z)=f(P_0)+f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)$$

- Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous.
  Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R.
  Then the error $E(x,y,z)=f(x,y,z)-L(x,y,z)$ in the approximation of f by L is bounded throughout R by the inequality:

$$|E(x,y)|\le\tfrac{1}{2}M\left(|x-x_0|+|y-y_0|+|z-z_0|\right)^2$$

- If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, *and* z_0 by small amounts dx, dy, and dz, the total differential:

$$df=f_x(P_0)dx+f_y(P_0)dy+f_z(P_0)dz$$

Example 10

Find the linearization $L(x,y,z)$ of $f(x,y,z)=x^2-xy+3\sin z$ at the point (2, 1, 0).
Find the upper bound for the error incurred in replacing f by L on the rectangle:

$$R:\quad |x-2|\le 0.01,\quad |y-1|\le 0.02,\quad |z|\le 0.01$$

Solution

$$f(2,1,0)=2^2-(2)(1)+3\sin 0=2$$

$f_x(2,1,0)=2x-y=3$	$f_{xx}=2$	$f_{xy}=-1$
$f_y(2,1,0)=-x=-2$	$f_{yy}=0$	$f_{xz}=0$
$f_z(2,1,0)=3\cos z=3$	$f_{zz}=-3\sin z$	$f_{yz}=0$

$$|-3\sin z|\le 3\sin 0.01\approx 0.03$$

Let $M=2$.

$$|E| \le \frac{1}{2} 2(0.01 + 0.02 + 0.01)^2$$

$$= 0.0016$$

Exercises *Section* 10.5 – Tangent Planes & Linear Approximation

$(\mathbf{1-40})$ Find the tangent plane and normal line of the surface at the given point(s).

1. $f(x,y)=4x^2+y^2+2$ @ $P_0(1,\ 1)$

2. $f(x,y)=x^2-y^2-16$ @ $P_0(5,\ 3)$

3. $f(x,y)=x^2-2y^2-7$ @ $P_0(1,\ 2)$

4. $f(x,y)=x^4-x^2+y^2+2$ @ $P_0\left(\frac{3}{2},\ \frac{\sqrt{3}}{4}\right)$

5. $f(x,y)=2x-y-e^{x-y}-2$ @ $P_0(1,\ 1)$

6. $16x^2+9y^2=144$ @ $P_0\left(\frac{3\sqrt{3}}{2},\ 2\right)$

7. $9x^2+4y^2=36$ @ $P_0\left(-1,\ -\frac{3\sqrt{3}}{2}\right)$

8. $4x^2+y^2+4z^2=16$ @ $P_0\left(1,\ 2,\ \sqrt{2}\right)$

9. $f(x,\ y,\ z)=z-x^2-y^2$ @ $P_0(1,\ 1,\ 2)$

10. $f(x,\ y,\ z)=x^2+y^2$ @ $P_0(0,\ 2,\ 4)$

11. $f(x,\ y,\ z)=2x+3y+z$ @ $P_0(2,\ 3,\ 1)$

12. $f(x,\ y,\ z)=y^2-z^2-x^2$ @ $P_0(1,\ 3,\ 2)$

13. $x^2+4y^2+9z^2=14$ @ $P_0(1,\ 1,\ 1)$

14. $2x^2-y^2+3z^2=5$ @ $P_0(1,\ -3,\ 2)$

15. $x^2-2y^2-4z^2=4$ @ $P_0(4,\ -2,\ -1)$

16. $x^2+2y^2+z^2=18$ @ $P_0(1,\ 2,\ 3)$

17. $x^2-y^2+z^2=4$ @ $P_0(1,\ -1,\ 2)$

18. $x^2+y^2+z^2=3$ @ $P_0(1,\ 1,\ 1)$

19. $x^2+2xy-y^2+z^2=7$ @ $P_0(1,\ -1,\ 3)$

20. $\cos\pi x-x^2y+e^{xz}+yz=4$ @ $P_0(0,\ 1,\ 2)$

21. $x^2-xy-y^2-z=0$ @ $P_0(1,\ 1,\ -1)$

22. $x^2 + y^2 - 2xy - x + 3y - z = -4$ @ $P_0(2, -3, 18)$

23. $z = \ln\left(x^2 + y^2\right)$ @ $P(1, 0, 0)$

24. $z = e^{-x^2 - y^2}$ at the point $(0, 0, 1)$

25. $z = \sqrt{y - x}$ at the point $(1, 2, 1)$

26. $z = 2x^2 + y^2$; $(1, 1, 3)$ *and* $(0, 2, 4)$

27. $x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 = 1$; $(0, 2, 0)$ *and* $\left(1, 1, \frac{3}{2}\right)$

28. $xy\sin z - 1 = 0$; $\left(1, 2, \frac{\pi}{6}\right)$ *and* $\left(-2, -1, \frac{5\pi}{6}\right)$

29. $yze^{xz} - 8 = 0$; $(0, 2, 4)$ *and* $(0, -8, -1)$

30. $z = x^2 e^{x-y}$; $(2, 2, 4)$ *and* $(-1, -1, 1)$

31. $z = \ln(1 + xy)$; $(1, 2, \ln 3)$ *and* $(-2, -1, \ln 3)$

32. $z = f(x, y) = \dfrac{1}{x^2 + y^2}$ at the point $\left(1, 1, \frac{1}{2}\right)$

33. $x^2 + y + z = 3$; $(1, 1, 1)$ *and* $(2, 0, -1)$

34. $x^2 + y^3 + z^4 = 2$; $(1, 0, 1)$ *and* $(-1, 0, 1)$

35. $xy + xz + yz = 12$; $(2, 2, 2)$ *and* $(2, 0, 6)$

36. $x^2 + y^2 - z^2 = 0$; $(3, 4, 5)$ *and* $(-4, -3, 5)$

37. $xy\sin z = 1$; $\left(1, 2, \frac{\pi}{6}\right)$ *and* $\left(-2, -1, \frac{5\pi}{6}\right)$

38. $yze^{xz} = 8$; $(0, 2, 4)$ *and* $(0, -8, -1)$

39. $z^2 - \frac{x^2}{16} - \frac{y^2}{9} = 1$; $\left(4, 3, -\sqrt{3}\right)$ *and* $\left(-8, 9, \sqrt{14}\right)$

40. $2x + y^2 - z^2 = 0$; $(0, 1, 1)$ *and* $(4, 1, -3)$

(41 – 44) Find parametric equation for the line tangent to the curve of intersection of the surfaces.

41. $x + y^2 + 2z = 4, \quad x = 1$ at the point $(1, 1, 1)$

42. $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$

43. $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1, 1, 3)$

44. $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$ at the point $\left(\sqrt{2}, \sqrt{2}, 4\right)$

45. Find an equation for the plane tangent to the level surface $f(x,y,z)=x^2-y-5z$ at the point $P_0(2,\ -1,\ 1)$. Also, find parametric equations for the line is normal to the surface at P_0.

46. By about how much will $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds=0.1$ unit in the direction of $3\hat{i}+6\hat{j}-2\hat{k}$?

47. By about how much will $f(x,y,z)=e^x\cos yz$ change if the point $P(x,y,z)$ moves from the origin at distance of $ds=0.1$ *unit* in the direction of $2\hat{i}+2\hat{j}-2\hat{k}$?

(48 – 55) Find the linearization $L(x,y)$ of:

48. $f(x,y)=x^2+y^2+1$ at the point (0, 0) and (1, 1)

49. $f(x,y)=(x+y+2)^2$ at the point (0, 0) and (1, 2)

50. $f(x,y)=x^3y^4$ at the point (1, 1) and (0, 0)

51. $f(x,y)=e^{2y-x}$ at the point (0, 0) and (1, 2)

52. $f(x,y,z)=x^2+y^2+z^2$ at the point (1, 1, 1)

53. $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ at the point (1, 2, 2)

54. $f(x,y,z)=\dfrac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

55. $f(x,y,z)=e^x+\cos(y+z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

(56 – 63) Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value.

56. $f(x,\ y)=4\cos(2x-y)$; $(a,\ b)=\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)$; estimate $f(0.8,\ 0.8)$

57. $f(x,\ y)=(x+y)e^{xy}$; $(a,\ b)=(2,\ 0)$; estimate $f(1.95,\ 0.05)$

58. $f(x,\ y)=xy+x-y$; $(a,\ b)=(2,\ 3)$; estimate $f(2.1,\ 2.99)$

59. $f(x,\ y)=12-4x^2-8y^2$; $(a,\ b)=(-1,\ 4)$; estimate $f(-1.05,\ 3.95)$

60. $f(x,\ y)=-x^2+2y^2$; $(a,\ b)=(3,\ -1)$; estimate $f(3.1,\ -1.04)$

61. $f(x,\ y)=\sqrt{x^2+y^2}$; $(a,\ b)=(3,\ -4)$; estimate $f(3.06,\ -3.92)$

62. $f(x,\ y)=\ln(1+x+y)$; $(a,\ b)=(0,\ 0)$; estimate $f(0.1,\ -0.2)$

63. $f(x, y) = \dfrac{x+y}{x-y}$; $(a, b) = (3, 2)$; estimate $f(2.95, 2.05)$

64. Find the direction numbers of the normal to the sphere $x^2 + y^2 + z^2 = 16$ at the point $\left(2, 2, 2\sqrt{2}\right)$ and show that it lies along the radius.

65. Estimate the change in the function $f(x, y) = -2y^2 + 3x^2 + xy$ when (x, y) changes from $(1, -2)$ to $(1.05, -1.9)$.

66. What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?

67. Show that the normal to the surface $z = \sqrt{xy}$ at any point on the surface is perpendicular to the line joining the point to the origin.

68. You plan to calculate the volume inside a stretch of pipeline that is about 36 *in*. in diameter and 1 *mile* long. With which measurement should you be more careful, the length or the diameter? Why?

69. The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

70. The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

71. A hemispherical tank with a radius of 1.50 *m* is filled with water to a depth of 1.00 *m*. Water level drops by 0.05 *m* (from 1.00 *m* to 0.95 *m*)

a) Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b) Approximate the change in the surface area of the water in the tank.

72. Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

a) Surface area

b) Volume

73. Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

The container is coated with a layer of ice $\frac{1}{2}$ *cm* thick. Use a differential to estimate the total volume of ice. (*Hint*: assume r is radius with $dr = \frac{1}{2}$ and h is height with $dh = 0$)

74. A standard 12-*fl-oz* can of soda is essentially a cylinder of radius $r = 1\ in$ and height $h = 5\ in.$

a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-*fl-oz*? What might its dimensions be? (There is more than one correct answer.)

75. Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown

a) Fill in the table showing the value of the directional derivative at points (a, b) in the direction given by the unit vectors $\vec{u}$, $\vec{v}$, *and* $\vec{w}$

	$(a,\ b)=(0,\ 0)$	$(a,\ b)=(2,\ 0)$	$(a,\ b)=(1,\ 1)$
$\vec{u}=\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$			
$\vec{v}=\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$			
$\vec{w}=\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$			

b) Interpret each of the directional derivatives computed in part(*a*) at the point $(2,\ 0)$.

76. Two spheres have the same center and radii *r* and *R*, where $0<r<R$. The volume of the region between the sphere is $V(r,\ R)=\frac{4\pi}{3}\left(R^3-r^3\right)$. $R=3$

a) First use your intuition. If *r* is held fixed, how does *V* change as *R* increases? What is the sign of V_R? If *R* is held fixed, how does *V* change as *r* increases (up to the value of *R*)? What is the sign of V_r?

b) Compute V_r and V_R. Are the results consistent with part (*a*)?

c) Consider spheres with and $r=1$. Does the volume change more if *R* is increased by $\Delta R=0.1$ (with *r* fixed) or if *r* is decreased by $\Delta r=0.1$ (with *R* fixed)?

77. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r=10\ cm$, a height of $h=50\ cm$, and a thickness of $t=0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r,h,t)=\pi ht(2r-t)$. Estimate maximum error in the volume of the tube.

78. The volume of a right circular cone with radius r and height h is $V = \frac{1}{3}\pi h r^2$

a) Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$

b) Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

79. The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

80. The Volume of a segment of a circular paraboloid with radius r and height h is $V = \frac{1}{2}\pi h r^2$.

Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

81. Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \geq 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \leq A \leq 1$.

a) Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

b) If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

c) Does the answer in part (*b*) depend on the current batting average? Explain.

82. A conical tank with radius 0.50 *m* and height 2.0 *m* is filled with water.

Water released from the tank, and the water level drops by 0.05 *m* (from 2.0 *m* to 1.95 *m*). Approximate the change in volume of water in the tank.
(***Hint***: When the water level drops, both the radius and height of the cone of water change).

83. Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon}\left(R^2 - r^2\right)$, where P is the difference in the pressure between the ends of the cylinder and υ is the viscosity of the fluid. Assuming that P and υ are constant, the velocity V along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where k is a constant that we will take to be $k = 1$.

a) Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3\ cm$ to $R = 3.05\ cm$ and the length increases from $L = 50\ cm$ to $L = 50.5\ cm$.

b) Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length increases by 2%.

84. Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in n random encounters, you will meet at least one person with flu is $P = f(n, r) = 1 - (1 - r)^n$. although n is a positive integer, regard it as a positive real number.

a) Compute f_r and f_n.

b) How sensitive is the probability P to the flu rate r? Suppose you meet $n = 20$ people. Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c) Approximately how much does the probability P increase the flu rate increases from $r = 0.9$ to $r = 0.91$

d) Interpret the results of parts (*b*) and (*c*).

85. When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

a) Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω and R_2 decreases from 3 Ω to 2.95 Ω.

b) Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c) Is it true that if R_1 and R_2 increase, then R increases? Explain.

d) Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

86. When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω, R_2 decreases from 3 Ω to 2.95 Ω, and R_3 increases from 1.5 Ω to 1.55 Ω

87. Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane P given by $Ax + By + Cz + 1 = 0$. Let $h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$ and $m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}$

a) Find the equation of the plane tangent to the ellipsoid at the point $(p,\ q,\ r)$.

b) Find the two points on the ellipsoid at which the tangent plane parallel to P and find equations of the tangent planes.

c) Show that the distance between the origin and the plane P is h.

d) Show that the distance between the origin and the tangent planes is hm.

e) Find a condition that guarantees the plane P does not intersect the ellipsoid.

Section 10.6 – Extreme Values / Optimization

10.6–1 *Definition* – Derivative Tests for Local Extreme Values

Let $f(x,y)$ be defined on a region R containing the point $(a,\ b)$. Then

- $f(a,\ b)$ is a local maximum value of f if $f(a,\ b) \geq f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.
- $f(a,\ b)$ is a local minimum value of f if $f(a,\ b) \leq f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.

$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$

$z = \frac{1}{2}\left(\left||x|-|y|\right|-|x|-|y|\right)$

10.6–2 *Theorem* – First derivative Test for Local Extreme Values

If $f(x,y)$ has a local maximum or minimum value at an interior point $(a,\ b)$ of its domain and if the first partial derivatives exist there, then $f_x(a,\ b) = 0$ *and* $f_y(a,\ b) = 0$

10.6–3 *Definition*

An interior point of the domain $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a ***critical point*** of $f.(x,y)$

10.6–4 *Definition*

A differentiable function $f(x,y)$ has a ***saddle point*** at a critical point $(a,\ b)$ if in every open disk centered at $(a,\ b)$ there are domain points where $f(x,\ y) > f(a,\ b)$ and domain points (x,y) where $f(x,\ y) < f(a,\ b)$. The corresponding point $(a,\ b,\ f(a,\ b))$ on the surface $z = f(x,\ y)$ is called a saddle point of the surface.

$$z = \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2}$$

$$z = y^2 - y^4 - x^2$$

Example 1

Find the local extreme values of $f(x,\ y) = x^2 + y^2 - 4y + 9$

Solution

The domain of *f* is the entire plane. The local extreme values occur:

$f_x = 2x = 0 \qquad f_y = 2y - 4 = 0$

Therefore, the critical point is $(0,\ 2)$ and the value $f(0,\ 2) = 0 + 2^2 - 8 + 9 \underline{= 5}$.

The critical point is a local *minimum*.

Example 2

Find the local extreme values of $f(x,\ y) = y^2 - x^2$

Solution

The domain of f is the entire plane.

$f_x = -2x = 0 \qquad f_y = 2y = 0$

Therefore, the local extreme is the origin $(0,\ 0)$ and the value $f(0,\ 0) = 0$.

$f(0,\ y) = y^2 \geq 0 \qquad f(x,\ 0) = -x^2 \leq 0$

The function has a saddle point at the origin and no local extreme values.

10.6–5 *Theorem* – Second Derivative Test for Local Extreme Values

Suppose that $f(x,\ y)$ and its first and second partial derivatives are continuous throughout a disk centered at $(a,\ b)$ and that $f_x(a,\ b) = f_y(a,\ b) = 0$. Then

- f has a ***local maximum*** at $(a,\ b)$ if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$.
- f has a ***local minimum*** at $(a,\ b)$ if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$.
- f has a ***saddle point*** at $(a,\ b)$ if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.
- ***The test is inconclusive*** at $(a,\ b)$ if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$. In this case, we must find some other way to determine the behavior of f at $(a,\ b)$.

Example 3

Find the local extreme values of $f(x,\ y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution

$f_x = y - 2x - 2 = 0 \qquad f_y = x - 2y - 2 = 0$

$$\begin{cases} -2x + y = 2 \\ x - 2y = 2 \end{cases} \rightarrow \underline{x = y = -2}$$

Therefore, the critical point is $(-2,\ -2)$

$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$

$$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0$$

$$f_{xx} = -2 < 0$$

The function f has a local maximum at $(-2,\ -2)$ and the value is

$$f(-2,\ -2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 8$$

Example 4

Find the local extreme values of $f(x,\ y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution

$f_x = -6x + 6y = 0 \quad and \quad f_y = 6y - 6y^2 + 6x = 0$

$$\begin{cases} -6x + 6y = 0 \\ 6y - 6y^2 + 6x = 0 \end{cases} \rightarrow \quad x = y$$

$$6y - 6y^2 + 6y = -6y(y-2) = 0$$

$$\begin{cases} y = 0 = x \\ y = 2 = x \end{cases}$$

$(0,\ 0)$ & $(2,\ 2)$ are the ***critical*** points.

$f_{xx} = -6 \qquad f_y = 6 - 12y \qquad f_{xy} = 6$

$$f_{xx} f_{yy} - f_{xy}^2 = (-6)(6 - 12y) - 6^2$$

$$= -36 + 72y - 36$$

$$= 72(y-1)$$

At $(0,\ 0)$

$$f_{xx} f_{yy} - f_{xy}^2 = -72 < 0$$

So, the function has a *saddle* point at the origin.

At $(2,\ 2)$

$$f_{xx} f_{yy} - f_{xy}^2 = 72 > 0 \text{ and } f_{xx} = -6 < 0$$

So, the function has a local *maximum* at $(2,\ 2)$ with a value of:

$$f(2,\ 2) = 12 - 16 - 12 + 24$$
$$= 8$$

10.6–6 Absolute Maxima and Minima on Closed Bounded Regions

The absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region *R* into three steps

1. *List the interior points of R* where *f* may have local maxima and minima and evaluate *f* at these points. These are the critical points of *f*.
2. *List the boundary points* of *R* where *f* may have local maxima and minima and evaluate *f* at these points.
3. *Look through the lists* for the maximum and minimum values of *f*. These will be the absolute maximum and minimum values of *f* on *R*. Since absolute maxima and minima are also local maxima and minima, the absolute maximum, and minimum values of appear somewhere in the lists made in Steps 1 and 2.

Example 5

Find the absolute maximum and minimum values of $f(x,\ y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0,\quad y = 0,\quad y = 9 - x$.

Solution

$$f_x = 2 - 2x = 0 \qquad f_y = 2 - 2y = 0$$

$$x = 1 \quad y = 1$$

The *critical point* is $(1,\ 1)$.

The value of *f* is:

$$f(1,\ 1) = 2 + 2 + 2 - 1 - 1$$
$$= 4$$

Boundary points:

i. On the segment *OA*, *y* = 0. The function

$$f(x,\ 0) = 2 + 2x - x^2$$

This function is defined on the closed interval $0 \le x \le 9$.

$$\begin{cases} x=0 & \to \underline{f(0,\ 0)=2} \\ x=9 & \to \underline{f(9,\ 0)=-61} \end{cases}$$

At the interior points where $f_x = 0$. The only point is $x=1$ where $f(1,\ 0)=3$

ii. On the segment *OB*, $x = 0$. The function:

$$f(0,\ y)=2+2y-y^2$$

$$f(0,\ 0)=2,\quad f(0,\ 9)=-61,\quad f(1,\ 0)=3$$

iii. Left the interior points of the segment *AB*. With $y=9-x$, then

$$f(x,\ y)=2+2x+2(9-x)-x^2-(9-x)^2$$

$$=2+2x+18-2x-x^2-81+18x-x^2$$

$$\underline{=-2x^2+18x-61}$$

$$f'(x,9-x)=-4x+18=0$$

$$\underline{x=\frac{9}{2}}$$

At $x=\frac{9}{2} \quad \Rightarrow \quad \underline{y=\frac{9}{2}}$

$$f\left(\frac{9}{2},\ \frac{9}{2}\right)=2+2\left(\frac{9}{2}\right)+2\left(9-\frac{9}{2}\right)-\left(\frac{9}{2}\right)^2-\left(9-\frac{9}{2}\right)^2$$

$$\underline{=-\frac{41}{2}}$$

$\therefore\ 4,\ 2,\ -61,\ 3,\ -\frac{41}{2}$.

The maximum is 4, which *f* assumes at (1, 1). The minimum is –61, which *f* assumes at (0, 9) and (9, 0).

Example 6

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 *in*. Find the dimensions of an acceptable box of largest volume.

Solution

Let *x*, *y*, and *z* represent the length, width, and height.

The girth is: $=2y+2z\,(=P)$

Volume: $V=xyz$

We want to maximize the volume of the box satisfying:

$$x+2y+2z=108$$
$$x=108-2y-2z$$

$$V(y,\ z)=(108-2y-2z)yz$$
$$=108yz-2y^2z-2yz^2$$
$$V_y(y,z)=108z-4yz-2z^2$$
$$=2z(54-2y-z)=0$$

$$V_z(y,z)=108y-2y^2-4yz$$
$$=2y(54-y-2z)=0$$

$$\begin{cases}2z(54-2y-z)=0 & \rightarrow \quad z=0 \quad 54-2y-z=0\\ 2y(54-y-2z)=0 & \rightarrow \quad y=0 \quad 54-y-2z=0\end{cases}$$

$$\begin{cases}2y+z=54\\ y+2z=54\end{cases} \rightarrow \underline{y=z=18}$$

$if\ \ y=0$

$$54-2y-z=0$$
$$z=54 \rightarrow (0,\ 54)$$

$if\ \ z=0$

$$54-y-2z=0$$
$$y=54 \rightarrow (54,\ 0)$$

$\therefore$ The critical points are: $(0,\ 0),\ (0,\ 54),\ (54,\ 0),\ (18,\ 18)$

At $(0,\ 0)$:

$$V(0,\ 0)=108yz-2y^2z-2yz^2\Big|_{(0,\ 0)}=0$$

At $(0,\ 54)$:

$$V(0,\ 54)=108yz-2y^2z-2yz^2\Big|_{(0,\ 54)}=0$$

At $(54,\ 0)$:

$$V(54,\ 0)=108yz-2y^2z-2yz^2\Big|_{(54,\ 0)}=0$$

At $(18,\ 18)$:

$$V(18,\ 18) = 108yz - 2y^2z - 2yz^2 \Big|_{(18,\ 18)} = 11{,}664$$

$$V_{yy} = -4z, \quad V_{zz} = -4y, \quad V_{yz} = 108 - 4y - 4z$$

$$V_{xx}V_{yy} - V_{xy}^2 = (-4z)(-4y) - (108 - 4y - 4z)^2$$

$$= 16yz - 16(27 - y - z)^2 \Big|_{(18,\ 18)}$$

$$= 16(18)(18) - 16(27 - 18 - 18)^2$$

$$= 3{,}888 > 0$$

$$V_{yy}(18,\ 18) = -4(18) < 0$$

That implies $(18,\ 18)$ give a maximum volume.

$$x = 108 - 2(18) - 2(18)$$

$$= 36$$

$$V = xyz = 36(18)(18)$$

$$= 11{,}664 \quad in^3$$

The dimensions of the package are $\boldsymbol{x}$ = 36 *in.*, $\boldsymbol{y}$ = 18 *in*, $\boldsymbol{z}$ = 18 *in*.

The maximum volume is $11{,}664 \ \ in^3$.

10.6–7 Summary of *Max–Min* Tests

The extreme values of $f(x, y)$ can occur only at

- ***i.*** ***Boundary points*** of the domain of *f*.
- ***ii.*** ***Critical points*** (interior points where $f_x = f_y = 0$ or points where f_x *or* f_y fail to exist)

If the first- and second-order partial derivatives of *f* are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the ***Second Derivative Test***:

- ***i.*** $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ ***Local Maximum***
- ***ii.*** $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ ***Local Minimum***
- ***iii.*** $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ ***Saddle Point***
- ***iv.*** $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ ***Test is inconclusive.***

Exercises *Section* 10.6 – Extreme Values / Optimization

$(\mathbf{1-45})$ Find all the local extreme and saddle points of the function.

1. $f(x,y)=x^2+xy+y^2+3x-3y+4$

2. $f(x,y)=x^2-xy+y^2+2x+2y-4$

3. $f(x,y)=x^3+y^3-3xy+15$

4. $f(x,y)=x^4-8x^2+3y^2-6y$

5. $f(x,y)=2xy-5x^2-2y^2+4x+4y-4$

6. $f(x,y)=x^2-4xy+y^2+6y+2$

7. $f(x,y)=2x^2+3xy+4y^2-5x+2y$

8. $f(x,y)=x^2-y^2-2x+4y+6$

9. $f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$

10. $f(x,y)=\sqrt{-9x^2+4y^2+36x+24y+36}$

11. $f(x,y)=1-\sqrt[3]{x^2+y^2}$

12. $f(x,y)=x^3+y^3+3x^2-3y^2-8$

13. $f(x,y)=4xy-x^4-y^4$

14. $f(x,y)=\dfrac{1}{x^2+y^2-1}$

15. $f(x,y)=\dfrac{1}{x}+xy+\dfrac{1}{y}$

16. $f(x,y)=y\sin x$

17. $f(x,y)=e^{2x}\cos y$

18. $f(x,y)=e^y-ye^x$

19. $f(x,y)=e^{-y}\left(x^2+y^2\right)$

20. $f(x,y)=2xy\,e^{-x^2-y^2}$

21. $f(x,y)=e^{-x^2-y^2-2x}$

22. $f(x,y)=e^{x^2y^2-2xy^2+y^2}$

23. $f(x,y)=2\ln x+\ln y-4x-y$

24. $f(x,y)=\ln(x+y)+x^2-y$

25. $f(x,y)=1+x^2+y^2$

26. $f(x,y)=x^2-6x+y^2+8y$

27. $f(x,y)=(3x-2)^2+(y-4)^2$

28. $f(x,y)=3x^2-4y^2$

29. $f(x,y)=x^4+y^4-16xy$

30. $f(x,y)=\frac{1}{3}x^3-\frac{1}{3}y^3+3xy$

31. $f(x,y)=x^4-2x^2+y^2-4y+5$

32. $f(x,y)=x^2+xy-2x-y+1$

33. $f(x,y)=x^2+6x+y^2+8$

34. $f(x,y)=x^2-4y^2+2xy+4x-6y+4$

35. $f(x,y)=4x^2+9y^2+8x-36y+24$

36. $f(x,y)=x^2+y^2+4x-6y$

37. $f(x,y)=4x^2+6y^2-2xy-8x+2y+3$

38. $f(x,y)=x^4+y^4-4xy$

39. $f(x,y)=5x^3+5y^3-xy$

40. $f(x,y)=3x^2+y^2-2xy-8y$

41. $f(x,y)=x^2+y^2-6x+4y-8$

42. $f(x,y)=3x^2+y^2+2xy$

43. $f(x,y)=\sqrt{x^2+y^2+1}$

44. $f(x,y)=\sqrt{x^2+y^2}+1$

45. $f(x,y)=x^3-y^3-3xy$

(46 – 49) Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

46. $f(x, y) = x^4 + y^4 - 16xy$

47. $f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$

48. $f(x, y) = xy(2 + x)(y - 3)$

49. $f(x, y) = 10 - x^3 - y^3 - 3x^2 + 3y^2$

(50 – 70) Find the absolute extreme values of the function on the specified region *R*.

50. $f(x, y) = 3x + 4y - 12$, *R* is the closed triangular region with vertices $(0, 0)$, $(3, 0)$, *and* $(3, 4)$

51. $f(x, y) = 2x^2 + y^2 - 4x - 2y + 3$ on the rectangle $R = \{(x, y):\ 0 \le x \le 3,\ 0 \le y \le 2\}$

52. $f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$ on the rectangle $R = \{(x, y):\ 0 \le x \le 3,\ -1 \le y \le 1\}$

53. $f(x, y) = x^4 + y^4 - 4xy + 1$ on the square $R = \{(x, y):\ -2 \le x \le 2,\ -2 \le y \le 2\}$

54. $f(x, y) = x^2y - y^3$ on the triangle $R = \{(x, y):\ 0 \le x \le 2,\ 0 \le y \le 2 - x\}$

55. $f(x, y) = xy$ on the semicircular disk $R = \left\{(x, y):\ -1 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\right\}$

56. $f(x, y) = x^2 + y^2 - 2y + 1;\quad R = \left\{(x, y):\ x^2 + y^2 \le 4\right\}$

57. $f(x, y) = 2x^2 + y^2;\quad R = \left\{(x, y):\ x^2 + y^2 \le 16\right\}$

58. $f(x, y) = 4 + 2x^2 + y^2;\quad R = \{(x, y):\ -1 \le x \le 1,\ -1 \le y \le 1\}$

59. $f(x, y) = 6 - x^2 - 4y^2;\quad R = \{(x, y):\ -2 \le x \le 2,\ -1 \le y \le 1\}$

60. $f(x, y) = 2x^2 - 4x + 3y^2 + 2;\quad R = \left\{(x, y):\ (x - 1)^2 + y^2 \le 1\right\}$

61. $f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1;\quad R = \left\{(x, y):\ (x - 1)^2 + (y + 1)^2 \le 1\right\}$

62. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2};\quad R = \left\{(x, y):\ x^2 + y^2 \le 4,\ y \ge 0\right\}$

63. $f(x, y) = \dfrac{-x^2 + 2y^2}{2 + 2x^2y^2}$; *R* is the closed region bounded by the lines $y = x$, $y = 2x$, and $y = 2$

64. $f(x, y) = \sqrt{x^2 + y^2}$; *R* is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$

65. $f(x, y) = x^2 + y^2 - 4;\quad R = \left\{(x, y):\ x^2 + y^2 < 4\right\}$

66. $f(x, y) = x + 3y;\quad R = \{(x, y):\ |x| < 1,\ |y| < 2\}$

67. $f(x, y) = 2e^{-x-y}$; $R = \{(x, y): x \geq 0, y \geq 0\}$

68. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.

69. $D(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 4$, $y = x$ in the first quadrant.

70. $T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the triangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.

71. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$

72. Find the minimum distance from the point $(2, -1, 1)$ to the plane $x + y - z = 2$

73. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

74. Find the absolute maxima and minima of the function $f(x,y) = \left(4x - x^2\right)\cos y$ on the triangular plate $1 \leq x \leq 3$, $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$.

75. Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length *a*, *b*, and *c* is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where 2*s* is the perimeter of the triangle.

76. Let *P* be a plane tangent to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a point in the first octane. Let *T* be the tetrahedron in the first octant bounded by *P* and the coordinate planes $x = 0$, $y = 0$, and $z = 0$. Find the minimum volume *T*. (the volume of a tetrahedron in one-third the area of the base times the height.)

77. Given three distinct noncollinear points *A*, *B*, and *C* in the plane, find the point *P* in the plane such the sum of the distances $|AP|+|BP|+|CP|$ is a minimum. Here is how to procees with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120° = \frac{2\pi}{3}\right)$.

a) Assume the coordinates of the threee given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b) Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.

c) The goal is to minimize $f(x,y,z) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d) Explain why part (c) implies that the optimal point *P* has the property the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e) What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?

f) Estimate the Steiner point for the three points $(0, 0)$, $(0, 1)$, $(2, 0)$.

(78 – 79) Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

78. $f(x, y) = -\left(x^2 - 1\right)^2 - \left(x^2 - e^y\right)^2$

79. $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$

80. A delivery company accepts cylindrical boxes with the sum of whose length and girth (perimeter of 130 *inches*).

a) Find the dimension of the cylindrical package of greatest volume that can be shipped.

b) What is the maximum volume of such a package.

81. The volume of an open rectangular box is $12\ ft^3$. If the material for its base costs three times as much (per square *foot*) as the material for its side.

What are the dimensions of the box that can be constructed at a minimum cost.

82. The volume of an open rectangular box is $24\ ft^3$. If the material for its base costs four times and the top twice as much (per square foot) as the material for its side.

What are the dimensions of the box that can be constructed at a minimum cost.

83. The volume of a closed rectangular box is $24\ ft^3$. If the material for its base and top costs twice as much (per square *foot*) as the material for its side.
What are the dimensions of the box that can be constructed at a minimum cost.

84. Find the dimensions of an open box of maximum volume that can be constructed from $36\ ft^2$ of cardboard.

85. Find the dimensions of a closed box of maximum volume that can be constructed from $48\ ft^2$ of cardboard.

Section 10.7 – Lagrange Multipliers

10.7-1 Constrained Maxima and Minima

We consider a problem where a constrained minimum can be found by eliminating a variable.

Example 1

Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Solution

$$\left|\overrightarrow{OP}\right| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

Subject to the constraint that $2x+y-z-5=0$

Since $\left|\overrightarrow{OP}\right|$ has a minimum value wherever the function $f(x,y,z)=x^2+y^2+z^2$ has a minimum value.

$$2x+y-z-5=0$$

$$z=2x+y-5$$

$$h(x,y)=f(x,\ y,\ 2x+y-5)$$
$$=x^2+y^2+(2x+y-5)^2$$

$h_x = 2x+2(2x+y-5)(2)$ $=10x+4y-20=0$	$h_y = 2y+2(2x+y-5)(1)$ $=4x+4y-10=0$

$$\begin{cases}10x+4y=20\\4x+4y=10\end{cases} \Rightarrow \quad \boxed{x=\frac{5}{3},\ y=\frac{5}{6}}$$

$$z=2\left(\frac{5}{3}\right)+\frac{5}{6}-5$$

$$\boxed{=-\frac{5}{6}}$$

Therefore, the closest point to the origin is: $P\left(\frac{5}{3},\ \frac{5}{6},\ -\frac{5}{6}\right)$

The distance from P to the origin is:

$$\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{5}{6}\right)^2+\left(-\frac{5}{6}\right)^2}=\frac{5\sqrt{6}}{6}$$

Example 2

Find the points on the hyperbolic cylinder $x^2-z^2-1=0$ that are closest to the origin.

Solution

The points closest to the origin are the points whose coordinates minimize the value of the function $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint that $x^2-z^2-1=0$

$$x^2-z^2-1=0$$
$$z^2=x^2-1$$

$$h(x,\ y)=f\left(x,\ y,\ \sqrt{x^2-1}\right)$$
$$=x^2+y^2+\left(x^2-1\right)$$
$$=2x^2+y^2-1$$

$h_x=4x=0$	$h_y=2y=0$

That is, at the point (0, 0) ????

The domain of *h* is the entire *xy*-plane, the domain from which we can select the first two coordinates of the points (x,y,z) on the cylinder is restricted to the shadow of the cylinder on the xy-plane; it does not include the band between the lines $x=-1$ and $x=1$.

$$x^2-z^2-1=0$$
$$x^2=z^2+1$$
$$k(y,z)=f\left(z^2+1,\ y,\ z\right)$$

$$= z^2 + 1 + y^2 + z^2$$
$$= y^2 + 2z^2 + 1$$

$k_y = 2y = 0$	$k_z = 4z = 0$

That implies to $y = z = 0$ and which leads to $x^2 = z^2 + 1 = 1 \quad \rightarrow \quad x = \pm 1$

The corresponding points on the cylinder are $(\pm 1,\ 0,\ 0)$.

$k(y,z) = y^2 + 2z^2 + 1 \geq 1$ gives a minimum value for *k*. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

Solution 2

Another way to find the points on the cylinder closet to the origin is to imagine a small sphere centered at the origin expanding until it touches the cylinder.

$$f(x,y,z) = x^2 + y^2 + z^2 - a^2 \quad and \quad g(x,y,z) = x^2 - z^2 - 1$$

$$\nabla f = \lambda \nabla g$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left(2x\hat{i} - 2z\hat{k}\right) \qquad f_x\hat{i} + f_y\hat{j} + f_z\hat{k} = \lambda g_x\hat{i} + \lambda g_y\hat{j} + \lambda g_z\hat{k}$$

$$\begin{cases} 2x = 2\lambda x \\ 2y = 0 \\ 2z = -2\lambda z \end{cases}$$

Since that no point on the surface has a zero *x*-coordinate to conclude that $x \neq 0$.
Hence, $2x = 2\lambda x$ only if:

$$2 = 2\lambda$$

$$\underline{\lambda = 1}$$

For $\lambda = 1$

$$2z = -2\lambda z = -2z$$

For this to satisfies, *z* must be zero.

Also $2y = 0 \Rightarrow \underline{y = 0}$

We conclude that the points have coordinates of the form $(x,\ 0,\ 0)$

$x^2 = z^2 + 1$

$\underline{= 1}$

$\underline{x = \pm 1}$

The points on the cylinder closest to the origin are the points $(\pm 1,\ 0,\ 0)$.

10.7-2 The Method of *Lagrange* Multipliers

The method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

For some scalar λ (called a ***Lagrange multiplier***)

10.7-3 *Theorem* – The orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve:

$$C: \quad \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$$

If P_0 is a point on *C* where *f* has a local maximum or minimum relative to the values on *C*, then ∇f is orthogonal to *C* at P_0.

10.7-4 *Corollary*

At the points on a smooth curve $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ where differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \bullet \vec{v} = 0$ where $\vec{v} = \frac{d\vec{r}}{dt}$.

10.7-5 The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of *f* subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of *x*, *y*, *z*, and λ that simultaneously satisfy the equations.

$$\nabla f = \lambda \nabla g \quad and \quad g(x,y,z) = 0$$

For functions of two independent variables, the condition is similar, but without the variables *z*.

Example 3

Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse:

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

Solution

$f(x,y) = xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

We need to find: $\nabla f = \lambda \nabla g \quad and \quad g(x,y,z) = 0$

$$y\hat{i} + x\hat{j} = \frac{1}{4}\lambda x\hat{i} + \lambda y\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$y = \frac{1}{4}\lambda x, \quad x = \lambda y$$

$$y = \frac{1}{4}\lambda(\lambda y) = \frac{1}{4}\lambda^2 y$$

$$y = 0 \quad or \quad 1 = \frac{1}{4}\lambda^2$$

$$\lambda^2 = 4$$

$$\underline{\lambda = \pm 2}$$

Consider these ***two*** cases:

***Case* 1**:

If $y = 0$, then $x = y = 0$. But (0, 0) is not on the ellipse.

Hence, $y \neq 0$.

***Case* 2**:

If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$.

$$g(x,y) = \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{y^2}{2} + \frac{y^2}{2} = 1$$

$$y^2 = 1$$

$\underline{y = \pm 1}$

Therefore, $f(x, y) = xy$ takes on its extreme values on the ellipse at the points $(\pm 2,\ \pm 1)$. The extreme values are $xy = 2$ *and* $xy = -2$.

The Geometry of the Solution: The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$.

At the point (2, 1):

$$\begin{cases} \nabla f = \hat{i} + 2\hat{j} \\ \nabla g = \frac{1}{2}\hat{i} + \hat{j} \end{cases} \quad \Rightarrow \quad \nabla f = 2\nabla g$$

At the point (–2, 1):

$$\begin{cases} \nabla f = \hat{i} - 2\hat{j} \\ \nabla g = -\frac{1}{2}\hat{i} + \hat{j} \end{cases} \quad \Rightarrow \quad \nabla f = -2\nabla g$$

Example 4

Find the maximum and minimum values that the function $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$.

Solution

$$f(x,y)=3x+4y \qquad g(x,y)=x^2+y^2-1=0$$

$$\nabla f=f_x\hat{i}+f_y\hat{j}=3\hat{i}+4\hat{j} \qquad \nabla g=2x\hat{i}+2y\hat{j}$$

$$\nabla f=\lambda\nabla g$$

$$3\hat{i}+4\hat{j}=2\lambda x\hat{i}+2\lambda y\hat{j}$$

$$2\lambda x=3,\quad 2\lambda y=4$$

$$x=\frac{3}{2\lambda},\quad y=\frac{2}{\lambda}$$

$$\left(\frac{3}{2\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2-1=0$$

$$\frac{9}{4\lambda^2}+\frac{4}{\lambda^2}=1$$

$$9+16=4\lambda^2$$

$$25=4\lambda^2$$

$$\lambda^2=\frac{25}{4}$$

$$\lambda=\pm\frac{5}{2}$$

$$\begin{cases} x=\dfrac{3}{2\lambda} & =\pm\dfrac{3}{5} \\ y=\dfrac{2}{\lambda} & =\pm\dfrac{4}{5} \end{cases}$$

Therefore, $f(x,y)=3x+4y$ has extreme values $\left(\pm\frac{3}{5},\ \pm\frac{4}{5}\right)$

$$f\left(\frac{3}{5},\ \frac{4}{5}\right)=3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)=5$$

$$f\left(-\frac{3}{5},\ -\frac{4}{5}\right)=3\left(-\frac{3}{5}\right)+4\left(-\frac{4}{5}\right)=-5$$

The Geometry of the solution:

The level curves of the function $f(x,y)=3x+4y$ are the lines $3x+4y=c$

At the point $\left(\frac{3}{5},\ \frac{4}{5}\right)$: $\quad \nabla f=3\hat{i}+4\hat{j},\quad \nabla g=\frac{6}{5}\hat{i}+\frac{8}{5}\hat{j},\quad \nabla f=\frac{5}{2}\nabla g$

10.7-6 Lagrange Multipliers with Two Constraints

To find the extreme values of a differentiable function $f(x,y,z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x,y,z)=0 \quad and \quad g_2(x,y,z)=0$$

g_1 *and* g_2 are differentiable, with ∇g_1 not parallel to ∇g_2

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z)=0, \quad g_2(x,y,z)=0$$

Example 5

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse.
Find the points on the ellipse that lie closet to and farthest from the origin.

Solution

$$f(x,y,z)=x^2+y^2+z^2$$

$$g_1(x,y,z)=x^2+y^2-1=0$$

$$g_2(x,y,z)=x+y+z-1=0$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda\left(2x\hat{i}+2y\hat{j}\right)+\mu\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=(2\lambda x+\mu)\hat{i}+(2\lambda y+\mu)\hat{j}+\mu\hat{k}$$

$2z=\mu$	$2x=2\lambda x+\mu$ $2(1-\lambda)x=\mu=2z$ $(1-\lambda)x=z$	$2y=2\lambda y+\mu$ $2(1-\lambda)y=\mu=2z$ $(1-\lambda)y=z$

$(1-\lambda)x=z=(1-\lambda)y$

These satisfy if either λ = 1 and $z = 0$ or λ ≠ 1 and $x=y=\frac{z}{1-\lambda}$.

If $z = 0$

$$\begin{cases} x^2+y^2-1=0 \\ x+y-1=0 \end{cases} \rightarrow \quad x=1-y$$

$$(1-y)^2+y^2-1=0$$

$$1-2y+y^2+y^2-1=0$$

$$2y(y-1)=0$$

$$\begin{cases} y=0 & \rightarrow x=1 \\ y=1 & \rightarrow x=0 \end{cases}$$

The points are: $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$

If $x=y$,

$x^2+y^2-1=0$ $x^2+x^2-1=0$ $2x^2=1$ $x^2=\frac{1}{2}$	$x+y+z-1=0$ $2x+z=1$ $z=1-2x$
$x=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$	$\begin{cases} x=-\frac{\sqrt{2}}{2} & \underline{z=1+\sqrt{2}} \\ x=\frac{\sqrt{2}}{2} & \underline{z=1-\sqrt{2}} \end{cases}$

The points are: $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 1-\sqrt{2}\right)$ and $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ 1+\sqrt{2}\right)$

The points on the ellipse closest to the origin are $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$. The point on the farthest from the origin is $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ 1+\sqrt{2}\right)$.

Exercises *Section* 10.7 – Lagrange Multipliers

$(\mathbf{1}-\mathbf{5})$ Find the extreme values of

1. On the ellipse $x^2+2y^2=1$ where $f(x,y)=xy$.

2. $f(x,y)=xy$ subject to the constraint $g(x,y)=x^2+y^2-10=0$.

3. $f(x,y)=x^3+y^2$ on the circle $x^2+y^2=1$

4. $f(x,y)=x^2+y^2-3x-xy$ on the circle $x^2+y^2\le 9$

5. $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$.

6. Find the points on the curve $x^2y=2$ nearest the origin.

7. Use the method of Lagrange multipliers to find:

a) The minimum value of $x+y$, subject to the constraints $xy=16,\quad x>0,\quad y>0$

b) The maximum value of xy, subject to the constraints $x+y=16$

8. Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius *a*. What is the largest surface area?

9. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ with sides parallel to the coordinate axes.

10. Find the maximum and minimum values of x^2+y^2 subject to the constraint $x^2-2x+y^2-4y=0$

11. The temperature at a point (x,y) on a metal plate is $T(x,y)=4x^2-4xy+y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

12. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

13. A closed rectangular box is to have volume $V\ cm^3$. The cost of the material used in the box is a $a\ cents/cm^2$ for top and bottom, $b\ cents/cm^2$ for front and back, and $c\ cents/cm^2$ for the remaining sides. What dimensions minimize the total cost of materials?

14. Find the extreme values of $f(x,y,z)=x(y+z)$ on the curve of intersection of the right circular cylinder $x^2+y^2=1$ and the hyperbolic cylinder $xz=1$.

15. Find the point closest to the origin on the curve of intersection of the plane $x+y+z=1$ and the cone $z^2=2x^2+2y^2$.

16. Find the point on the plane $x+2y+3z=13$ closest to the point $(1,\ 1,\ 1)$.

17. Find the point on the sphere $x^2+y^2+z^2=4$ farthest from the point $(1,\ -1,\ 1)$.

18. Find the minimum distance from the surface $x^2-y^2-z^2=1$ to the origin.

19. Find the maximum and minimum values of $f(x,y,z)=x-2y+5z$ on the sphere $x^2+y^2+z^2=30$.

20. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

21. A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters Earth's atmosphere and its surface begins to heat. After 1 *hour*, the temperature at the point (x,y,z) on the probe's surface is $T(x,y,z)=8x^2+4yz-16z+600$.

Find the hottest point on the probe's surface.

22. What point on the plane $x+y+4z=8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

(23 – 58) Use ***Lagrange multipliers*** to find the maximum and minimum values of *f* (when they exist) subject to the given constraint.

23. $f(x,\ y)=2x+y+10$ *subject to* $2(x-1)^2+4(y-1)^2=1$

24. $f(x,\ y)=x^2y^2$ *subject to* $2x^2+y^2=1$

25. $f(x,\ y)=x+2y$ *subject to* $x^2+y^2=4$

26. $f(x,\ y)=xy^2$ *subject to* $x^2+y^2=1$

27. $f(x,\ y)=x+y$ *subject to* $x^2-xy+y^2=1$

28. $f(x,\ y)=x^2+y^2$ *subject to* $2x^2+3xy+2y^2=7$

29. $f(x, y) = xy \quad subject\ to \quad x^2 + y^2 - xy = 9$

30. $f(x, y) = x - y \quad subject\ to \quad x^2 - 3xy + y^2 = 20$

31. $f(x, y) = e^{2xy} \quad subject\ to \quad x^2 + y^2 = 16$

32. $f(x, y) = x^2 + y^2 \quad subject\ to \quad x^6 + y^6 = 1$

33. $f(x, y) = y^2 - 4x^2 \quad subject\ to \quad x^2 + 2y^2 = 4$

34. $f(x, y) = xy + x + y \quad subject\ to \quad x^2y^2 = 4$

35. $f(x, y) = xy \quad subject\ to \quad x^2 + y^2 = 1$

36. $f(x, y) = x^2 + (y-2)^2 \quad subject\ to \quad x^2 - y^2 = 1$

37. $f(x, y) = x^2 + y^2 \quad subject\ to \quad xy = 1$

38. $f(x, y) = xy^2 \quad subject\ to \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

39. $f(x, y) = xy \quad subject\ to \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

40. $f(x, y) = x + y \quad subject\ to \quad x^4 + y^4 = 1.$

41. $f(x, y) = x^2 + y^2 \quad subject\ to \quad x^4 + 7x^2y^2 + y^4 = 1$

42. $f(x, y, z) = xyz \quad subject\ to \quad x^2 + y^2 + z^2 = 1$

43. $f(x, y, z) = xyz \quad subject\ to \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

44. $f(x, y, z) = x + 2y + 4z \quad subject\ to \quad x^2 + y^2 + z^2 = 21$

45. $f(x, y, z) = 2x + 3y + 5z \quad subject\ to \quad x^2 + y^2 + z^2 = 19$

46. $f(x, y, z) = x^4 + y^4 + z^4 \quad subject\ to \quad x + y + z = 1$

47. $f(x, y, z) = x + 3y - z \quad subject\ to \quad x^2 + y^2 + z^2 = 4$

48. $f(x, y, z) = xyz \quad subject\ to \quad x^2 + 2y^2 + 4z^2 = 9$

49. $f(x, y, z) = x \quad subject\ to \quad x^2 + y^2 + z^2 - z = 1$

50. $f(x, y, z) = x - z \quad subject\ to \quad x^2 + y^2 + z^2 - y = 2$

51. $f(x, y, z) = x^2 + y^2 + z^2 \quad subject\ to \quad x^2 + y^2 + z^2 - 4xy = 1$

52. $f(x, y, z) = x + y + z \quad subject\ to \quad x^2 + y^2 + z^2 - 2x - 2y = 1$

53. $f(x, y, z) = 2x + z^2 \quad subject\ to \quad x^2 + y^2 + 2z^2 = 25$

54. $f(x, y, z) = x^2 + y^2 - z$ *subject to* $z = 2x^2y^2 + 1$

55. $f(x, y, z) = \sqrt{xyz}$ *subject to* $x + y + z = 1$ *with* $x \geq 0,\ y \geq 0,\ z \geq 0$

56. $f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $x^2 + y^2 + z^2 - 4xy = 1$

57. $f(x, y, z) = x + 2y - z$ *subject to* $x^2 + y^2 + z^2 = 1$

58. $f(x, y, z) = x^2y^2z$ *subject to* $2x^2 + y^2 + z^2 = 25$

59. Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

60. Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of $32\pi\ in^3$.

61. Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1,\ 3,\ 1)$. Give an argument showing you have found an absolute minimum of the distance function.

62. Let $P_0(a, b, c)$ be a fixed point in $\mathbb{R}^3$ and let $d(x, y, z)$ be the distance between P_0 and a variable point $P(x, y, z)$.

a) Compute $\nabla d(x, y, z)$.

b) Show that $\nabla d(x, y, z)$ points in the direction from P_0 to P and has magnitude 1 for all (x, y, z).

c) Describe the level surfaces of *d* and give the direction of $\nabla d(x, y, z)$ relative to the level surfaces of *d*.

d) Discuss $\lim\limits_{P \to P_0} \nabla d(x, y, z)$.

63. A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (the girth is the perimeter of the smallest base of the box).

64. Find the rectangular box with a volume of 16 ft^3 that has minimum surface area.

65. Find the minimum and maximum distances between the ellipse $x^2 + xy + y^2 = 1$ and the origin.

66. Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$.

67. Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$.

68. Find the point on the plane $2x + 3y + 6z - 10 = 0$ *closest to the point* $(-2,\ 5,\ 1)$.

69. Find the point on the surface $4x + y - 1 = 0$ *closest to the point* $(1,\ 2,\ -3)$.

70. Find the points on the cone $z^2 = x^2 + y^2$ *closest to the point* $(1,\ 2,\ 0)$.

71. Find the minimum and maximum distances between the sphere:
$x^2 + y^2 + z^2 = 9$ *closest to the point* $(2,\ 3,\ 4)$

72. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ *subject to* $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

73. Find the maximum value of $x_1 + x_2 + \cdots + x_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$.

74. Find the maximum value of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for the given positive real numbers $a_1, a_2, \cdots, a_n$.

75. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line *L*. Find the point on *L* nearest the origin.

76. Find the maximum and minimum values of:
$f(x,\ y,\ z) = xyz$ *subject to* $x^2 + y^2 = 4$ *and* $x + y + z = 1$

77. The paraboloid $z = x^2 + 2y^2 + 1$ and the plane $x - y + 2z = 4$ intersect in a curve *C*. Find the points on *C* that have minimum and maximum distance from the origin.

78. Find the maximum and minimum values of $f(x,\ y,\ z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

79. The temperature of points on an elliptical plate $x^2 + y^2 + xy \le 1$ is given by $T(x,\ y) = 25\left(x^2 + y^2\right)$. Find the hottest and coldest temperatures on the edge of the elliptical plate.

80. A triangle inscribed in a circle of radius *R*. Find the triangle shape that has the largest perimeter.

81. A metal cistern in the shape of a right circular cylinder with volume $V = 50\ m^3$ needs to be painted each year to reduce corrosion. The paint is applied only to surfaces exposed to the elements (the outside cylinder wall and the circular top).

Find the dimensions r and h of the cylinder that minimize the area of the painted surfaces.

82. A piece of cardboard measures 10-*in*. by 15-*in*. Two equal squares are removed from the corners of 10-*in*. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

Find the maximum volume and the dimensions *x*, *base*, and *lid*.

83. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 *inches*.

Find the dimensions of the package of maximum volume that can be sent.

84. A rectangle is to be inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$

What should be the dimension of the rectangle be to maximize its area?
What is the maximum area?

85. A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone.

Find the radius, height, and volume of the cone of greatest volume that can be made this way.

86. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 84 *inches*.

a) Find the dimensions of the package of maximum volume that can be sent.
b) What is the maximum volume of the package?

87. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.

88. The figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base.

What values of r and h will give the smaller cone the largest possible volume?

89. Sometimes the solution of a *max-min* problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.

90. A rectangular box with a square base of height *y feet* and the square with length *x feet* is inscribed in a right circular cone with a height of *h feet* and a base with radius *r feet*.

Find the volume of the rectangular box, where *h* and *r* are constants.

91. A rectangular sheet of perimeter 36 *cm* and dimensions *x cm* and *y cm* is to be rolled into a cylinder as shown below.

What values of *x* and *y* give the largest volume?

92. A cylindrical container can with a volume of $500\ cm^3$.

a) What dimensions will minimize the material needed to produce each can?
b) What is minimize the surface area?

93. Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 *cm*. the three resulting flaps are bent up, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.

94. A cone is constructed by cutting a sector of angle θ from a circular sheet of metal with radius 20 *cm*. the cut sheet is then folded up and wheeled.

What angle θ maximizes the volume of the cone?
Find the radius and height of the cone with maximum volume that can be formed in this way.

95. A box is inscribed in a sphere with a radius *R*. Find the dimension of the box that maximize the volume.

Summary

$\frac{\partial^2 f}{\partial y \partial x}$ *or* $f_{xy} = \left(f_x\right)_y$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \quad \frac{dz}{dx} = -\frac{F_x}{F_z}, \quad \frac{dz}{dy} = -\frac{F_y}{F_z}$$

Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$\left(\frac{df}{ds}\right)_{\vec{u},\, P_0} = \left(\nabla f\right)_{P_0} \bullet \vec{u}$$

Directional Derivative: $D_{\vec{u}} f = \nabla f \bullet \vec{u} = |\nabla f|\,|\vec{u}|\cos\theta = |\nabla f|\cos\theta$

Tangent Plane: $f_x\left(P_0\right)\left(x - x_0\right) + f_y\left(P_0\right)\left(y - y_0\right) + f_z\left(P_0\right)\left(z - z_0\right) = 0$

Normal Line: $x = x_0 + f_x\left(P_0\right)t, \quad y = y_0 + f_y\left(P_0\right)t, \quad z = z_0 + f_z\left(P_0\right)t$

Linearization: $L(x,y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$

f has a local *maximum* at $(a,\ b)$ if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$.

f has a local *minimum* at $(a,\ b)$ if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$.

f has a *saddle* point at $(a,\ b)$ if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.

The test is inconclusive at $(a,\ b)$ if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$.

Lagrange Multipliers: $\nabla f = \lambda \nabla g \quad \textit{and} \quad g(x,y,z) = 0$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z) = 0, \quad g_2(x,y,z) = 0$$

Exercises *Lecture* 10 – Review

(1 – 3) For the given function:

a) Find the function's domain.
b) Find the function's range.
c) Find the boundary of the function's domain.

1. $f(x,y)=3x+5y+2$

2. $f(x,y)=\sqrt{4-x^2-y^2}$

(3 – 16) Find the limit.

3. $\lim\limits_{(x,y)\to(-1,\,2)}\left(x^2-2xy+y^2+3x-4y+2\right)$

4. $\lim\limits_{(x,y)\to(1,\,-2)}\dfrac{3x+y}{2x-3y}$

5. $\lim\limits_{(x,y)\to(5,\,-2)}\sqrt{\dfrac{x^3-y}{x^2+2y}}$

6. $\lim\limits_{\substack{(x,y)\to(0,\,0)\\ x=y}}\dfrac{2xy}{x^2+y^2}$

7. $\lim\limits_{(x,y)\to(2,\,1)}\dfrac{3xy^2}{x^2+3y^4}$

8. $\lim\limits_{(x,y)\to(0,\,1)}\dfrac{y^2\sin x}{x}$

9. $\lim\limits_{(x,y)\to\left(\frac{\pi}{4},\,1\right)}\dfrac{y^2\tan x}{y+1}$

10. $\lim\limits_{(x,y)\to(0,\,0)}\dfrac{xy-y^2}{\sqrt{x}-\sqrt{y}}$

11. $\lim\limits_{(x,y)\to(0,\,0)}\dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

12. $\lim\limits_{(x,y,z)\to(-1,\,-2,\,3)}\sqrt{9-x^2-2y^2+z^2}$

13. $\lim\limits_{(x,y,z)\to(1,\,2,\,-3)}\dfrac{xz^2-y^2z}{1-xyz}$

14. $\lim\limits_{(x,y,z)\to(-1,\,-2,\,3)}\sqrt{z^2-x^2-2y^2}$

15. $\lim\limits_{(x,y,z)\to(1,\,2,\,-3)}\dfrac{x^2-y+z}{2x+3y-z}$

16. $\lim\limits_{(x,y,z)\to(1,\,2,\,0)}\dfrac{x^2-4xy+z^2}{2x+y-z}$

(17 – 24) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

17. $f(x,y)=x^3-2x^2y+xy-2y^2$

18. $f(x,y)=\left(x^3-y^2\right)^5$

19. $f(x,y)=\dfrac{x^2y-2x}{x-2y}$

20. $f(x,y)=\sin\left(xy^2-2x\right)+\cos\left(x^2+2xy\right)$

21. $f(x,y)=\sec\left(x^2y\right)+\tan\left(x-y^2\right)$

22. $f(x,y)=e^{x^2+y^2}$

23. $f(x,y)=\ln\left(x^2+y^2\right)$

24. $f(x,y)=\tan^{-1}\left(x^2+y^2\right)$

$(\mathbf{25-30})$ Find f_x, f_y, *and* f_z.

25. $f(x,y,z)=x^2-2xy+xzy+z^2$

26. $f(x,y,z)=\dfrac{x^2y-2xz+z^2}{x-2yz}$

27. $f(x,y,z)=\sin\left(y^2-xz\right)+\cos\left(x^2+yz\right)$

28. $f(x,y,z)=2\sin(2\pi x)\cos(3\pi y)\tan(4\pi z)$

29. $f(x,y,z)=2\sin(3x)\cos(2y)\,e^{xz^2}$

30. $f(x,y,z)=\ln(xy)+\cos(3xy)\,e^{-z}$

$(\mathbf{31-36})$ Find *second*-order partial derivatives of the function.

31. $f(x,y)=x^2+3xy+y^2$

32. $f(x,y)=x^3-2x^2y+3xy^2+y^3$

33. $f(x,y)=\ln\left(x^2+y^2\right)$

34. $f(x,y,z)=3x^3+2y^3+5z^3+10x^2y^2z^2$

35. $f(x,y,z)=\ln\left(x^2+y^2+z^2\right)$

36. $f(x,y,z)=e^{x^2+y^2+z^2}$

$(\mathbf{37-38})$ Verify that the function satisfies ***Laplace's equation*** $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$

37. $f(x,y)=\sin x\,\sinh y$

38. $f(x,y)=\ln\sqrt{(x-a)^2+(y-b)^2}$

$(\mathbf{39-40})$ Verify that the function satisfies ***Laplace's equation*** $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$

39. $f(x,y,z)=e^{-2x}\sin y\,\cos z$

40. $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

$(\mathbf{41-44})$ Express $\dfrac{dw}{dt}$ as a function of t, then evaluate $\dfrac{dw}{dt}$ at the given value of t.

41. $w=y\,e^{-x}$, $x=t$, $y=\sin t$, $t=0$

42. $w=x^3+y^3$, $x=\cos t$, $y=\sin t$, $t=\dfrac{\pi}{4}$

43. $w=x^2+y^2+z^2$, $x=\cos t$, $y=\sin t$, $z=t$, $t=\dfrac{\pi}{2}$

44. $w=\cos(xy)-z^2$, $x=e^t$, $y=\ln t$, $z=t$, $t=1$

45. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ when $r=1$ and $s=\dfrac{\pi}{2}$ if

$w=\sin(2x+y)$, $x=rs$, $y=r+\cos(s)$

46. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = \tan^{-1}\frac{x}{y}$, $x = 2u + v$, $y = 3u - v$

(47 – 54) Evaluate $\frac{dy}{dx}$, $\frac{dx}{dy}$

47. $x^2 = y^2 + \sin xy$

48. $x^3 \sin y + y = 3x + 2$

49. $\cos^2\left(y^2 - 3x\right) = y + 3$

50. $y\sqrt{x+3} = xy + 9$

51. $4x^5 + \tan y = x^2 + 5y$

52. $\ln(2 + xy) = x^2 - y$

53. $\sqrt{x^2 + 2xy + y^4} = 1$

54. $x\,e^y + y\,e^x + 2\ln x - 2 - 3\ln 2 = 0$

(55 – 60) Evaluate $\frac{dy}{dx}$, $\frac{dy}{dz}$, $\frac{dz}{dx}$.

55. $x^2 = y^2 + \sin z$

56. $2x^2 + 3xyz + 4y^2 - 5xy + 2yz = 0$

57. $z = x\,y\,e^{-\frac{1}{2}\left(x^2+y^2\right)}$

58. $e^{xyz}\ln xyz = 5$

59. $x^3y^3z^3 = x^2 + y^2 + z^2$

60. $z = \sqrt{3x^2 + y^2}$

(61 – 66) Find ∇f at the given point.

61. $f(x, y) = x^2 + y^2$, @ $(1, 1)$

62. $f(x, y) = e^{x^2+y^2}$, @ $(2, -1)$

63. $f(x, y) = x^3y^2$, @ $(3, 1)$

64. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, @ $(-1, 1, 1)$

65. $f(x, y, z) = \frac{x^2y^3}{z^4}$, @ $(1, 1, 1)$

66. $f(x, y, z) = \tan\left(x^2 + y^2 + z^2\right)$, @ $(1, 0, 1)$

67. Find the direction in which $f(x, y) = x^2y$ increases and decreases most rapidly at $P_0(-1, -1)$ and find the derivative of f in each direction.

68. Find an equation for the plane tangent and normal line to the level surface $f(x, y, z) = y\,e^{-x^2}\sin z$ at the point $\left(0, 1, \frac{\pi}{3}\right)$.

$(\mathbf{69-74})$ Find all the local maxima, local minima, and saddle points of the function.

69. $f(x, y) = x^2 + 2y^2 - 4x + 4y$

70. $f(x, y) = xy + x - y$

71. $f(x, y) = x + \dfrac{3x^2 - 6y + 12y^2}{x^2 - 2y + y^2}$

72. $f(x, y) = \dfrac{x^3 - y^3}{x^2 - y^2}$

73. $f(x, y) = e^{x^2 - 2x - 4y^2 + 5}$

74. $f(x, y) = x + \dfrac{4y^2}{x}$

75. Find the extreme, and saddle points of the function:

$$f(x, y) = xy - x^3y^2 \qquad R = \{(x, y)\ \ 0 \le x \le 1,\ 0 \le y \le 1\}$$

$(\mathbf{76-81})$ Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint.

76. $f(x, y) = xy - x^3y^2$ *subject to* $0 \le x \le 1,\quad 0 \le y \le 1$

77. $f(x, y) = x + 2y$ *subject to* $x^2 + y^2 \le 1$

78. $f(x, y) = xy^2 - xy^2 - y^3 + y$ *subject to* $x^2 + y^2 = 1$

79. $f(x, y) = (x + y)\, e^{-x^2 - y^2}$ *subject to* $x^2 + y^2 = 1$

80. $f(x, y, z) = x + y - z$ *subject to* $x^2 + y^2 + z^2 = 1$

81. $f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $xyz^2 = 1$

82. A manufacturer wants to design an open box having square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

83. A rectangular box with a square base, an open top, and a volume of $216\ in^3$ is to be constructed. Suppose the cost of the material for the base is $0.20\ \ \$/in^2$ and the cost of the material for the sides is $0.30\ \ \$/in^2$.

a) What dimensions of the box should be to minimize the cost?
b) What is the cost of the box?

84. A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

What values of r and h will give the cylinder the largest possible volume?

85. A rectangular box with a square base of height y *feet* and the square with length x *feet* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

Find the volume of the rectangular box.

86. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

Practice *Exam*

Find the limit:

1. $\lim\limits_{(x,\,y)\to(-1,\,1)} \dfrac{x^2-y^2}{x^2-xy-2y^2}$

2. $\lim\limits_{P\to(\pi,\,0,\,2)} ze^{-2y}\cos 2x$

3. $\lim\limits_{(x,\,y,\,z)\to\left(\frac{\pi}{2},\,0,\,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz}$

4. $\lim\limits_{(x,\,y)\to(1,\,2)} \dfrac{x^2y}{x^4+2y^2}$

Find the first partial derivatives of

5. $f(x,y)=5xy-7x^2-y^2+3x-6y+2$

6. $f(x,\,y)=x\cos y-y\sin x$

7. $f(x,\,y,\,z)=e^{x+2y+3z}$

8. $f(x,y,z)=e^{-\left(x^2+y^2+z^2\right)}$

Find ***second-order*** partial derivatives of the function:

9. $w=x^2\tan(xy)$

10. $g(x,y)=x^2y+\cos y+y\sin x$

11. Verify that the function satisfies Laplace's equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

$$u(x,\,y)=y\left(3x^2-y^2\right)$$

12. Evaluate $\dfrac{dy}{dx}$ $\quad x^3+3xy^2-y^5=0$

13. Find $\dfrac{\partial w}{\partial r}$ when $r=1,\ s=-1$ if $w=(x+y+z)^2,\quad x=r-s,\quad y=\cos(r+s),\quad z=\sin(r+s)$

14. Evaluate the derivatives $w'(t)$, where $w=xy\sin z,\quad x=t^2,\quad y=4t^3,\quad and\quad z=t+1$

15. Find ∇f at the given point $\quad f(x,\,y,\,z)=xy+yz+xz+4;\quad P=(2,\ -2,\ 1)$

16. Find all the ***local maxima***, ***local minima***, and ***saddle points*** of the function

$$f(x,\,y)=10-x^3-y^3-3x^2+3y^2$$

17. Find the maximum and minimum values of

$f(x,\,y,\,z)=x+2y-z\quad$ *subject to* $\quad x^2+y^2+z^2=1$

(Use the method of ***Lagrange multipliers***)

Lecture Eleven

Multiple Integrals

Section 11.1 – Double Integrals over Rectangular Regions

11.1−1 Double Integrals

Consider a function .. $f(x,y)$ .. defined on a rectangular region *R*,

$$R: \quad a \le x \le b, \quad c \le y \le d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.

To form a Riemann sum over *R*, select a point (x_k, y_k) in the kth small rectangle, multiply the value of *f* at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

As the rectangles get narrow and short, their number *n* increases, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

Then the function f is said to be integrable and the limit is called double integral of f over *R*,

$$\iint_R f(x,y)\,dA \quad \textit{or} \quad \iint_R f(x,y)\,dx\,dy$$

Example 1

Evaluate $\displaystyle\int\int e^{4x}y^3\,dy\,dx$

Solution

$$\int\int e^{4x}y^3\,dy\,dx = \int e^{4x}dx \int y^3\,dy$$

$$= \frac{1}{4}e^{4x}\left(\frac{1}{4}y^4\right)+C$$

$$= \frac{1}{16}y^4 e^{4x}+C$$

Example 2

Calculate $\displaystyle\iint_R f(x,y)\,dA$ for $f(x,y)=100-6x^2y$ *and* $R:\ 0\le x\le 2,\quad -1\le y\le 1$

Solution

$$\int_{-1}^{1}\int_{0}^{2}\left(100-6x^2y\right)dx\,dy = \int_{-1}^{1}\left(100x-2x^3y\right|_0^2\,dy$$

$$= \int_{-1}^{1}(200-16y)\,dy$$

$$= 200y-8y^2\Big|_{-1}^{1}$$

$$= 200-8-(-200-8)$$

$$= 400$$

Example 3

Calculate $\displaystyle\iint_R \left(x^2y^2+\cos\pi x+\sin\pi y\right)dA \qquad R:\ [-2,\ -1]\times[0,\ 1]$

Solution

$$\iint_R \left(x^2y^2+\cos\pi x+\sin\pi y\right)dA = \int_0^1\int_{-2}^{-1}\left(x^2y^2+\cos\pi x+\sin\pi y\right)dx\,dy$$

$$= \int_0^1 \left(\frac{1}{3}y^2x^3 + \frac{1}{\pi}\sin \pi x + x \sin \pi y\right)\Big|_{-2}^{-1} dy$$

$$= \int_0^1 \left(-\frac{1}{3}y^2 - \sin \pi y + \frac{8}{3}y^2 + 2\sin \pi y\right) dy$$

$$= \int_0^1 \left(\frac{7}{3}y^2 + \sin \pi y\right) dy$$

$$= \frac{7}{9}y^3 - \frac{1}{\pi}\cos \pi y \Big|_0^1$$

$$= \frac{7}{9} + \frac{1}{\pi} + \frac{1}{\pi}$$

$$= \frac{7}{9} + \frac{2}{\pi}$$

$$\iint_R \left(x^2y^2 + \cos \pi x + \sin \pi y\right) dA = \int_{-2}^{-1}\int_0^1 \left(x^2y^2 + \cos \pi x + \sin \pi y\right) dy\, dx$$

$$= \int_{-2}^{-1} \left(\frac{1}{3}x^2y^3 + y \cos \pi x - \frac{1}{\pi}\cos \pi y\right)\Big|_0^1 dx$$

$$= \int_{-2}^{-1} \left(\frac{1}{3}x^2 + \cos \pi x + \frac{2}{\pi}\right) dx$$

$$= \frac{1}{9}x^3 + \frac{1}{\pi}\sin \pi x + \frac{2}{\pi}x \Big|_{-2}^{-1}$$

$$= -\frac{1}{9} - \frac{2}{\pi} + \frac{8}{9} + \frac{4}{\pi}$$

$$= \frac{7}{9} + \frac{2}{\pi}$$

Example 4

Evaluate $\displaystyle\int_0^1 \int_y^1 y\, e^{-x^3}\, dx\, dy$

Solution

$$y \le x \le 1 \quad \rightarrow \quad \begin{cases} y = x \\ x = 1 \end{cases}$$

$0 \le y \le x \quad \& \quad 0 \le x \le 1$

$$\int_0^1 \int_y^1 y\, e^{-x^3}\, dx\, dy = \int_0^1 \int_0^x y\, e^{-x^3}\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 e^{-x^3} y^2 \Big|_0^x \, dx$$

$$= \frac{1}{2} \int_0^1 x^2 e^{-x^3} \, dx$$

$$= -\frac{1}{6} \int_0^1 e^{-x^3} \, d\left(-x^3\right)$$

$$. = -\frac{1}{6} e^{-x^3} \Big|_0^1 .$$

$$= -\frac{1}{6}\left(e^{-1} - 1\right)$$

$$= \frac{1}{6}\left(1 - \frac{1}{e}\right)$$

$$= \frac{e-1}{6e}$$

Example 5

Calculate $\iint_R \frac{\sin x}{x}\, dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution

$$\int_0^1 \int_0^x \left(\frac{\sin x}{x}\right) dy\, dx = \int_0^1 \frac{\sin x}{x} \left(y\right)\Big|_0^x dx$$

$$= \int_0^1 \sin x\, dx$$

$$= -\cos x \Big|_0^1$$

$$= 1 - \cos 1$$

❖ $\int_0^1 \int_y^1 \left(\frac{\sin x}{x}\right) dx\, dy$, we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

11.1–2 Finding Limits on Integration

1. Sketch the region of Integration and label the bounding curves

2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).

$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$$

3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral is

$$\iint_R f(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

Example 6

Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$ and write an equivalent integral with the order of integration reversed.

Solution

The given inequalities are: $x^2 \le y \le 2x \quad and \quad 0 \le x \le 2$

$$\begin{cases} y = x^2 & x = \sqrt{y} \\ y = 2x & x = \dfrac{y}{2} \end{cases}$$

$$\begin{cases} x = 0 & y = 0 \\ x = 2 & y = 4 \end{cases}$$

The integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$

- If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region *R*, then the following properties hold

1. *Constant Multiple*: $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$
2. *Sum and Difference*: $\iint_R \left(f(x,y) \pm g(x,y)\right)dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
3. *Domination*:

   a) $\iint_R f(x,y)\,dA \ge 0 \quad if \quad f(x,y) \ge 0 \;\; on\ R$

   b) $\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA \quad if \quad f(x,y) \ge g(x,y) \;\; on\ R$
4. *Additivity*: $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

   If *R* is the union of two non-overlapping regions R_1 and R_2 .

11.1–3 *Definition*

The area of a closed, bounded plane region R is

$$Area = \iint_R dA$$

Example 7

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

$$y = x = x^2$$

$$x = 0,\ 1$$

$$Area = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 y \Big|_{x^2}^{x} dx$$

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ unit}^2$$

Example 8

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution

$$y = x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$x = -1,\ 2$

$$Area = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

$$= \int_{-1}^{2} y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \left(x + 2 - x^2\right) dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= \frac{1}{2}(4) + 4 - \frac{1}{3}(8) - \left(\frac{1}{2}(-1)^2 - 2 + \frac{1}{3}\right)$$

$$= \frac{9}{2} \quad unit^2$$

Example 9

Find the area of the region R between $y = x^2$ and $y^2 = x$.

Solution

$$y = x^2 = \left(y^2\right)^2$$

$$y = y^4$$

$y = 0,\ 1$

$0 \le y \le 1$

$$y = x^2$$

$$x = \sqrt{y}$$

$y^2 \le x \le \sqrt{y}$

$$Area = \int_{0}^{1}\int_{y^2}^{\sqrt{y}} dx\,dy$$

$$= \int_0^1 x \Big|_{y^2}^{\sqrt{y}} \, dy$$

$$= \int_0^1 \left(y^{1/2} - y^2 \right) dy$$

$$= \frac{2}{3} y^{3/2} - \frac{1}{3} y^3 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3} \quad unit^2$$

11.1–4 Average values

Average values of f **over** $\boldsymbol{R}$ $= \dfrac{1}{area\ of\ R} \displaystyle\iint_R f dA$

❖ Average value of f over $R = \dfrac{1}{area\ of\ R} \displaystyle\iint_R f\ dA$

$$= \frac{2}{\pi}$$

Example 10

Find the average value of $f(x, y) = x \cos xy$ over the rectangle R: $0 \le x \le \pi, \quad 0 \le y \le 1$.

Solution

$$\int_0^{\pi} \int_0^1 x \cos xy \ dy \, dx = \int_0^{\pi} \sin xy \Big|_0^1 \ dx \qquad \int x \cos xy \ dy = \sin xy + C$$

$$= \int_0^{\pi} \sin x \ dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= 2$$

11.1–5 *Double Integrals as Volumes*

$$Volume = \lim_{n\to\infty} S_n = \iint_R f(x,y)\, dA, \text{ where } \Delta A_k \to 0 \text{ as } n \to \infty$$

$n = 16$ $n = 64$ $n = 256$

As n increases, the ***Riemann sum*** approximations approach the total volume of the solid.

If $f(x,y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)dA$.

If R is a region in the xy-plane, bounded ***above*** and ***below*** by the curves $y = g_1(x)$ and $y = g_2(x)$ and on the sides by the lines $x = a,\ x = b$. Calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy$$

Then integrate $A(x)$ from $x = a$ *to* $x = b$ to get the volume as an iterated integral

$$V = \int_a^b A(x)\, dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Similarly, if R is a region bounded by the curves $x = h_1(y)$, $x = h_2(y)$, and the lines $y = c,\ y = d$, then the volume calculated by slicing is given by the iterated integral .

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

$$Volume = \lim_{k \to n} \sum f(x_k, y_k)\, \Delta A_k$$

$$= \iint_R f(x, y)\, dA$$

Example 11

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region

$R:\quad 0 \le x \le 2,\quad 0 \le y \le 1$ in the xy-plane.

Solution

$$Volume = \int_{x=0}^{x=2} A(x)\, dx$$

$$= \int_0^2 \int_{y=0}^{y=1} (4 - x - y)\, dy\, dx$$

$$= \int_0^2 \left(4y - xy - \frac{1}{2}y^2\right)\Big|_0^1\, dx$$

$$= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2}\right) dx$$

$$= \int_0^2 \left(\frac{7}{2} - x\right) dx$$

$$= \frac{7}{2}x - \frac{1}{2}x^2 \Big|_0^2$$

$$= 7 - 2$$

$$= 5 \quad unit^3$$

$$Volume = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy$$

Example 12

Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$, and $x = 1$, and whose top lies in the plane $z = f(x, y) = 3 - x - y$

Solution

$$0 \le x \le 1, \quad 0 \le y \le x$$

$$Volume = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

$$= \int_0^1 \left(3y - xy - \frac{1}{2} y^2 \right|_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{1}{2} x^2 \right) dx$$

$$= \int_0^1 \left(3x - \frac{3}{2} x^2 \right) dx$$

$$= \frac{3}{2} x^2 - \frac{1}{2} x^3 \Big|_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= 1 \quad unit^3$$

$$Volume = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy$$

$$= \int_0^1 \left(3x - \frac{1}{2} x^2 - yx \right|_y^1 dy$$

$$= \int_0^1 \left(3y - \frac{3}{2}y^2\right) dy$$

$$= \frac{3}{2}y^2 - \frac{1}{2}y^3 \Big|_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= 1 \quad unit^3$$

11.1–6 *Theorem* – Fubini's Theorem

Let $f(x, y)$ is continuous on a region R,

1. If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

$$= \int_a^b \int_c^d f(x, y)\, dy dx$$

2. If R is defined by: $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on [a, b], then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

3. If R is defined by: $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on [c, d], then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example 13

Find the volume of the wedge like solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution

$$y = 2\sqrt{x} \quad \rightarrow \quad \boxed{x = \frac{y^2}{4}}$$

$$y = 4x - 2 \quad \rightarrow \quad \boxed{x = \frac{y+2}{4}}$$

$$y = 4\frac{y^2}{4} - 2 = y^2 - 2$$

$$y^2 - y - 2 = 0$$

$$\boxed{y = -1,\ 2}$$

$$Volume = \int_0^2 \int_{y^2/4}^{(y+2)/4} \left(16 - x^2 - y^2\right) dx\, dy$$

$$= \int_0^2 \left(16x - \frac{1}{3}x^3 - y^2 x\right)\Bigg|_{y^2/4}^{(y+2)/4} dy$$

$$= \int_0^2 \left[\left(16\frac{y+2}{4} - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2\frac{y+2}{4}\right) - \left(16\frac{y^2}{4} - \frac{1}{3}\frac{y^6}{64} - \frac{y^4}{4}\right)\right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{1}{192}\left(y^3 + 6y^2 + 12y + 8\right) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right] dy$$

$$= \int_0^2 \left(4y + 8 - \frac{1}{192}y^3 - \frac{1}{32}y^2 - \frac{1}{16}y - \frac{1}{24} - \frac{1}{4}y^3 - \frac{9}{2}y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right) dy$$

$$= \int_0^2 \left(\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24}\right) dy$$

$$= \frac{1}{1344}y^7 + \frac{1}{20}y^5 - \frac{49}{768}y^4 - \frac{145}{96}y^3 + \frac{63}{32}y^2 + \frac{191}{24}y \Bigg|_0^2$$

$$= \frac{2}{21} + \frac{8}{5} - \frac{49}{48} - \frac{145}{12} + \frac{63}{8} + \frac{191}{12}$$

$$= \frac{178}{105} + \frac{513}{48}$$

$$= \frac{62,409}{5,040} \quad unit^3$$

Example 14

Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below the rectangle $R:\ 0 \le x \le 1, \quad 0 \le y \le 2$

Solution

$$Volume = \int_0^1 \int_0^2 \left(10 + x^2 + 3y^2\right) dy\, dx$$

$$= \int_0^1 \left(10y + yx^2 + y^3 \right|_0^2 dx$$

$$= \int_0^1 \left(2x^2 + 28\right) dx$$

$$= \frac{2}{3}x^3 + 28x \Big|_0^1$$

$$= \frac{2}{3} + 28$$

$$= \frac{86}{3} \quad unit^3$$

Exercises *Section* 11.1 – Double Integrals over Rectangular Regions

(1 – 91) Evaluate the integral.

1. $\iint \left(x^3 + y^3\right) dy\, dx$

2. $\iint y^3 e^{4x}\, dy\, dx$

3. $\int_0^1 \int_y^1 y\, e^{-x^3}\, dx\, dy$

4. $\int_1^2 \int_0^4 2xy\, dy\, dx$

5. $\int_0^2 \int_{-1}^1 (x - y)\, dy\, dx$

6. $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\, dy$

7. $\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dy\, dx$

8. $\int_0^1 \int_0^1 \frac{y}{1 + xy}\, dx\, dy$

9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\, dy\, dx$

10. $\int_0^1 \int_1^2 x\, y\, e^x\, dy\, dx$

11. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy$

12. $\int_1^2 \int_1^4 \frac{xy}{\left(x^2 + y^2\right)^2}\, dx\, dy$

13. $\int_1^3 \int_1^{e^x} \frac{x}{y}\, dy\, dx$

14. $\int_1^2 \int_0^{\ln x} x^3 e^y\, dy\, dx$

15. $\int_1^{10} \int_0^{1/y} y\, e^{xy}\, dx\, dy$

16. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\, dx\, dy$

17. $\int_0^1 \int_{x^2}^x \sqrt{x}\, dy\, dx$

18. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\, dx\, dy$

19. $\int_0^2 \int_0^{4-x^2} 2x\, dy\, dx$

20. $\int_0^1 \int_{2y}^2 4\cos\left(x^2\right) dx\, dy$

21. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2}\, dx\, dy$

22. $\displaystyle\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx$

23. $\displaystyle\int_0^2 \int_0^{\sqrt{9-y^2}} \frac{2}{\sqrt{9-y^2}}\, dx\, dy$

24. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y)\, dx\, dy$

25. $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, dy$

26. $\displaystyle\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\, dy$

27. $\displaystyle\int_{-2}^2 \int_0^{4-y^2} dx\, dy$

28. $\displaystyle\int_0^\infty \int_0^\infty xy\, e^{-x^2-y^2}\, dx\, dy$

29. $\displaystyle\int_1^\infty \int_1^\infty \frac{1}{xy}\, dx\, dy$

30. $\displaystyle\int_0^3 \int_0^\infty \frac{x^2}{1+y^2}\, dy\, dx$

31. $\displaystyle\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)}\, dx\, dy$

32. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \frac{x}{y} e^y\, dx\, dy$

33. $\displaystyle\int_0^1 \int_0^x \frac{xy}{x^4+1}\, dy\, dx$

34. $\displaystyle\int_0^a \int_y^a \sqrt{a^2-y^2}\, dx\, dy$

35. $\displaystyle\int_0^1 \int_0^x e^{-x^2}\, dy\, dx$

36. $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

37. $\displaystyle\int_0^{\pi/2} \int_0^x \frac{\sin x}{x}\, dy\, dx$

38. $\displaystyle\int_0^1 \int_x^1 \frac{y^2}{x^2+y^2}\, dy\, dx$

39. $\displaystyle\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\, dy\, dx$

40. $\displaystyle\int_0^1 \int_0^x \left(1-x^2\right) dy\, dx$

41. $\displaystyle\int_0^1 \int_0^y \left(1-x^2\right) dx\, dy$

42. $\displaystyle\int_0^1 \int_{x^2}^{x^2} \left(x^2+y^2\right) dy\, dx$

43. $\displaystyle\int_0^{\ln 4} \int_0^{\ln 3} e^{x+y}\, dx\, dy$

44. $\displaystyle\int_0^1 \int_{y^4}^{y^2} \sqrt{\frac{y}{x}}\, dx\, dy$

45. $\displaystyle\int_0^1 \int_{2x}^{5x} y^2\, dy\, dx$

46. $\int_0^2 \int_0^1 (x+y)\, dy\, dx$

47. $\int_{1/2}^1 \int_0^y y\, dx\, dy$

48. $\int_0^{\sqrt{2}} \int_{x^3-x}^{3x-x^2} dy\, dx$

49. $\int_0^1 \int_{\sqrt{2}}^{2-y} y^2\, dx\, dy$

50. $\int_0^1 \int_y^{10y} \sqrt{xy-y^2}\, dx\, dy$

51. $\int_0^1 \int_{x^2}^{x} y\, dy\, dx$

52. $\int_1^2 \int_y^{3y} (x+y)\, dx\, dy$

53. $\int_0^1 \int_{-1}^{3} (3-x+4y)\, dy\, dx$

54. $\int_0^2 \int_0^3 \left(3x^2-y\right) dy\, dx$

55. $\int_1^2 \int_3^5 \left(2-3x^2+y^2\right) dx\, dy$

56. $\int_0^{\pi} \int_1^2 x \sin(xy)\, dy\, dx$

57. $\int_0^1 \int_0^{\ln 5} x e^{xy}\, dy\, dx$

58. $\int_{-1}^2 \int_{2x^2-2}^{x^2+x} x\, dy\, dx$

59. $\int_2^4 \int_1^2 \left(x^2+y^2\right) dy\, dx$

60. $\int_1^2 \int_0^{y^{3/2}} \frac{x}{y^2}\, dx\, dy$

61. $\int_1^2 \int_4^6 \frac{x}{y^2}\, dx\, dy$

62. $\int_0^1 \int_0^{x^2} x e^y\, dy\, dx$

63. $\int_1^2 \int_0^3 (x+y)\, dx\, dy$

64. $\int_0^3 \int_0^{\sqrt{9-x^2}} (x+y)\, dy\, dx$

65. $\int_0^1 \int_x^{\sqrt{x}} (4x+10y)\, dy\, dx$

66. $\int_0^5 \int_0^{\frac{5-y}{2}} (5-2x-y)\, dx\, dy$

67. $\int_0^1 \int_{1-x}^{1-x^2} xy\, dy\, dx$

68. $\int_0^1 \int_{x^2}^{x} xy^2\, dy\, dx$

69. $\int_0^3 \int_{\frac{2}{3}y}^{2} x^2\, dx\, dy$

70. $\displaystyle\int_1^3 \int_{4x}^{5x} \frac{y^2}{x}\, dy\, dx$

71. $\displaystyle\int_0^1 \int_x^{\sqrt{x}} \left(y + y^3\right) dy\, dx$

72. $\displaystyle\int_1^4 \int_x^{x^2} \left(x + y^3\right) dy\, dx$

73. $\displaystyle\int_2^4 \int_y^{8-y} y\, dx\, dy$

74. $\displaystyle\int_0^{2\pi} \int_0^{\pi} \sin(x+y)\, dx\, dy$

75. $\displaystyle\int_{-1}^2 \int_0^{\pi/2} \left(y \sin 3x - y^3 \cos x\right) dx\, dy$

76. $\displaystyle\int_0^1 \int_0^{\pi/2} y \sin x\, dx\, dy$

77. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \int_0^{x^2} \frac{1}{x} \cos\frac{y}{x}\, dy\, dx$

78. $\displaystyle\int_0^1 \int_x^1 \sin\left(\pi y^2\right) dy\, dx$

79. $\displaystyle\int_0^2 \int_{e^y}^{e^2} \left(x^2 + e^y\right) dx\, dy$

80. $\displaystyle\int_0^2 \int_1^3 \left(x^2 y^3 - x e^y\right) dy\, dx$

81. $\displaystyle\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}}\, dx\, dy$

82. $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

83. $\displaystyle\int_0^3 \int_{x^2}^9 x e^{y^2}\, dy\, dx$

84. $\displaystyle\int_0^4 \int_0^{4-x} (4 - x - y)\, dy\, dx$

85. $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} (2 - x - y)\, dx\, dy$

86. $\displaystyle\int_0^{\sqrt[4]{\pi}} \int_0^y x^2 \sin\left(y^4\right) dx\, dy$

87. $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{x^2 + y^2 + 1}\, dy\, dx$

88. $\displaystyle\int_0^2 \int_1^4 y\sqrt{x}\, dy\, dx$

89. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} x\, dx\, dy$

90. $\displaystyle\int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$

91. $\displaystyle\int_0^{\pi} \int_0^{\pi} \cos(x+y)\, dy\, dx$

$(\mathbf{92-130})$ Evaluate the double integral over the given region *R*.

92. $\iint \left(2xy - x^2\right) dxdy$ R: rectangle bounded by $x=-1$, $x=2$, $y=0$, $y=4$.

93. $\iint_R dA$ R: is the region in the first quadrant bounded by $y^2 = x^3$ and the line $y = x$.

94. $\iint_R dA$ R: is the region between $y = x^2$ and the line $y = 2x$ lying to the left of $x = 1$.

95. $\iint_R (4x+2)\ dA$ R: region enclosed by the curves $y = x^2$ *and* $y = 2x$.

96. $\iint_R x^2\ dA$ R: is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines.

97. $\iint_R 6xy^2\ dA$ R: $[2,\ 4] \times [1,\ 2]$

98. $\iint_R \left(2x - 4y^3\right) dA$ R: $[-5,\ 4] \times [0,\ 2]$

99. $\iint_R \left(3x^2 - y\right) dA$ R: $[0,\ 2] \times [0,\ 1]$

100. $\iint_R \dfrac{1}{(2x+3y)^2}\ dA$ R: $[0,\ 1] \times [1,\ 2]$

101. $\iint_R x e^{xy}\ dA$ R: $[-1,\ 2] \times [0,\ 1]$

102. $\iint_R x \cos^2 y\ dA$ R: $[-2,\ 3] \times \left[0,\ \frac{\pi}{2}\right]$

103. $\iint_R e^y \cos x \, dA \qquad R: \left\{(x, y) \mid 0 \le x \le \frac{\pi}{4}, \ 0 \le y \le 1\right\}$

104. $\iint_R \left(6y^2 - 2x\right) dA \qquad R: \ 0 \le x \le 1, \ 0 \le y \le 2$

105. $\iint_R \frac{\sqrt{x}}{y^2} \, dA \qquad R: \ 0 \le x \le 4, \ 1 \le y \le 2$

106. $\iint_R y \sin(x + y) \, dA \qquad R: \ -\pi \le x \le 0, \ 0 \le y \le \pi$

107. $\iint_R e^{x-y} \, dA \qquad R: \ 0 \le x \le \ln 2, \ 0 \le y \le \ln 2$

108. $\iint_R \frac{y}{x^2y^2 + 1} \, dA \qquad R: \ 0 \le x \le 1, \ 0 \le y \le 1$

109. $\iint_R e^{x/y} \, dA \qquad R = \left\{(x, y) \mid 1 \le y \le 2, \ y \le x \le y^2\right\}$

110. $\iint_R \left(4xy - y^3\right) dA \qquad R$: region bounded by $y = \sqrt{x}$ *and* $y = x^3$

111. $\iint_R \left(6x^2 - 40y\right) dA \qquad R$: region bounded by the triangle vertices $(0, 3)$, $(1, 1)$, *and* $(5, 3)$

112. $\iint_R \left(xy - y^3\right) dA \qquad R$: region consisting of the square $\left\{(x, y) \mid -1 \le x \le 0, \ 0 \le y \le 1\right\}$ together with the triangle $\left\{(x, y) \mid 0 \le x \le 1, \ x \le y \le 1\right\}$

113. $\iint_R x^{-1/2} e^y \, dA \qquad R$ is the region bounded by $x = 1$, $x = 4$, $y = \sqrt{x}$, and $y = 0$

114. $\iint\limits_R \left(x^2+y^2\right) dA$ R is the region $\{(x,y):\ 0\le x\le 2,\ 0\le y\le x\}$

115. $\iint_R \frac{2y}{\sqrt{x^4+1}}\, dA$ *R is the region bounded by* $x=1,\quad x=2,\quad y=x^{3/2},\quad y=0$

116. $\iint\limits_R y\, dA$ $R=\left\{(x,\ y):\quad 0\le x\le \frac{\pi}{3},\quad 0\le y\le \sec x\right\}$

117. $\iint\limits_R (x+y)\, dA$ R is the region bounded by $y=\frac{1}{x}$ *and* $y=\frac{5}{2}-x$

118. $\iint\limits_R \frac{xy}{1+x^2+y^2}\, dA$ $R=\{(x,\ y):\quad 0\le y\le x,\quad 0\le x\le 2\}$

119. $\iint\limits_R x\sec^2 y\, dA$ $R=\left\{(x,\ y):\quad 0\le y\le x^2,\quad 0\le x\le \frac{\sqrt{\pi}}{2}\right\}$

120. $\iint\limits_R \left(3xy-x^2-y^2+6\right) dA$ R: is the triangle bounded by $x=0,\ y=0$ & $y=1-\frac{1}{2}x$

121. $\iint\limits_R \left(xy+e^y\right) dA$ R: is the square with corners $(3,\ 1)$ & $(4,\ 2)$

122. $\iint\limits_R \frac{\sin x}{x}\, dA$ $R=\{\ (x,y)\ |\ 0\le x\le \pi,\ 0\le y\le x\}$

123. $\iint\limits_R x\, dA$ R: *Region bounded by the curves of* $y=x^2$ *and* $y=x+6$

124. $\iint\limits_R (x-y)\, dA$ R: is the region bounded left by $x-axis$ and bounded by

$y^2=3x$ *and* $y^2=4-x$

125. $\iint_R \left(xy - 3xy^2\right) dA \qquad R = \{ (x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2 \}$

126. $\iint_R \left(x^3 + y^3\right) dA \qquad R = \{ (x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2 \}$

127. $\iint_R \left(6x^2 + 4xy^3\right) dA \qquad R = \{ (x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4 \}$

128. $\iint_R \left(4e^{x^2} - 5\sin y\right) dA \qquad R:$ *the region bounded by* $y = x$, $y = 0$, *and* $x = 4$

129. $\iint_R \left(x^2 + 6y\right) dA \qquad R:$ *the region bounded by* $y = \cos x$ *and* $y = x^2$

130. $\iint_R \left(2xy^2 + 2y\cos x\right) dA \qquad R:$ *the region bounded by* $y = \sqrt{x}$, $y = 3$, *and* $x = 0$

(131 – 138) Sketch the region of integration and evaluate the integral.

131. $\int_0^{\pi} \int_0^{x} x \sin y \, dy\, dx$

132. $\int_0^{\pi} \int_0^{\sin x} y \, dy\, dx$

133. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx\, dy$

134. $\int_1^{4} \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy\, dx$

135. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy\, dx$

136. $\int_0^{2} \int_x^{2} 2y^2 \sin xy \, dy\, dx$

137. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) dx\, dy$

138. $\int_0^{2} \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy\, dx$

139. Evaluate $f(x,y) = \frac{1}{xy}$ over the ***square*** $1 \le x \le 2, \quad 1 \le y \le 2$

140. Evaluate $f(x,y) = y\cos xy$ over the ***rectangle*** $0 \le x \le \pi, \quad 0 \le y \le 1$

141. Find the area between: $y = x^2$ & $y = x^3$.

142. Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the *xy*-plane.

143. Find the area of the region enclosed by the coordinate axes and the line $x = 0$ and $x + y = 2$.

144. Find the area of the region enclosed by the lines $y = 2x$, and $y = 4$.

145. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

146. Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$.

147. Find the area of the region enclosed by the curve $y = \ln x$, $y = 2\ln x$, and the line $x = e$ in the first quadrant.

148. Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$.

149. Find the area of the region enclosed by the lines $y = x - 2$, $y = -x$, and the curve $y = \sqrt{x}$.

150. Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$.

151. Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Make a sketch of the region.

152. Find the area of the region bounded by the lines $y = |x|$ and $y = 20 - x^2$. Make a sketch of the region.

153. Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

154. Find the area of the region *R* bounded by $xy = 1$ *and* $y + 2x = 3$

$(\mathbf{155} - 158)$ Find the area of the region.

155. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\,dy$

156. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx$

157. $\displaystyle\int_{-1}^{2} \int_{y^2}^{y+2} dx\,dy$

158. $\displaystyle\int_0^2 \int_{x^2-4}^{0} dydx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx$

159. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

160. Find the average height of $f(x,y)=\frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$.

161. Find the volume below the surface $z = x^2 \sin\left(\frac{\pi}{6}y\right)$ and above the rectangle:

$R = \{ (x,y) \mid 0 \le x \le 6,\ 0 \le y \le 6 \}$.

162. Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and three coordinate planes.

163. Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$, and $z = 0$.

164. Find the volume of the solid bounded by the graphs of $z = x^2 + 1$, $z = 2$, $y = 0$, and $x + y = 2$

165. Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x$, $z = x$, $y = 3$, $y = 0$, and $z = 0$.

166. Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 64 - x^2$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$.

167. Find the volume under surface $f(x,y) = xy$ lying over the area in first quadrant between the parabolas $y = x^2$ & $y = 18 - x^2$.

168. Find the volume of the solid bounded by the elliptic paraboloid $f(x,y) = 27 - 2x^2 - y^2$ and the planes $y = 3$ & $x = 3$, and three coordinate planes.

169. Find the volume bounded by the elliptic paraboloid $f(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ and xy–plane.

170. Find the volume of the solid bounded lies below the surface $f(x,y) = 16xy + 200$ and lies above the region xy–plane bounded by $y = x^2$ & $y = 8 - x^2$.

171. Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$ and $y = 3x, \quad x = 0, \quad z = 0$.

172. Find the volume of the solid bounded above by the graph $f(x,y) = xy \sin\left(x^2 y\right)$ and below by xy–plane on the rectangular region $(0,\ 1)$ & $(0,\ \pi)$.

173. Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R:\ -1 \le x \le 1,\ \ -1 \le y \le 1$.

174. Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle $R:\ 0 \le x \le 4,\ \ 0 \le y \le 2$.

175. Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle $R:\ 0 \le x \le 1,\ \ 0 \le y \le 2$.

176. Find the volume of the region bounded above the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R:\ 0 \le x \le 2,\ \ 0 \le y \le 2$.

177. Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the *xy*-plane.

178. Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the *xy*-plane.

179. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$.

180. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

181. Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the *xy*-plane.

182. Find the Volume of the solid bounded under the surface $z = xy$ and above the rectangular region in the $xy - plane$ bounded by the lines $y = 6 - x$, $y = 2x$, and $y = 0$.

183. Find the Volume of the solid bounded under the surface $z = xy$ and above the rectangular region in the $xy - plane$ bounded by the lines $y = 6 - x$, $y = 2x$, and $x = 0$.

184. Find the Volume of the solid bounded under the plane $z = 4 - 2x - y$ and above the triangular region $R = \{(x, y) \mid 0 \le x \le 1,\ \ 0 \le y \le 2\}$.

185. Find the Volume of the solid bounded under the surface $z = x^3 + 4y$ and above the $xy - plane$, and bounded by the line $y = 2x$ and the parabola $y = x^2$.

186. Find the Volume of the under the paraboloid $z = x^2 + y^2$ and bounded above the bounded by the line $y = x$ and the parabola $y = x^2$.

187. Find the Volume of the under the cylinder $y^2 + z^2 = 9$ and the plane $x = 0$, $y = 0$, $z = 0$, and $y = 2 - 2x$.

188. Evaluate $\int_0^{1/2} \left(\sin^{-1}(2x) - \sin^{-1} x\right) dx$ by converting it to a double integral.

189. Integrate $f(x, y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines $y = x, \quad y = 2x, \quad x = 1, \quad and \quad x = 2$

190. Integrate $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$ *and* $(0,1)$.

191. Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

192. Evaluate $\int_{-2}^{0} \int_{v}^{-v} 2 \, dp \, dv$

193. Evaluate $\int_{-\pi/3}^{\pi/3} \int_{0}^{\sec t} 3\cos t \, du \, dt$

194. Consider the region $R = \{(x, y): \ |x| + |y| \le 1\}$

a) Use a double integral to show that the area of *R* is 2.

b) Find the volume of the square column whose base is *R* and whose upper surface is $z = 12 - 3x - 4y$.

c) Find the volume of the solid above *R* and beneath the cylinder $x^2 + z^2 = 1$.

d) Find the volume of the pyramid whose base is *R* and whose vertex is on the *z*-axis at $(0, 0, 6)$.

195. Find the volume of the tetrahedron bounded by the plane $2x+2y+z=a$ and three points $\left(\frac{a}{2},\ 0,\ 0\right)$, $(0,\ 0,\ a)$, and $\left(\frac{a}{6},\ \frac{a}{3},\ 0\right)$.

196. Find the volume of the solid beneath the cylinder $f(x,\ y)=e^{-x}$ and above the region $R=\{(x,\ y):\quad 0\le x\le \ln 4,\quad -2\le y\le 2\}$

197. Find the volume of the solid beneath the plane $f(x,\ y)=6-x-2y$ and above the region $R=\{(x,\ y):\quad 0\le x\le 2,\quad 0\le y\le 1\}$

198. Find the volume of the solid beneath the plane $f(x,\ y)=24-3x-4y$ and above the region $R=\{(x,\ y):\quad -1\le x\le 3,\quad 0\le y\le 2\}$

199. Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): \ 1 \le x \le 2, \quad 0 \le y \le 1\}$

200. Find the volume bounded above by the surface $x + y + z = 6$ and below in the first quadrant by $y = x$ and $y = 2$.

201. Find the volume bounded below in the first quadrant by the lines $x = 4$ and $y = 2$, cuts above by the plane $2y + z = 6$

202. Find the volume bounded above by the plane $z=6$ and below the lines $y=x$ and $x=2$.

203. Find the volume of the solid of a tetrahedron $2x+3y+4z=12$

204. Find the Volume of the solid bounded under the ellipse paraboloid $z=8-2x^2-y^2$ and above the rectangular region $R=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 2\}$

205. Find the Volume of the solid bounded under the surface $z=6-y$ and above the $xy-plane$, and bounded by the lines $y=4$ and $x=3$ in the first quadrant.

206. Find the Volume of the solid bounded under the surface $z = 4 - x + \frac{1}{2}y$ and above the $xy-plane$, and bounded by the lines $y = 4$ and $x = 3$ in the first quadrant.

207. Find the Volume of the solid bounded under the surface $z = 4 - x^2$ and above the $xy-plane$, and bounded by the lines $y = 4$ and $x = 2$ in the first quadrant.

208. Find the Volume of the solid inside the paraboloid $z = 4 - x^2 - y^2$ and bounded above the $xy-plane$.

Section 11.2 – Double Integrals in Polar Coordinates

11.2–1 Integrals in Polar Coordinates

If f is continuous throughout R, this sum will approach a limit as Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

$$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k\right)\Delta A_k$$

$$\lim_{n\to\infty} S_n = \iint_R f\left(r,\theta\right)\, dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is:

$$Area = \frac{1}{2}\theta \cdot r^2$$

Inner radius: $\frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

outer radius: $\frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

$$\Delta A_k = \begin{pmatrix} area\ of \\ large\ \ sector \end{pmatrix} - \begin{pmatrix} area\ of \\ small\ \ sector \end{pmatrix}$$

Leads to the formula: $\Delta A_k = r_k\ \Delta r\, \Delta\theta$

11.2–2 Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is:

$$Area = \iint_R r\ dr\ d\theta$$

Example 1

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution

From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion since it has a form of symmetry.

$$Area = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\ dr\ d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2}r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} (2\cos 2\theta)\, d\theta$$

$$= 4\int_0^{\pi/4} \cos 2\theta\ d(2\theta)$$

$$= 4\ \sin 2\theta \Big|_0^{\pi/4}$$

$$= 4\sin\frac{\pi}{2}$$

$$= 4 \quad unit^2$$

11.2–3 Changing Cartesian Integrals into Polar Integrals

$$\iint_R f(x,\ y)\ dx\ dy = \iint_G f(r\cos\theta,\ r\sin\theta)\ r\ dr d\theta$$

Example 2

Evaluate $\displaystyle\iint_R e^{x^2+y^2}\ dy\ dx$

Where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

Solution

$$y = \sqrt{1-x^2} \quad > 0$$

$$0 \le \theta \le \pi \quad \& \quad 0 \le r \le 1$$

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta \qquad d\left(r^2\right) = 2rdr$$

$$= \frac{1}{2}\int_0^{\pi} d\theta \int_0^1 e^{r^2}\,d\left(r^2\right)$$

$$= \frac{1}{2}\,\theta\Big|_0^{\pi}\left(e^{r^2}\Big|_0^1\right.$$

$$= \frac{\pi}{2}(e-1)$$

Example 3

Evaluate the integral $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dy\,dx$

Solution

Since: $0 \le x \le 1 \;\rightarrow$ interior of $x^2+y^2=1$ *and* in QI

Let: $r^2 = x^2+y^2$ *with* $0 \le r \le 1$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dydx = \int_0^{\pi/2}\int_0^1\left(r^2\right)rdrd\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^1 r^3\,dr$$

$$= \frac{\pi}{2}\;\frac{1}{4}r^4\Big|_0^1$$

$$= \frac{\pi}{8}$$

- Or we can use the integral table to solve it

$$\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dy\,dx = \int_0^1\left(x^2\sqrt{1-x^2}+\frac{1}{3}\left(1-x^2\right)^3\right)dx$$

?

Example 4

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the *xy*-plane.

Solution

The region of integration *R* is the unit circle:

$x^2 + y^2 = 1$

$0 \le r \le 1 \quad \& \quad 0 \le \theta \le 2\pi$

$$Volume = \int_0^{2\pi} \int_0^1 \left(9 - r^2\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(9r - r^3\right) dr$$

$$= 2\pi \left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right|_0^1$$

$$= 2\pi\left(\frac{9}{2} - \frac{1}{4}\right)$$

$$= \frac{17\pi}{2} \text{ unit}^3$$

Example 5

Using the polar integration, find the area of the region *R* in the *xy*-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan\theta$

$$\theta = \frac{\pi}{3}$$

Line $y = 1$ intersects $x^2 + y^2 = 4$

when $x^2 + 1 = 4$

$$x = \sqrt{3}$$

A line from origin to $\left(\sqrt{3}, 1\right)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan\theta$$

$$\theta = \frac{\pi}{6}$$

$$\therefore \quad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

The polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r\sin\theta$ for y:

$$y = 1$$

$$r\sin\theta = 1$$

$$r = \frac{1}{\sin\theta}$$

$$= \csc\theta$$

The radius of the circle is 2.

$$\therefore \quad \csc\theta \le r \le 2$$

$$Area = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r \, dr \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2} r^2 \Big|_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2} \int_{\pi/6}^{\pi/3} \left(4 - \csc^2\theta\right) d\theta$$

$$= \frac{1}{2} \left(4\theta + \cot\theta\right) \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2} \left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left(\frac{4\pi}{6} + \sqrt{3}\right)\right)$$

$$= \frac{1}{2} \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3}\right)$$

$$= \frac{1}{2} \left(\frac{2\pi - 2\sqrt{3}}{3}\right)$$

$$= \frac{\pi - \sqrt{3}}{3} \quad unit^2$$

Example 6

Evaluate the double integral: $\iint e^{-x^2-y^2}\,dA$

In the first quadrant and bounded by the circle $x^2+y^2=a^2$ and the coordinate axes.

Solution

$x^2+y^2=r^2$

$0 \le r \le a$

In QI: $0 \le \theta \le \frac{\pi}{2}$

$$\iint e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2}\, r\, dr\, d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^a e^{-r^2}\, d\left(-r^2\right)$$

$$= -\frac{\pi}{4}\, e^{-r^2}\Big|_0^a$$

$$= -\frac{\pi}{4}\left(e^{-a^2}-1\right)$$

$$= \frac{\pi}{4}\left(1-e^{-a^2}\right)$$

Exercises *Section* 11.2 – Double Integrals in Polar Coordinates

(1 – 5) Evaluate the integral.

1. $\displaystyle\int_0^{\tan^{-1}\frac{3}{2}} \int_0^{2\sec\theta} r\ dr\, d\theta$

2. $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\tan\theta\ \sec\theta} r^3 \cos^2\theta\ dr\, d\theta$

3. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{2} r^2 \cos\theta\ dr\, d\theta$

4. $\displaystyle\int_0^{2\pi} \int_0^{1-\cos\theta} r^3 \cos^2\theta\ dr\, d\theta$

5. $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} r^3\ dr\, d\theta$

5. $\displaystyle\int_0^{\frac{\pi}{2}} \int_1^{\infty} \frac{\cos\theta}{r^3}\ r\ dr\, d\theta$

(6 – 19) Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral.

6. $\displaystyle\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy\, dx$

7. $\displaystyle\int_0^{1} \int_0^{\sqrt{1-y^2}} \left(x^2+y^2\right) dx\, dy$

8. $\displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

9. $\displaystyle\int_0^{6} \int_0^{y} x\, dx\, dy$

10. $\displaystyle\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx$

11. $\displaystyle\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$

12. $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right) dx\, dy$

13. $\displaystyle\int_1^{2} \int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}\, dy\, dx$

14. $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\, dy\, dx}{\left(1+x^2+y^2\right)^2}$

15. $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right) dx\, dy$

16. $\displaystyle\int_0^{\infty} \int_0^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}\, dx\, dy$

17. $\displaystyle\int_0^{3} \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dy\, dx$

18. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\, dy\, dx$

19. $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} \left(16-x^2-y^2\right)\, dx\, dy$

$(\mathbf{20-24})$ Evaluate the integral.

20. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4-x^2-y^2\right)\, dy\, dx$

21. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} 4y\, dx\, dy$

22. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2\left(x^2+y^2\right)^2\, dy\, dx$

23. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}\, dx\, dy$

24. $\displaystyle\int_{0}^{1}\int_{0}^{a} \frac{1}{\left(1+x^2+y^2\right)^2}\, dy\, dx$

$(\mathbf{25-39})$ Evaluate the double integral over the given region R.

25. $\displaystyle\iint_R \frac{xy}{1+x^2+y^2}\, dA \qquad R=\left\{(x,\, y):\quad 0\le y\le x,\quad 0\le x\le 2\right\}$

26. $\displaystyle\iint_R \left(x^2+y^2+3\right)\, dA; \qquad R:\ \textit{in the circle of radius } 2 \textit{ centered at the origin}$

27. $\displaystyle\iint_R \left(1-x^2-y^2\right)\, dA \quad R:$ bounded by unit circle

28. $\displaystyle\iint_R xy\, dA \qquad R:$ bounded by unit circle and lies in the first quadrant.

29. $\displaystyle\iint_R x\, dA \qquad R=\left\{(r,\, \theta)\ |\ 1\le r\le 2,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$

30. $\displaystyle\iint_R \left(x^2+y^2\right) dA; \qquad R=\left\{(r,\, \theta):\quad 0\le r\le 4,\quad 0\le\theta\le 2\pi\right\}$

31. $\iint\limits_R 2xy\,dA;\qquad R=\left\{(r,\ \theta):\ \ 1\le r\le 3,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

32. $\iint\limits_R 2xy\ dA;\qquad R=\left\{(x,\ y):\ \ x^2+y^2\le 9,\quad y\ge 0\right\}$

33. $\iint\limits_R \frac{dA}{1+x^2+y^2};\qquad R=\left\{(r,\ \theta):\ \ 1\le r\le 2,\quad 0\le\theta\le\pi\right\}$

34. $\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}};\qquad R=\left\{(x,\ y):\ \ x^2+y^2\le 4,\quad y\ge 0\right\}$

35. $\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}};\qquad R=\left\{(x,\ y):\ \ x^2+y^2\le 4,\quad x,\ y\ge 0\right\}$

36. $\iint\limits_R e^{-x^2-y^2}dA;\qquad R=\left\{(x,\ y):\ \ x^2+y^2\le 9\right\}$

37. $\iint\limits_R \sqrt{x^2+y^2}\ dA;\qquad R=\left\{(x,\ y):\ \ y\le x\le 1,\quad 0\le y\le 1\right\}$

38. $\iint\limits_R \sqrt{x^2+y^2}\ dA;\qquad R=\left\{(x,\ y):\ \ 1\le x^2+y^2\le 2\right\}$

39. $\iint\limits_R \frac{dA}{\left(x^2+y^2\right)^{5/2}};\qquad R=\left\{(r,\ \theta):\ \ 1\le r\le\infty,\quad 0\le\theta\le 2\pi\right\}$

40. $\iint\limits_R e^{-x^2-y^2}dA;\qquad R=\left\{(r,\ \theta):\ \ 0\le r\le\infty,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

41. $\iint\limits_R \frac{dA}{\left(1+x^2+y^2\right)^2};\qquad R\in QI$

42. $\iint\limits_R \sin\left(x^2+y^2\right)\ dA\quad R=\left\{\ (x,y)\ \mid\ \frac{\pi}{2}\le x^2+y^2\le\pi\right\}$

43. Evaluate the integral $\iint_R \sqrt{x^2+y^2}\, dA$ over the region R in the first quadrant bounded by the spiral $r=\theta$, the circle $r=1$, and $x-axis$.

44. Find the area of the region: $f(\theta)=\sqrt{\sin\theta}$, $0 \le \theta \le \frac{\pi}{4}$

45. Find the area of the region: $f(\theta)=\sin\theta$, $-\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

46. Find the area of the region enclosed by $r=\sin\left(\frac{\theta}{2}\right)$

47. Find the area of the region: $f(\theta)=\sqrt{\cos\theta}$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

48. Find the area of the region inside the curve $r=\sqrt{\cos\theta}$

49. Find the area of the region inside the right lobe of $r=\sqrt{\cos 2\theta}$

50. Find the area of the region: $f(\theta)=\sin\theta+\cos\theta$, $0 \le \theta \le \frac{\pi}{2}$

51. Find the area of the region inside the oval limaçon $r=4+2\sin\theta$

52. Find the area of the region inside the cardioid $r=a(1+\cos\theta)$, $a>0$

53. Find the area of the region inside the six-leaved rose $r^2=2\sin 3\theta$

54. Find the area of the region enclosed by $r=\cos\left(\frac{\theta}{2}\right)$

55. Find the area enclosed by one leaf of the rose. Given by the equation $r=\sin 3\theta$.

56. Find the area of the region lies inside the cardioid $r=a\,(1+\sin\theta)$ and outside the circle $r=a$

57. Find the area of the upper half of the cardioid $r=2a\,(1-\cos\theta)$

58. Find the area inside the cardioid $r=2-2\sin\theta$

59. Find the area of the region lies inside the circle $r=a\cos\theta$ and outside the cardioid $r=a\,(1-\cos\theta)$

60. Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

61. Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

62. Find the area enclosed by one leaf of the rose $r=12\cos 3\theta$

63. Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

64. Find the area of the region bounded by all leaves of the rose $r=3\cos 2\theta$

65. Find the area of the region inside both the circles $r = 2$ and $r = 4\cos\theta$

66. Find the area of the region that lies inside both the cardioids $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$

67. Find the area of the annular region $\{(r,\ \theta):\ \ 1 \le r \le 2,\ \ \ 0 \le \theta \le 2\pi\}$

68. Find the area of the region bounded by the cardioid $r = 2(1 - \sin\theta)$

69. Find the area of the region bounded by all leaves of the rose $r = 2\cos 3\theta$

70. Find the area of the region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

71. Find the area of the region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

72. Find the area of the region bounded by the spiral $r = 2\theta$, for $.0 \le \theta \le \pi$., and the *x*-axis.

73. Find the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

74. Find the area of the region bounded by $r = 2\sin 2\theta$

75. Find the area of the region bounded by $r^2 = 2\sin 2\theta$

76. Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in QI

77. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = 1 + \cos\theta$

78. Find the area of the region inside the circle $r = 6$ above the line $r = 3\csc\theta$

79. Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

80. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = \cos\theta$

81. Find the area of the region outside the circle $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\cos\theta}$

82. Find the area of the region inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

83. Find the area of the region in the plane enclosed by the four-leaf rose $f(\theta) = 2\cos 2\theta$

84. Find the area of the region shared by the circles $r = 2\cos\theta$ *and* $r = 2\sin\theta$

85. Find the area of the given region Inner loop of $r = 2 - 4\cos\theta$

86. Find the area of the given region between the loops $r = 1 + 2\cos\theta$

87. Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Use a double integral to compute the area of the region.

88. Find the area enclosed by $y = x^2$ and $x + y = 2$.

89. Find the area enclosed by $y^3 = 2x$ and $y = 2x$.

90. Find the area enclosed by $y^2 = 4x$ and $y = 2x$.

91. Find the area in the first quadrant bounded by the ellipse $x^2 + 4y^2 = 5$, and the hyperbolic $xy = 1$.

92. Find the area of the portion of the cylinder $x^2 + z^2 = a^2$, lying inside the cylinder $x^2 + y^2 = a^2$

93. Integrate $f(x, y) = \dfrac{\ln\left(x^2 + y^2\right)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$

94. The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

95. Evaluate $\iint\limits_R (x + y)\, dA$; R is the disk bounded by circle $r = 4\sin\theta$

96. Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \left\{ (r,\ \theta):\ \ 2 \le r \le 4,\ \ 0 \le \theta \le 2\pi \right\}$. (An *annulus* is the region between two concentric circles.)

97. Find the volume of a solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and outside the cylinder $x^2 + y^2 = 4$ and above the $xy - plane$.

98. Find the volume cut out of the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $x^2 + y^2 = 2y$.

99. Find the volume cut out of the sphere $x^2 + y^2 + z^2 = 9$ by the cylinder $x^2 + y^2 = 3x$.

100. Find the volume cut out of the sphere $x^2 + y^2 + z^2 = 9$ by the cylinder $x^2 + y^2 = 2x$.

101. Find the volume of the solid bounded by $z = 8 - x^2 - y^2$ *and* $z = x^2 + y^2$.

102. Find the volume of a solid bounded by the paraboloid $z = 4 - x^2 - y^2$ below by the $xy - plane$.

103. Find the volume of a solid bounded by the paraboloid $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

104. Find the volume of the solid bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the plane $z = 1$

105. Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - 3y^2$ and below by the hyperbolic paraboloid $z = x^2 - y^2$.

106. Find the volume of the solid bounded below the surface $z = e^{-x^2-y^2}$ and inside the cylinder $x^2 + y^2 = a^2$.

107. Find the volume in the first quadrant of the solid bounded below the surface $z = e^{-x^2-y^2}$ and inside the cylinder $x^2 + y^2 = 4$.

108. Which bowl holds more water if it is filled to a depth of four units?

a) The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$

b) The cone $z = \sqrt{x^2 + y^2}$, for $0 \le z \le 4$

c) The hyperboloid $z = \sqrt{1 + x^2 + y^2}$, for $1 \le z \le 5$

d) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$

109. Consider the surface $z = x^2 - y^2$

a) Find the region in the *xy*-plane in polar coordinates for which $z \ge 0$.

b) Let $R = \left\{(r,\ \theta):\ \ 0 \le r \le a,\ \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region *R*.

110. A cake is shaped like a hemisphere of radius 4 with its based on the *xy*-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the *xy*-plane and separated by an angle of φ.

a) Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

b) Suppose the cake is sliced by a plane perpendicular to the *xy*-plane at $x = a > 0$. Let *D* be the smaller of the two pieces produced. For what value of *a* is the volume of *D* equal to the volume in part (*a*)?

111. Suppose the density of a thin plate represented by the region *R* is $\rho(r,\ \theta)$ (in units of mass per area). The mass of the plate is $\iint_R \rho(r,\ \theta)\,dA$. Find the mass of the thin half annulus $R = \{(r,\ \theta):\ \ 1 \le r \le 4,\ \ 0 \le \theta \le \pi\}$ with a density $\rho(r,\ \theta) = 4 + r\sin\theta$

112. An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It is evaluated in the following steps.

a) Assume that $I = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$

Where we have chosen the variables of integration to be *x* and *y* and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

b) Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} xe^{-x^2}\,dx$, and $\int_0^{\infty} x^2e^{-x^2}\,dx$.

113. For what values of *p* does the integral $\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA$ exist in the following cases?

a) $R = \{(r,\ \theta):\ \ 1 \le r \le \infty,\ \ \ 0 \le \theta \le 2\pi\}$

b) $R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ \ 0 \le \theta \le 2\pi\}$

114. Consider the integral $\iint_R \frac{k}{\left(1+x^2+y^2\right)^2}\,dA$ where $R = \{(x,\ y):\ \ 0 \le x \le 1,\ \ \ 0 \le y \le a\}$

a) Evaluate *I* for $a = 1$.

b) Evaluate *I* for arbitrary $a > 0$.

c) Let $a \to \infty$ in part (b) to find *I* over the infinite strip $R = \{(x,\ y):\ \ 0 \le x \le 1,\ \ \ 0 \le y \le \infty\}$

115. In polar coordinates an equation of an ellipse with eccentricity $0 < e < 1$ and semimajor axis *a* is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is πab, where $b^2 = a^2\left(1-e^2\right)$

$(\mathbf{104-107})$ Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above:

116. $R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ 0 \le \theta \le 2\pi\}$

117. $R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ 0 \le \theta \le 2\pi\}$

118. $R = \{(r,\ \theta):\ \ 1 \le r \le 2,\ \ 0 \le \theta \le 2\pi\}$

119. $R = \left\{(r,\ \theta):\ \ 1 \le r \le 2,\ \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$

$(\mathbf{108-111})$ Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above:

120. $R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ 0 \le \theta \le 2\pi\}$

121. $R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ 0 \le \theta \le \pi\}$

122. $R = \{(r,\ \theta):\ \ \sqrt{3} \le r \le 2\sqrt{2},\ \ 0 \le \theta \le 2\pi\}$

123. $R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le \sqrt{15},\ \ -\frac{\pi}{2} \le \theta \le \pi\right\}$

124. Find the volume of the solid between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

125. Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$

126. Find the volume of island $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$

127. Find the volume of island $z = 100 - 4\left(x^2 + y^2\right)$.

128. Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$

129. Find the volume in the first octant between the plane $z = 0$ and $z = x + y + 2$, and inside the cylinder $x^2 + y^2 = 16$

130. Find the volume of one of the wedges cut from the cylinder $4x^2 + y^2 = a^2$ by the $xy - plane$ and $z = my$

131. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the $xy - plane$ and $y + z = 4$

132. Find the volume bounded by the paraboloid $z = x^2 + 4y^2$, below by the $xy - plane$ and laterally by the cylinder $y^2 = x$ and $y = x^2$

133. Find the volume bounded by the paraboloid $z = \frac{1}{4}\left(x^2 + y^2\right)$, and the cylinder $x^2 + y^2 = 8y$, and the plane $z = 0$

134. Find the volume of the solid bounded below the hemisphere $x^2 + y^2 + z^2 = 9$, and above the $xy - plane$, and inside the cylinder $x^2 + y^2 = 1$

135. Find the volume of the solid bounded below the paraboloid $z = 4 - x^2 - y^2$, and above the $xy - plane$, and inside the cylinder $(x-1)^2 + y^2 = 1$

136. Find the volume of the solid bounded below the paraboloid $z = x^2 + y^2$, and above the $xy - plane$, and inside the cylinder $x^2 + y^2 = 4$.

137. Find the volume of the solid bounded below the paraboloid $z = 9 - 2x^2 - 2y^2$, and the plane $z = 1$.

138. Find the volume of the solid bounded below the cone $z=\sqrt{x^2+y^2}$, and above the $xy-plane$, and inside the cylinder $x^2+y^2=4$.

139. Find the volume of the solid bounded below the plane $3x+4y+z=12$, and above the $xy-plane$, and inside the cylinder $x^2+y^2=2x$

140. Find the volume of the solid bounded inside the sphere $x^2+y^2+z^2=2$, and above the cone $z=\sqrt{x^2+y^2}$

141. Find the volume of the region inside right circular with an equation $r=4\sin\theta$ and bounded above by the sphere with radius 4 centered at the origin and bounded below by the $xy-plane$.

Section 11.3 – Triple Integrals

11.3–1 Triple Integrals

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such a solid ball or a lump of clay, then the integral of F over D may be defined in the following way.

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \rightarrow \quad S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

The limit of this summation is the triple integral of F over D.

$$\lim_{n \to !} S_n = \iiint_D F(x, y, z)\, dV \qquad or \qquad \lim_{\|P\| \to !} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz$$

11.3–2 Volume of a region in Space – *Definition*

The volume of a closed, bounded region D in space is

$$V = \iiint_D dV$$

11.3–3 Find Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint_D F(x, y, z)\, dV$

1. ***Sketch***: Sketch the region *D* along with its "shadow" *R* (vertical projection) in the *xy*-plane. Label the upper and lower bounding surfaces of *D* and *R*.

2. ***Find the z-limits of integration***: Draw a line *M* passing through (x, y) in *R* parallel to the *z*-axis. As *z* increases, *M* enters *D* at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$.

3. ***Find the y-limits of integration***: Draw a line *L* passing through (x, y) parallel to the *y*-axis. As increases, *L* enters *R* at $y = g_1(x)$ and leaves at $y = g_2(x)$.

4. ***Find the x-limits of integration***: Choose *x*-limits that include all lines through *R* parallel to the *y*-axis $(x = a \;\; and \;\; x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx$$

Example 1

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution

***z*-limits**: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

***y*-limits**: $z = x^2 + 3y^2 = 8 - x^2 - y^2$

$$2x^2 + 4y^2 = 8$$

$$x^2 + 2y^2 = 4$$

$$y^2 = \frac{4 - x^2}{2}$$

$$y = \pm\sqrt{\frac{4 - x^2}{2}}$$

$$-\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}}$$

***x*-limits**: $x^2 + 2y^2 = 4$

$$y = 0 \quad \rightarrow \quad \underline{x = \pm 2}$$

$$-2 \le x \le 2$$

$$Volume = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} z \Big|_{x^2+3y^2}^{8-x^2-y^2} dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - x^2 - y^2 - x^2 - 3y^2\right) dy\, dx$$

$$= \int_{-2}^{2} \left(\left(8 - 2x^2\right)y - \frac{4}{3}y^3\right) \Bigg|_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} + \left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left(2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left(2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left(8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left(8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right) dx$$

$$= \frac{16}{3}\int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \frac{16}{3(2)^{3/2}}\int_{-2}^{2} \left(4-x^2\right)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}}\,\frac{2^{1/2}}{2^{1/2}} = \frac{16\sqrt{2}}{3\cdot 4} = \frac{4\sqrt{2}}{3}$$

$$x = 2\sin u \quad dx = 2\cos u\, du$$

$$4 - x^2 = 4\cos^2 u$$

$$\begin{cases} x = 2 & \to u = \sin^{-1}\frac{x}{2} = \sin^{-1} 1 = \frac{\pi}{2} \\ x = -2 & \to u = \sin^{-1}(-1) = -\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(4\cos^2 u\right)^{3/2} \left(2\cos u\, du\right)$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 16(\cos u)^3 \left(\cos u\right) du$$

$$= \frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \cos^4 u\, du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \cos^2 2u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2u + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \left(\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u\right) \Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3} \left[\frac{3\pi}{4} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin\pi - \frac{1}{8}\sin 2\pi\right)\right]$$

$$= \frac{16\sqrt{2}}{3} \left(\frac{3\pi}{2}\right)$$

$$= 8\pi\sqrt{2} \quad unit^3$$

Example 2

Integrate $F(x, y, z) = 1$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1) in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution

z-limits of integration:

A line M parallel to the z-axis through a typical point $(x,\ y)$ in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exists through the upper plane where $z = y - x$.

$0 \le z \le y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$(1,\ 1,\ 0)$

$$a + b = 0 \quad \Rightarrow \quad a = -b$$

and $(0,\ 1,\ 1)$

$$b + c = 0$$

$$\underline{c = -b}$$

$$-bx + by - bz = 0$$

$$-x + y - z = 0$$

$$z = y - x$$

y-limits of integration:

On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exists at $y = 1$. $x \le y \le 1$

x-limits of integration: A line L parallel to the y-axis through a typical point (x, y) in the xy-plane sweeps out the shadow, where $0 \le x \le 1$ at the point $(1,\ 1,\ 0)$

The integral is: $\int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\, dz\, dy\, dx$

$$Volume = \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx$$

$$= \int_0^1 \int_x^1 z \Big|_0^{y-x} dy\, dx$$

$$= \int_0^1 \int_x^1 (y-x)\, dy\, dx$$

$$= \int_0^1 \left(\tfrac{1}{2}y^2 - xy\right) \Big|_x^1 dx$$

$$= \int_0^1 \left(\tfrac{1}{2} - x - \left(\tfrac{1}{2}x^2 - x^2\right)\right) dx$$

$$= \int_0^1 \left(\tfrac{1}{2} - x + \tfrac{1}{2}x^2\right) dx$$

$$= \left(\tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right) \Big|_0^1$$

$$= \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}$$

$$= \tfrac{1}{6} \quad unit^3$$

$$Volume = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} y \Big|_{x+z}^1 dz\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-z)\, dz\, dx$$

$$= \int_0^1 \left(z - xz - \frac{1}{2}z^2 \right|_0^{1-x} dx$$

$$= \int_0^1 \left(1 - x - x(1-x) - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 \left((1-x)(1-x) - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{6} \quad unit^3$$

Example 3

Evaluate the integral $\displaystyle\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \, dz \, dy \, dx$

Solution

$$\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \, dz \, dy \, dx = \int_0^1 \int_x^{x^2} xy \, z \Big|_{xy}^{x^2y^3} dy \, dx$$

$$= \int_0^1 \int_x^{x^2} xy \left(x^2y^3 - xy \right) dy \, dx$$

$$= \int_0^1 \int_x^{x^2} \left(x^3y^4 - x^2y^2 \right) dy \, dx$$

$$= \int_0^1 \left(\frac{1}{5}x^3y^5 - \frac{1}{3}x^2y^3 \right) \Big|_x^{x^2} dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{1}{3}x^8 - \frac{1}{5}x^8 + \frac{1}{3}x^5\right)\ dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{8}{15}x^8 + \frac{1}{3}x^5\right)\ dx$$

$$= \left(\frac{1}{70}x^{14} - \frac{8}{135}x^9 + \frac{1}{18}x^6\right|_0^1$$

$$= \frac{1}{70} - \frac{8}{135} + \frac{1}{18}$$

$$= \frac{2}{189}$$

Example 4

Evaluate the integral $\displaystyle\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dxdydz$

Solution

$$\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dx\,dy\,dz = \int_0^a \int_0^{a-z} yz\ x\ \Big|_0^{a-y-z}\ dy\,dz$$

$$= \int_0^a \int_0^{a-z} \left(ayz - y^2z - yz^2\right)\ dydz$$

$$= \int_0^a \left(\frac{1}{2}azy^2 - \frac{1}{3}zy^3 - \frac{1}{2}z^2y^2\right|_0^{a-z}\ dz$$

$$= \int_0^a \left(\frac{1}{2}\left(az - z^2\right)(a-z)^2 - \frac{1}{3}z(a-z)^3\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2\ \left(3\left(az - z^2\right) - 2z(a-z)\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2\ \left(3az - 3z^2 - 2az + 2z^2\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2 \left(az - z^2\right) dz$$

$$= \frac{1}{6}\int_0^a z\,(a-z)^3 \, dz$$

$$= \frac{1}{6}\int_0^a \left(a^3 z - 3a^2 z^2 + 3az^3 - z^4\right) dz$$

$$= \frac{1}{6}\left(\frac{1}{2}a^3 z^2 - a^2 z^3 + \frac{3}{4}az^4 - \frac{1}{5}z^5\right|_0^a$$

$$= \frac{1}{6}\left(\frac{1}{2}a^5 - a^5 + \frac{3}{4}a^5 - \frac{1}{5}a^5\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{20}\right)$$

$$= \frac{a^5}{120}$$

Example 5

Find the volume bounded by the cylinder $z = \dfrac{4}{y^2+1}$, bounded by the planes

$y = x, \quad y = 3, \quad x = 0, \quad z = 0$

Solution

$$0 \le z \le \frac{4}{y^2+1}$$
$$0 \le x \le y$$
$$0 \le y \le 3$$

$$Volume = \int_0^3 \int_0^y \int_0^{\frac{4}{y^2+1}} dz\, dx\, dy$$

$$= \int_0^3 \int_0^y z \Big|_0^{\frac{4}{y^2+1}} dxdy$$

$$= \int_0^3 \int_0^y \frac{4}{y^2+1}\, dx\, dy$$

$$= \int_0^3 \frac{4}{y^2+1}\, x \Big|_0^y \, dy$$

$$= \int_0^3 \frac{4y}{y^2+1}\, dy$$

$$= 2 \int_0^3 \frac{1}{y^2+1}\, d\left(y^2+1\right)$$

$$= 2\, \ln\left(y^2+1\right) \Big|_0^3$$

$$= 2\, \ln(10) \quad unit^3$$

11.3–4 Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{\textit{Average value}} \text{ of } F \text{ over } D = \frac{1}{volume\ of\ D} \iiint_D F\ dV$$

Example 7

Find the average of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution

$$Volume = 2 \cdot 2 \cdot 2$$

$$= 8 \quad unit^3$$

The value of the integral of F over the cube is

$$Volume = \int_0^2 \int_0^2 \int_0^2 xyz\ dx\ dy\ dz$$

$$= \int_0^2 z\,dz \quad \int_0^2 y\,dy \quad \int_0^2 x\,dx$$

$$= \left(\frac{1}{2}z^2\right|_0^2 \quad \left(\frac{1}{2}y^2\right|_0^2 \quad \left(\frac{1}{2}x^2\right|_0^2$$

$$= \frac{1}{8}(4)(4)(4)$$

$$= 8 \quad unit^3$$

Average value of xyz over $cube$ $= \dfrac{1}{\text{volume of D}} \displaystyle\iiint_{cube} xyz\ dV$

$$= \left(\frac{1}{8}\right)\ (8)$$

$$= 1$$

Exercises *Section* 11.3 – Triple Integrals

(1 – 49) Evaluate the integral.

1. $\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right) z\,dz\,dy\,dx$

2. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

3. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\;dx\,dy\,dz$

4. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dy\,dx\,dz$

5. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

6. $\int_0^1\int_0^{1-x^2}\int_0^{4-x^2-y} x\,dz\,dy\,dx$

7. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(u+v+w)\;du\,dv\,dw$

8. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv$

9. $\int_0^1\int_{-z}^{z}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx\,dz$

10. $\int_0^{\pi}\int_0^{y}\int_0^{\sin x} dz\,dx\,dy$

11. $\int_0^9\int_0^1\int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}\,dx\,dy\,dz$

12. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x+y+z)\,dx\,dy\,dz$

13. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\,dy\,dz\,dx$

14. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dz\,dy\,dx$

15. $\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)\,dz\,dy\,dx$

16. $\int_{-2}^2\int_3^6\int_0^2 dx\,dy\,dz$

17. $\int_{-1}^1\int_{-1}^2\int_0^1 6xyz\;dy\,dx\,dz$

18. $\int_{-2}^2\int_1^2\int_1^e \frac{xy^2}{z}\,dz\,dx\,dy$

19. $\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z}\;dx\,dy\,dz$

20. $\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\frac{\pi}{2}} \sin\pi x\cos y\sin 2z\;dy\,dx\,dz$

21. $\int_0^2\int_1^2\int_0^1 yze^x\,dx\,dz\,dy$

22. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

23. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \; dzdydx$

24. $\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

25. $\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y} dxdzdy$

26. $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz$

27. $\int_0^{\pi} \int_0^{\pi} \int_0^{\sin x} \sin y \; dzdxdy$

28. $\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z} \; dx\, dy\, dz$

29. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz \; dz\, dy\, dx$

30. $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x} \; dz\, dx\, dy$

31. $\int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy \; dzdxdy$

32. $\int_0^1 \int_2^4 \int_{-1}^5 \left(x + yz^2\right) dx\, dy\, dz$

33. $\int_0^1 \int_{x^2}^{x} \int_0^{xy} dz\, dy\, dx$

34. $\int_0^6 \int_0^{12-2y} \int_0^{4-\frac{1}{3}x-\frac{2}{3}y} dz\, dx\, dy$

35. $\int_0^1 \int_1^y \int_0^{1-x^4-y^4} \ln x \; dz\, dx\, dy$

36. $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz \; dz\, dy\, dx$

37. $\int_{-1}^1 \int_0^1 \int_{-y^2}^{\sqrt{y}} \left(x + yz\right) dx\, dy\, dz$

38. $\int_0^6 \int_0^{2-\frac{1}{3}z} \int_0^{\frac{1}{2}(6-3y-z)} dx\, dy\, dz$

39. $\int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{16-4x^2-y^2} dz\, dy\, dx$

40. $\int_0^3 \int_0^2 \int_0^{4xy} \left(x^2 + y^2\right) dz\, dy\, dx$

41. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_0^{5-z} dx\, dz\, dy$

42. $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_{\frac{1}{4}\left(x^2+y^2\right)}^4 dz\, dy\, dx$

43. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

44. $\int_1^2 \int_0^z \int_0^{\sqrt{y}} x\, e^{x^2} \; dx\, dy\, dz$

45. $\int_0^2 \int_0^{\frac{\pi}{2}} \int_0^y e^x \; dx\, dy\, dz$

46. $\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \int_0^{xy} \sin \frac{z}{y} \; dz\, dx\, dy$

47. $\displaystyle\int_0^{\ln 8}\int_0^{\ln 4}\int_0^{\ln 2} e^{-x-y-2z}\,dx\,dy\,dz$

48. $\displaystyle\int_0^{9}\int_0^{1}\int_{2y}^{2} \frac{4\sin x^2}{\sqrt{z}}\,dx\,dy\,dz$

49. $\displaystyle\int_0^{4}\int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}}\int_0^{16-\frac{x^2}{4}-y^2} dz\,dx\,dy$

50. Evaluate the integral (with different differentia) $\displaystyle\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx$

a) $dydzdx$ ***b***) $dydxdz$ ***c***) $dxdydz$ ***d***) $dxdzdy$ ***e***) $dzdxdy$

$(\mathbf{51-55})$ Use another order to evaluate:

51. $\displaystyle\int_0^{5}\int_{-1}^{0}\int_0^{4x+4} dydxdz$

52. $\displaystyle\int_0^{1}\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} dzdydx$

53. $\displaystyle\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dydzdx$

54. $\displaystyle\int_0^{4}\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-z^2}} dydzdx$

55. $\displaystyle\int_1^{4}\int_z^{4z}\int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\,dydxdz$

$(\mathbf{56-64})$ Evaluate:

56. $\displaystyle\iiint_D (xy+xz+yz)\,dV\,;\quad D=\{(x,\,y,\,z):\ -1\le x\le 1,\ \ -2\le y\le 2,\ \ -3\le z\le 3\}$

57. $\displaystyle\iiint_D xyz\,e^{-x^2-y^2}\,dV\,;\quad D=\{(x,\,y,\,z):\ 0\le x\le\sqrt{\ln 2},\ \ 0\le y\le\sqrt{\ln 4},\ \ 0\le z\le 1\}$

58. $\displaystyle\iiint_D y^2\,dx\,dy\,dz$

D: is the tetrahedron in the first octant bounded by the coordinate planes and the plane $2x+3y+z=6$

59. $\iiint_D (5x-3y)\ dV$

D: over the solid tetrahedron bounded by the planes $x=0,\ \ y=0,\ \ z=0,\ \ and\ \ x+y+z=1$

60. $\iiint_D xyz\ dV \quad D=\left\{(x,y,z)\ |\ \sqrt{y}\le x\le 1,\ \ 0\le y\le 1,\ \ 0\le z\le y^2\right\}$

61. $\iiint_D x^2yz\ dV \qquad D=\left\{(x,y,z)\ |\ -2\le x\le 1,\ \ 0\le y\le 3,\ \ 1\le z\le 5\right\}$

62. $\iiint_D xyz^2\ dV \qquad D=\left\{(x,y,z)\ |\ 0\le x\le 1,\ \ -1\le y\le 2,\ \ 0\le z\le 3\right\}$

63. $\iiint_D dV$ $\quad D$: is in the first octant bounded by the planes $y=0,\ \ z=0,\ \ x+y=2,$

$x+2y=6$, and the cylinder $y^2+z^2=4$

64. $\iiint_D z\ dV$ $\quad D$: is in the first octant bounded by the planes $y=0,\ \ z=0,\ \ x+y=2,$

$x+2y=6$, and the cylinder $y^2+z^2=4$

65. Let $D=\left\{(x,y,z):\ 0\le x\le y^2,\ 0\le y\le z^3,\ 0\le z\le 2\right\}$

a) Use a triple integral to find the volume of *D*.

b) In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

c) What is the volume of the region $D=\left\{(x,y,z):\ 0\le x\le y^p,\ 0\le y\le z^q,\ 0\le z\le 2\right\}$, where p and q are positive real numbers?

66. Find the volume bounded by the coordinate planes and $6x+4y+3z=12$

67. For the function $f(x,y,z)=xy$, find the volume enclose by the planes $z=x+y$ and $z=0$. And between the surfaces $y=x^2$ and $x=y^2$

68. Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

69. Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y = 2$.

70. Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

71. Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the *z*-direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

72. Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

73. Find the volumes of the region between the cylinder $z = y^2$ and the *xy*-plane that is bounded by the planes $x = 0,\quad x = 1,\quad y = -1,\quad y = 1$

74. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1,\quad y + 2z = 2$

75. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y+z=2$, and the cylinder $x=4-y^2$

76. Find the volumes of the wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y, \quad z=0$

77. Find the volumes of the region cut from the cylinder $x^2+y^2=4$ by the plane $z=0$ and the plane $x+z=3$

78. Find the volumes of the region common to the interiors of the cylinders $x^2+y^2=1$ and $x^2+z^2=1$, one-eighth of which is shown below

79. Find the volume of the solid in the first octant bounded by the plane $2x+3y+6z=12$ and the coordinate planes

80. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$

81. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$

82. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$

83. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$

84. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$

85. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$

86. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$

87. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$

88. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$

89. Find the volume of the solid bounded by $x=0$, $x=1-z^2$, $y=0$, $z=0$, and $z=1-y$

90. Find the volume of the solid bounded by $x=0$, $x=2$, $y=0$, $y=e^{-z}$, $z=0$, and $z=1$

91. Find the volume of the solid bounded by $x=0$, $x=2$, $y=z$, $y=z+1$, $z=0$, and $z=4$

92. Find the volume of the solid bounded by $x=0$, $y=z^2$, $z=0$, and $z=2-x-y$

93. Find the volume of the solid common to the cylinders $z=\sin x$ and $z=\sin y$ over the square $R=\{(x,y):\ 0\le x\le \pi,\ 0\le y\le \pi\}$

94. Find the volume of the wedge of the square column $|x|+|y|=1$ created by the planes $z=0$ and $x+y+z=1$

95. Find the volume of a right circular cone with height h and base radius r.

96. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

97. Find the volume of a truncated cone of height h whose ends have radii r and R.

98. Find the volume of the solid that is bounded above by the cylinder $z=4-x^2$, on the sides by the cylinder $x^2+y^2=4$, and below by the xy-plane.

99. Find the volume of the prism in the first octant bounded by the planes $y=3-3x$ *and* $z=2$

100. Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$

101. Find the volume of the solid in the first octant bounded above by the plane $y + z = 1$ and cut by $y = \sqrt{x}$ as it is shown in the figure

102. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

103. Find the volume of the pyramid that has a square base with a side length of 2, in the $xy-plane$ and the vertex point at $(0, 0, 1)$.

104. Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the cylinder $z = 4 - y^2$

105. Find the volume of the solid that is enclosed by the cylinder $y^2 + z^2 = 4$, and the planes $z = 5 - x$, $x = 0$, and $z = 0$.

Section 11.4 – Triple Integrals in Cylindrical and Spherical Coordinates

11.4–1 Integration in Cylindrical Coordinates – *Definition*

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r *and* θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

11.4–2 Transformation Between Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$= \iiint_D f \; dz \; r\,dr\,d\theta$$

11.4−3 How to integrate in Cylindrical Coordinates

To evaluate $\iiint_D F(r,\ \theta,\ z)\ dV$

1. ***Sketch***: Sketch the region *D* along with its projection *R* on the *xy*-plane. Label the upper and lower bounding surfaces of *D* and *R*.

2. ***Find the z–limits of integration***: Draw a line *M* passing through (r,θ) in *R* // *z*-axis. As *z* increases, *M* enters *D* at $z = g_1(r,\theta)$ to $z = g_2(r,\theta)$.

3. ***Find the r–limits of integration***: Draw a line *L* passing through (r,θ) from the origin. From $r = h_1(\theta)$ to $r = h_2(\theta)$.

4. ***Find the θ–limits of integration***: As *L* sweeps across *R*, the angle θ it makes with the positive *x*-axis runs from $\theta = \alpha$ *and* $\theta = \beta$.

$$\iiint_D F(r,\ \theta,\ z)\ dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} F(r,\theta,z)\, dz\ r\ dr\, d\theta$$

Example 1

Find the volume bounded by the sphere $x^2+y^2+z^2=9$ and the paraboloid $x^2+y^2=8z$

Solution

$$x^2+y^2=r^2$$

$$\begin{cases} r^2+z^2=9 \\ r^2=8z \end{cases}$$

$$\begin{cases} z=\sqrt{9-r^2} \\ z=\frac{1}{8}r^2 \end{cases}$$

$$\frac{1}{8}r^2 \le z \le \sqrt{9-r^2}$$

$$r^2=9-z^2=8z$$

$$z^2+8z-9=0 \quad \rightarrow \quad \underline{z=1,\ \cancel{-9}}$$

$$z=1 \quad \Rightarrow \quad r^2=8z=8$$

$$r=2\sqrt{2}$$

$$\underline{0 \le r \le 2\sqrt{2}}$$

$$\underline{0 \le \theta \le 2\pi}$$

$$Volume=\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_{\frac{1}{8}r^2}^{\sqrt{9-r^2}} r\ dz\,dr\,d\theta$$

$$=\int_0^{2\pi} d\theta \int_0^{2\sqrt{2}} r\ z\ \Big|_{\frac{1}{8}r^2}^{\sqrt{9-r^2}}\ dr$$

$$=2\pi\int_0^{2\sqrt{2}} r\left(\sqrt{9-r^2}-\frac{1}{8}r^2\right)dr$$

$$=2\pi\int_0^{2\sqrt{2}} r\left(9-r^2\right)^{1/2}dr-\frac{\pi}{4}\int_0^{2\sqrt{2}} r^3\ dr$$

$$= -\pi \int_0^{2\sqrt{2}} \left(9 - r^2\right)^{1/2} d\left(9 - r^2\right) - \frac{\pi}{16} r^4 \Big|_0^{2\sqrt{2}}$$

$$= -\frac{2\pi}{3}\left(9 - r^2\right)^{3/2} \Big|_0^{2\sqrt{2}} - \frac{\pi}{16}(64)$$

$$= -\frac{2\pi}{3}(1 - 27) - 4\pi$$

$$= \frac{52\pi}{3} - 4\pi$$

$$= \frac{40\pi}{3} \quad \text{unit}^3$$

11.4–4 *Definition*

Spherical coordinates represent a point P in space by ordered triple $(\rho,\ \phi,\ \theta)$ in which

1. ρ is the distance from P to the origin
2. ϕ is the angle $\overrightarrow{OP}$ makes with positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from the cylindrical coordinates $(0 \le \theta \le 2\pi)$

11.4–5 Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinate

$$r = \rho \sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Example 2

Find a spherical coordinate equation for the sphere $z = \sqrt{x^2 + y^2}$

Solution

The cone is symmetric with respect to the *z*-axis and cuts the first quadrant of the *yz*-plane along the line $z = y$. The angle between the cone and the positive *z*-axis is therefore $\frac{\pi}{4}$ rad. The cone consists of the points whose spherical coordinates have $\varphi = \frac{\pi}{4}$.

$$z = \sqrt{x^2 + y^2}$$

$$\rho\cos\varphi = \sqrt{\rho^2 \sin^2\varphi}$$

$$\rho\cos\varphi = \rho\sin\varphi$$

$$\cos\varphi = \sin\varphi \quad \rightarrow \quad \boxed{\varphi = \frac{\pi}{4}}$$

11.4–6 Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$dV = d\rho \cdot \rho d\phi \cdot \rho \sin\phi d\theta$$
$$= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

11.4–7 How to integrate in Spherical Coordinates

To evaluate $\iiint_D F(\rho, \phi, \theta) \, dV$

1. ***Sketch***: Sketch the region *D* along its projection *R* on the *xy*-plane. Label the surface that bound of *D*.

2. ***Find the ρ–limits of integration***: Draw a ray *M* from the origin through *D* making an angle ϕ with the positive *z*-axis. Also draw the projection of *M* on the *xy*-plane (call the projection *L*). The ray *L* makes an angle θ with the positive *x*-axis. As ρ increases, *M* enters *D* at $\rho = g_1(\phi, \theta)$ to $\rho = g_2(\phi, \theta)$.

3. ***Find the ϕ–limits of integration***: For the given θ, the angle ϕ that M makes with the z-axis runs $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$.

4. ***Find the θ–limits of integration***: As L sweeps over R as θ runs from α to β.

5. $$\iiint_D f(\rho, \phi, \theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\,\rho^2 \sin\phi\, d\rho d\phi d\theta$$

Example 3

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$

Solution

$$f(\rho, \phi, \theta) = 1$$

$$V = \iiint_D \rho^2 \sin\phi\, d\rho d\phi d\theta$$

ρ–limits: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. And L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D form $\rho = 0$ to $\rho = 1$

ϕ–limits: The cone $\phi = \frac{\pi}{3}$ makes with the positive z-axis. $0 \le \phi \le \frac{\pi}{3}$

θ–limits: $0 \le \theta \le 2\pi$

$$Volume = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\phi \, d\phi \quad \left(\frac{1}{3}\rho^3 \right|_0^1$$

$$= \frac{2\pi}{3} \quad \left(-\cos\phi \right|_0^{\pi/3}$$

$$= -\frac{2\pi}{3}\left(\frac{1}{2}-1\right)$$

$$= -\frac{2\pi}{3}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{3} \quad unit^3$$

Example 4

Find the volume cut from the cone $x^2 + y^2 - z^2 = 0$, by the sphere $x^2 + y^2 + (z-2)^2 = 4$

Solution

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 = 4 - (z-2)^2 = z^2$$

$$4 - z^2 + 4z - 4 = z^2$$

$$2z^2 - 4z = 0 \quad \rightarrow \quad z = 0,\ 2$$

$$0 + y^2 = z^2 \quad \rightarrow \quad y = z = 2$$

$$\phi = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$$

$$0 \le \phi \le \frac{\pi}{4}$$

$$0 \le \theta \le 2\pi$$

$$\cos\phi = \frac{\rho}{4} \quad \rightarrow \quad \rho = 4\cos\phi$$

$$0 \le \rho \le 4\cos\phi$$

$$Volume = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} (\sin\phi)\, \rho^3 \Big|_0^{4\cos\phi} d\phi$$

$$= \frac{128\pi}{3}\int_0^{\frac{\pi}{4}} (\sin\phi)\, \cos^3\phi \, d\phi$$

$$= -\frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \cos^3\phi \, d(\cos\phi)$$

$$= -\frac{32\pi}{3} \cos^4\phi \Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{32\pi}{3}\left(\frac{1}{4}-1\right)$$

$$= -\frac{32\pi}{3}\left(-\frac{3}{4}\right)$$

$$= 8\pi \quad unit^3$$

Example 5

Evaluate the integral $\iiint_D \left(x^2+y^2+z^2\right)^{-3/2} dV$ over the region D.

Where the region D in the first octant between 2 spheres of radius 1 and 2 centered at the origin.

Solution

$$D = \left\{(\rho,\ \varphi,\ \theta):\ 1 \le \rho \le 2;\quad 0 \le \varphi \le \frac{\pi}{2};\quad 0 \le \theta \le \frac{\pi}{2}\right\}$$

$$\iiint_D \left(x^2+y^2+z^2\right)^{-3/2} dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^2 \left(\rho^2\right)^{-3/2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi \int_1^2 \rho^{-1} \, d\rho$$

$$= \frac{\pi}{2}\left(-\cos\varphi\right|_0^{\frac{\pi}{2}} \left(\ln\rho\right|_1^2$$

$$= \frac{\pi}{2}(1)\left(\ln 2\right)$$

$$= \frac{\pi \ln 2}{2}$$

Example 6

Evaluate the integral $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} \sqrt{x^2+y^2+z^2}\, dxdydz$

Solution

$$\sqrt{x^2+y^2+z^2} = \rho$$

$$0 \le x \le \sqrt{a^2-y^2-z^2}$$

$$0 \le y \le \sqrt{a^2-z^2}$$

$$\Rightarrow \quad 0 \le \theta \le \frac{\pi}{2}$$

$$a = \rho \quad \rightarrow \quad 0 \le \rho \le a$$

$$z = a \quad \rightarrow \quad \varphi = \frac{\pi}{2}$$

$$\Rightarrow \quad 0 \le \varphi \le \frac{\pi}{2}$$

$$\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} \sqrt{x^2+y^2+z^2}\, dxdydz = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho\, \rho^2 \sin\varphi\, d\rho d\varphi d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^a \rho^3\, d\rho$$

$$= \frac{\pi}{2}\left(-\cos\varphi\right|_0^{\frac{\pi}{2}} \left(\frac{1}{4}\rho^4\right|_0^a$$

$$= \frac{\pi a^4}{8}$$

11.4–8 Coordinate Conversion Formulas

Cylindrical to Rectangular	*Spherical to Rectangular*	*Spherical to Cylindrical*
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,rdr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Exercises *Section* 11.4 – Triple Integrals in Cylindrical & Spherical Coordinates.

$(\mathbf{1-25})$ Evaluate the cylindrical coordinate integral.

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, rdr\, d\theta$

2. $\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_r^{3+24r^2} dz\, rdr\, d\theta$

3. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\, rdr\, d\theta$

4. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right)dz\, rdr\, d\theta$

5. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\, dz\, d\theta$

6. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right)r\, d\theta\, drdz$

7. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}\left(r\sin\theta+1\right)r\, d\theta\, dz\, dr$

8. $\int_{-1}^5\int_0^{\frac{\pi}{2}}\int_0^3 r\cos\theta\, dr\, d\theta\, dz$

9. $\int_0^{\frac{\pi}{4}}\int_0^6\int_0^{6-r} rz\, dz\, dr\, d\theta$

10. $\int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\, dz\, dr\, d\theta$

11. $\int_0^4\int_0^z\int_0^{\frac{\pi}{2}} re^r\, d\theta drdz$

12. $\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{e^{-r^2}} r\, dzdrd\theta$

13. $\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{5-r^2} r\, dz\, dr\, d\theta$

14. $\int_0^{\pi}\int_0^{\cos\theta}\int_{2r^2}^{2r\cos\theta} r\, dz\, dr\, d\theta$

15. $\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta$

16. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta$

17. $\int_0^{\frac{\pi}{2}}\int_0^4\int_0^{\sqrt{16-z^2}} \sqrt{16-r^2}\, rz\, dr\, dz\, d\theta$

18. $\int_0^{\frac{\pi}{2}}\int_0^1\int_0^2 r^2 z\sin\theta\, dz\, dr\, d\theta$

19. $\int_0^{\pi}\int_0^1\int_0^4 r^2 z\sin\theta\, dz\, dr\, d\theta$

20. $\int_0^{2\pi}\int_0^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\, dz\, dr\, d\theta$

21. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^3\int_0^{r\sin\theta} r\csc^3\theta\, dz\, dr\, d\theta$

22. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^1\int_0^{\sin\theta} r\cos\theta\sin\theta\, dz\, dr\, d\theta$

23. $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^1\int_0^{\frac{1}{e}} r\, dz\, dr\, d\theta$

24. $\displaystyle\int_0^{\frac{\pi}{3}}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\, dz\, dr\, d\theta$

25. $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta}\int_0^{r\cos\theta} r\, dz\, dr\, d\theta$

26. Convert $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\, rdrd\theta, \quad r\geq 0$

a) Rectangular coordinates with order of integration dz*dx*d*y*.
b) Spherical coordinates
c) Evaluate one of the integrals.

27. Convert the integral $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^x \left(x^2+y^2\right)dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

28. Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3\left(\sin\theta\cos\theta\right)z^2\, dzdrd\theta$$

Arrange the order of integration to be *z* first, then *y*, then *x*.

(29 – 31) Evaluate the integrals in cylindrical coordinates.

29. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3 \left(x^2+y^2\right)^{3/2} dzdydx$

30. $\displaystyle\int_{-2}^{2}\int_{-1}^{1}\int_0^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2} dxdzdy$

31. $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$

$(32-54)$ Evaluate the spherical coordinate integral.

32. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

33. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

34. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

35. $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

36. $\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

37. $\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

38. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

39. $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi \, d\rho d\phi d\theta$

40. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^3 \cos\phi \sin\phi \, d\rho d\phi d\theta$

41. $\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\phi \, \rho^2 \, d\rho d\theta d\phi$

42. $\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} e^{-\rho^3} \rho^2 \, d\rho d\theta d\varphi$

43. $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} \rho^2 \sin\phi \, d\rho d\phi d\theta$

44. $\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} \rho^2 \sin\varphi \cos\varphi \, d\rho d\theta d\varphi$

45. $\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

46. $\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

47. $\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \, \rho^2 \, d\rho d\theta d\varphi$

48. $\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_1^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

49. $\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^4 \sin\varphi \, d\rho \, d\varphi \, d\theta$

50. $\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\varphi} \sin 2\varphi \, d\rho \, d\varphi \, d\theta$

51. $\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\varphi} \sec^2\varphi \, d\rho \, d\varphi \, d\theta$

52. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

53. $\int_0^{\frac{\pi}{2}}\int_0^{\sin\varphi}\int_0^{\frac{\pi}{4}} \rho^2 \sin\varphi \, d\theta \, d\rho \, d\varphi$

54. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\sin\varphi} \rho^2 \cos\varphi \sin\varphi \, d\rho \, d\theta \, d\varphi$

(55 – 65) Evaluate the integral.

55. $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\ dz\,dx\,dy$

56. $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{3} \left(x^2+y^2\right)^{3/2} dz\,dy\,dx$

57. $\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2}\,dx\,dz\,dy$

58. $\int_{0}^{4}\int_{0}^{\frac{\sqrt{2}}{2}}\int_{x}^{\sqrt{1-x^2}} e^{-x^2-y^2}\ dy\,dx\,dz$

59. $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4} dz\,dy\,dx$

60. $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} dz\,dy\,dx$

61. $\int_{-1}^{1}\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\ dx\,dy\,dz$

62. $\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2} dxdzdy$

63. $\int_{0}^{4}\int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}}\int_{0}^{16-\frac{x^2}{4}-y^2} dz\,dx\,dy$

64. $\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2}\ dx\,dz\,dy$

65. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} z \, dz \, dy \, dx$

66. Evaluate $\iiint_D r^2 \, dV$ D: is bounded by the paraboloid $r^2 = 9 - z$, and the plane $z = 0$

67. Evaluate $\iiint_D \sqrt{3x^2 + 3z^2} \, dV$ D: is the solid bounded by $y = 2x^2 + 2z^2$, and the plane $y = 8$

68. Evaluate: $\iiint_D e^{\left(x^2+y^2+z^2\right)^{3/2}} \, dx \, dy \, dz$

D: is the solid bounded below by the half–cone: $z = \sqrt{x^2 + y^2}$ and above by the spherical surface.

69. Evaluate $\iiint_D \left(x^2 + y^2 + z^2\right)^{5/2} dV$; D is the unit ball.

70. Evaluate $\iiint_D e^{-\left(x^2+y^2+z^2\right)^{3/2}} \, dV$; D is the unit ball.

71. Evaluate $\iiint_D \frac{1}{\left(x^2 + y^2 + z^2\right)^{3/2}} dV$; D is the solid between the spheres of radius 1 and 2 centered at the origin.

72. Evaluate $\iiint_D \left(x^2 + y^2 + z^2\right) dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

73. Evaluate $\iiint_D x^2 dV$; $D = \left\{(r, \theta, z): \quad 0 \le r \le 1, \quad 0 \le z \le 2r, \quad 0 \le \theta \le 2\pi\right\}$

74. Evaluate $\iiint\limits_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ -\sqrt{4-r^2}\le z\le\sqrt{4-r^2},\ \ 0\le\theta\le 2\pi\right\}$

75. Evaluate $\iiint\limits_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ r\le z\le\sqrt{2-r^2},\ \ 0\le\theta\le 2\pi\right\}$

76. Evaluate $\iiint\limits_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ r^2\le z\le\sqrt{2-r^2},\ \ 0\le\theta\le 2\pi\right\}$

77. Evaluate $\iiint\limits_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 4,\ \ 2r\le z\le 24-r^2,\ \ 0\le\theta\le 2\pi\right\}$

78. Evaluate $\iiint\limits_D y^2z^2dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 1,\ \ 0\le\varphi\le\frac{\pi}{3},\ \ 0\le\theta\le 2\pi\right\}$

79. Evaluate $\iiint\limits_D \left(x^2+y^2\right)dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 2\le\rho\le 3,\ \ 0\le\varphi\le\pi,\ \ 0\le\theta\le 2\pi\right\}$

80. Evaluate $\iiint\limits_D y^2dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 3,\ \ 0\le\varphi\le\pi,\ \ 0\le\theta\le\pi\right\}$

81. Evaluate $\iiint\limits_D xe^{x^2+y^2+z^2}dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 1,\ \ 0\le\varphi\le\frac{\pi}{2},\ \ 0\le\theta\le\frac{\pi}{2}\right\}$

82. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}\ dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 1\le\rho\le 2,\ \ 0\le\varphi\le\frac{\pi}{4},\ \ 0\le\theta\le 2\pi\right\}$

83. Evaluate $\iiint\limits_D \frac{1}{\rho}\ dV$

D: is in the first octant bounded by the cone $\varphi=\frac{\pi}{4}$, *and* $\varphi=\tan^{-1}2$, and the sphere $\rho=\sqrt{6}$

84. Find the volume of the solid in the first octant bounded by the cylinder $y=e^x$ and the planes $x=0$, $x=1$, $z=0$, and $y=z$.

85. Find the volume under the paraboloid $x^2 + y^2 = a\,z$, above the $xy - plane$, and inside the cylinder $x^2 + y^2 = 2a\,x$.

86. Find the volume in the first octant bounded by the planes $x = 0$, $z = 0$, and $y = 2 - x$, the parabolic cylinder $y = x^2$, and the surface $z = x^2 + 2y^2$.

87. Find the volume bounded by the cone $r = z$, the cylinder $r = a\cos\theta$, and the plane $z = 0$.

88. Find the volume bounded by the surfaces $x^2 + y^2 = z$ and $x^2 + y^2 = a^2$, and the $xy - plane$.

89. Find the volume bounded by the 2 cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

90. Find the volume bounded by the sphere $x^2 + y^2 + z^2 = 9$ and the paraboloid $x^2 + y^2 = 8x$

91. Find the volume cut from by the paraboloid $z = x^2 + 4y^2$ by the plane $z = 1$.

92. Find the volume inside the circle $x^2 + y^2 = 9$, above $z = 0$, and below $x + z = 4$

93. Find the volume inside the circle $x^2 + y^2 = 4x$, above $z = 0$, and below $x^2 + y^2 = 4z$

94. Find the volume within the cylinder $z = 4\cos\theta$ bounded above by the sphere $r^2 + z^2 = 16$ and below by the plane $z = 0$

95. Find the volume of the solid whose height is 4 and whose base is the disk $\{(r,\ \theta):\ \ 0 \le r \le 2\cos\theta\}$

96. Find the volume of the solid in the first octant bounded by the cylinder $r = 1$ and the plane $z = x$

97. Find the volume of the solid bounded by the cylinder $r = 1$ and $r = 2$ and the planes $z = 4 - x - y$ and $z = 0$

98. Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$

99. Find the volume over the region D that is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$

100. Find the volume over the region D which is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$

101. Find the volume over the region D which is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$

102. Find the volume over the region D which is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$, and whose top lies in the plane $z = 2 - x$

103. Find the volume of the solid region D that lies inside the cone $\phi = \frac{\pi}{6}$ and inside the sphere $\rho = 4$.

104. Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2,\ z \geq 0$.

105. Find the volume of the solid bounded below by the hemisphere $\rho = 1,\ z \geq 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$.

106. Find the volume of the solid

107. Find the volume of the solid

108. Find the volume of the solid

109. Find the volume of the solid

110. Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

111. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

112. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

113. Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$

114. Evaluate the integral in cylindrical coordinates

$$\int_0^{2\pi}\int_0^1\int_{-1}^1 r\, dz dr d\theta$$

115. Evaluate the integral in cylindrical coordinates

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dzdxdy$$

116. Evaluate the integral in cylindrical coordinates

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^1 \left(x^2+y^2\right)^{3/2} dzdxdy$$

117. Evaluate the integral in cylindrical coordinates

$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2} dzdydx$$

118. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z=0$ and the hyperboloid $z=\sqrt{17}-\sqrt{1+x^2+y^2}$

$z=\sqrt{17}-\sqrt{1+x^2+y^2}$

119. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$

120. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$

121. Evaluate the integral

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

122. Evaluate the integral

$$\int_0^{\pi} \int_0^{\pi/6} \int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

123. Evaluate the integral

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi}\left(\rho^{-3}\right)\rho^2\sin\varphi\, d\rho d\varphi d\theta$$

124. Evaluate the integral

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta$$

125. Use the spherical coordinates to find the volume of a ball of radius $a > 0$

126. Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \geq 0$

127. Use the spherical coordinates to find the volume of the solid of revolution

$$D = \left\{(\rho,\varphi,\theta):\ 0 \leq \rho \leq 1+\cos\varphi,\ 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi\right\}$$

128. Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$

129. Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$

130. dUse the spherical coordinates to find the volume of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$

131. Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$

132. The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.

Find the volume of the solid that is common to the two cylinders.

133. The coordinate axes from the axes of three right circular cylinders with radius 1.

Find the volume of the solid that is common to the three cylinders.

134. Find the volume of one of the wedges formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = z$

135. Find the volume of the region inside the parabolic cylinder $y = x^2$ between the planes $z = 3 - y$ and $z = 0$

136. Find the volume of the tetrahedron with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(1,\ 1,\ 0)$, and $(1,\ 1,\ 1)$

137. Find the volume of the region bounded by the plane $z=\sqrt{29}$ and the hyperboloid $z=\sqrt{4+x^2+y^2}$. Use integration in cylindrical coordinates.

138. Find the volume of the solid cylinder whose height is 4 and whose base is the disk $\{(r,\theta):\ 0\le r\le 2\cos\theta\}$. Use integration in cylindrical coordinates.

139. Use integration in spherical coordinates to find the volume of the rose petal of revolution $D=\left\{(\rho,\varphi,\theta):\ 0\le\rho\le 4\sin 2\varphi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\right\}$

140. Use integration in spherical coordinates to find the volume of the region above the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$.

141. Find the volume of the cardioid of revolution

$$D = \left\{ (\rho, \varphi, \theta) : \ 0 \le \rho \le \frac{1-\cos\varphi}{2}, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi \right\}$$

142. Find the volume inside the surface $z = \frac{20}{1+x^2+y^2} - 2$ bounded below by $xy - plane$.

143. Find the volume of the cap of a sphere of radius *R* with height *h*.

144. Find the volume of a solid lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

145. Find the volume of the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$

146. Find the volume bounded above by the semi-sphere $x^2 + y^2 + z^2 = 4$, and on the side the cylinder $x^2 + y^2 = 1$

147. Find the volume of the region inside right circular with an equation $r = 2\sin\theta$ and bounded above by the sphere with radius 2 centered at the origin and bounded below by the $xy - plane$.

148. Find the volume of the region bounded above by the sphere $z = x^2 + y^2 + z^2$ and below by the cone $z = \sqrt{x^2 + y^2}$

149. Find the volume of the region bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the cone $z = \sqrt{3x^2 + 3y^2}$

150. Find the volume above the surface $\sqrt{x^2 + z^2}$, bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

151. A cake is shaped like a solid cone with radius 4 and height 2, with its based on the *xy*-plane. A wedge of the cake is removed by making two slices from the axis of the cone outward, perpendicular to the *xy*-plane separated by an angle of *Q* radians, where $0 < Q < 2\pi$

a) Find the volume of the slice for $Q = \frac{\pi}{4}$. Use geometry to check your answer.

b) Find the volume of the slice for $0 < Q < 2\pi$. Use geometry to check your answer.

152. A spherical fish tank with a radius of 1 *ft* is filled with water to a level 6 *in*. below the top of the tank.

a) Determine the volume and weight of the water in the fish tank. (The weight density of water is about 62.5 lb / ft^3.)

b) How much additional water must be added to completely fill the tank?

153. A spherical cloud of electric charge has known charge density $Q(\rho)$, where ρ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a) $Q(\rho)=\dfrac{2\times10^{-4}}{\rho^4},\quad 1\le\rho<\infty$

b) $Q(\rho)=\dfrac{2\times10^{-4}}{1+\rho^3},\quad 1\le\rho<\infty$

c) $Q(\rho)=2\times10^{-4}e^{-0.01\rho^3},\quad 0\le\rho<\infty$

154. A point mass *m* is a distance *d* from the center of a thin spherical shell of mass *M* and radius *R*. The magnitude of the gravitational force on the point mass is given by the integral

$$F(d)=\frac{GMm}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(d-R\cos\phi)\sin\phi}{\left(R^2+d^2-2Rd\cos\phi\right)^{3/2}}\,d\phi\,d\theta$$

Where *G* is the gravitational constant.

a) Use the change of variable $x=\cos\phi$ to evaluate the integral and show that if $d>R$, then $F(d)=\dfrac{GMm}{d^2}$, which means the force is the same as if the mass of the shell were concentrated at its center.

b) Show that is $d<r$ (the point mass is inside the shell), then $F=0$.

155. Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 *ft* and a radius of 1 *ft*. If the water level is 6 *in*. above the lowest part of the tank, determine how much water must be drained from the tank.

156. You are designing a special capsule for your company, consisting of two materials. The two hemispheres, 1-*unit* radius on both sides, have one substance, the second substance in a cylinder with a 1-*unit* radius and length of 4 *units*.

a) Find the volume for the substance that needed to fill up the cylinder part.

b) Find the volume for each substance that needed to fill up both hemispheres.

c) Find the total volume of a capsule.

Note: use all methods.

Section 11.5 – Integrals for Mass Calculations

11.5–1 One-Dimensional Center of Mass

If two objects with masses m_1 and m_2 sit at distances d_1 and d_2 from the pivot point (with no mass), then the balances provided $m_1 d_1 = m_2 d_2$.

Solving the equation for $\bar{x}$, the balance point or center of mass of the two-mass system is located at

$$\bar{x} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

The quantities $m_1 d_1$ and $m_2 d_2$ are called moments about the origin (or just moments). The location of the center of mass is the sum of moments divided by the sum of the masses.

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i d_i}{\sum_{i=1}^{n} m_i}$$

11.5–2 Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ representing mass per unit length. When a curve C is parametrized by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, $a \le t \le b$, the density is the function $\delta(x(t), y(t), z(t))$, and then the arc length differential is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

The formula of mass is

$$m = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $m = \int_C \delta\, ds$ $\quad \delta = \delta(x, y, z)$ *is the* density at (x, y, z)

First moments about the coordinates planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}$$

Moments of inertia about axes and other lines:

$$I_x = \iint_R \delta\, y^2\, ds, \quad I_y = \iint_R \delta\, x^2\, ds, \qquad I = I_x + I_y$$

$$I_x = \int_C \left(y^2 + z^2\right) \delta\, ds, \quad I_y = \int_C \left(x^2 + z^2\right) \delta\, ds, \quad I_z = \int_C \left(x^2 + y^2\right) \delta\, ds$$

$$I_L = \int_C r^2 \delta\, ds$$ $r(x, y, z) =$ distance from the point (x, y, z) to the line L

Units: $\begin{cases} kg \cdot m^2 \\ lbf \cdot ft \cdot s^2 \end{cases}$

Example 1

A slender metal arch, denser at the bottom than top, lies along the semicircle $z^2 + y^2 = 1,\ \ z \geq 0$, in the *yz*-plane. Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$

Solution

$\bar{x} = 0$ *and* $\bar{y} = 0$, because the arch lies in the *yz*-plane with its mass distributed symmetrically about the *z*-axis.

$$\vec{r}(t) = (\cos t)\,\hat{j} + (\sin t)\hat{k}, \quad 0 \leq t \leq \pi$$

$$|v(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1}$$

$$\Rightarrow ds = |v|\,dt = dt$$

$$m = \int_0^{\pi} (2 - z)\ dt$$

$$= \int_0^{\pi} (2 - \sin t)\ dt$$

$$= 2t + \cos t \Big|_0^{\pi}$$

$$= 2\pi + \cos\pi - \cos 0$$

$$\underline{= 2\pi - 2}$$

$$M_{xy} = \int_C z\,\delta\,ds$$

$$= \int_C z\,(2 - z)\,ds$$

$$= \int_0^{\pi} (\sin t)(2 - \sin t)\ dt$$

$$= \int_0^{\pi} \left(2\sin t - \sin^2 t\right)\, dt$$

$$= \left(-2\cos t - \frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_0^{\pi}$$

$$= -2(-1) - \frac{\pi}{2} + 2$$

$$= 4 - \frac{\pi}{2}$$

$$= \frac{8-\pi}{2}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{8-\pi}{2} \cdot \frac{1}{2\pi - 2}$$

$$= \frac{8-\pi}{4\pi - 4}$$

$\therefore$ The ***center mass*** is $\left(0,\ 0,\ \frac{8-\pi}{4\pi-4}\right)$ *unit*

11.5–3 Two-Dimensional Objects – *Definition*

Let ρ be an integrable area density function defined over a closed bounded region $\boldsymbol{R}$ in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by $\boldsymbol{R}$ are:

$$m = \iint_R \delta(x, y)\, dA \qquad\qquad m = \iint_R \delta(r, \theta)\, r\, dr\, d\theta$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\,\delta(x, y)\, dA \qquad\qquad \bar{x} = \frac{1}{m}\iint_R r\cos\theta\ \delta(r,\theta)\, r\, dr\, d\theta$$

and *and*

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\,\delta(x, y)\, dA \qquad\qquad \bar{y} = \frac{1}{m}\iint_R r\sin\theta\ \delta(r,\theta)\, r\, dr\, d\theta$$

Where m is the mass, and M_y & M_x are the moments with respect to the *y*-axis and *x*-axis, respectively. If δ is constant, the center of mass is called the ***centroid*** and is independent of the density,

Example 2

Find the centroid (center of mass) of the constant density, dart-shaped region bounded by the *y*-axis and the curves $y = e^{-x} - \frac{1}{2}$ *and* $y = \frac{1}{2} - e^{-x}$

Solution

Assume: $\delta = 1$

$$y = e^{-x} - \frac{1}{2} = \frac{1}{2} - e^{-x}$$

$$2e^{-x} = 1$$

$$-x = \ln\frac{1}{2}$$

$$\underline{x = \ln 2}$$

$$m = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} 1 \, dy \, dx$$

$$= \int_0^{\ln 2} y \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} \left(2e^{-x} - 1\right) dx$$

$$= \left(-2e^{-x} - x\right. \Big|_0^{\ln 2}$$

$$= -2e^{-\ln 2} - \ln 2 + 2$$

$$= -2\left(\frac{1}{2}\right) - \ln 2 + 2$$

$$= 1 - \ln 2$$

$$M_y = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} x \, dy dx$$

$$= \int_0^{\ln 2} xy \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} x\left(e^{-x} - \frac{1}{2} - \frac{1}{2} + e^{-x}\right) dx$$

$$= \int_0^{\ln 2} x\left(2e^{-x} - 1\right) dx$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= -2xe^{-x} - 2e^{-x} - \frac{1}{2}x^2 \Big|_0^{\ln 2}$$

$$= -2(\ln 2)\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) - \frac{1}{2}(\ln 2)^2 + 0 + 2 + 0$$

$$= 1 - \ln 2 - \frac{1}{2}(\ln 2)^2$$

$$\bar{x} = \frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{1 - \ln 2} \qquad \bar{x} = \frac{M_y}{m}$$

$$= \frac{1}{2}\frac{2 - 2\ln 2 - (\ln 2)^2}{1 - \ln 2}$$

The center of mass is located approximately at $\left(\frac{2 - 2\ln 2 - (\ln 2)^2}{2 - 2\ln 2}, \quad 0\right)$ $(0.217, \ 0)$

11.5–4 Three-Dimensional Objects – *Definition*

Let δ be an I ntegrable area density function defined over a closed bounded region ***D*** in $\mathbb{R}^3$. The coordinates of the center of mass of the object represented by ***D*** are:

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\ \delta(x, y, z)\, dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\ \delta(x, y, z)\, dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\ \delta(x, y, z)\, dV$$

Where $m = \iiint_D \delta(x, y, z)\ dV$ is the mass.

M_{yz}, M_{xz}, *and* M_{xy} are the moments with respect to the coordinate's planes.

Example 4

Find the center of mass of the constant density solid cone ***D*** bounded by the surface

$$z = 4 - \sqrt{x^2 + y^2} \quad and \quad z = 0$$

Solution

The one is symmetric about the *z*-axis and has uniform density, the center of mass lies on the *z*-axis, that is, $\bar{x} = 0$ *and* $\bar{y} = 0$.

The disk has a radius of 4 and centered at the origin. Therefore, the cone has height 4 and radius 4; by the volume formula is $\frac{1}{3}\pi h r^2 = \frac{1}{3}\pi 4\left(4^2\right) = \frac{64\pi}{3}$.

The cone has a constant density, so we assume that $\delta = 1$ and its mass is $m = \frac{64\pi}{3}$

$$z = 4 - \sqrt{x^2 + y^2} = 4 - r$$

$$M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{4-r} z\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 r \left(\frac{1}{2} z^2 \right|_0^{4-r} dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^4 r(4-r)^2\, dr\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^4 \left(16r - 8r^2 + r^3\right) dr\, d\theta$$

$$= \frac{1}{2} \left(8r^2 - \frac{8}{3} r^3 + \frac{1}{4} r^4 \right|_0^4 \quad (\theta \Big|_0^{2\pi}$$

$$= \frac{1}{2} \left(128 - \frac{512}{3} + 64 \right) (2\pi)$$

$$= \frac{64\pi}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{64\pi/3}$$

$$= 1$$

∴ The ***center of mass*** is located at $(0,\ 0,\ 1)$

Example 5

Find the center of mass of the interior of the hemisphere ***D*** of a radius *a* with its base on the *xy*-plane. The density of the objects is $f(\rho,\ \phi,\ \theta) = 2 - \frac{\rho}{a}$ (heavy near the center and light near the outer surface.)

Solution

The one is symmetric about the *z*-axis and has uniform density, the center of mass lies on the *z*-axis, that is $\bar{x} = 0$ *and* $\bar{y} = 0$.

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2}\sin\phi d\phi \int_0^{a}\left(2\rho^2-\frac{1}{a}\rho^3\right)d\rho$$

$$= \theta\Big|_0^{2\pi}\left(-\cos\phi\Big|_0^{\pi/2}\left(\frac{2}{3}\rho^3-\frac{1}{4a}\rho^4\Big|_0^{a}\right.\right.$$

$$= (2\pi)(1)\left(\frac{2}{3}a^3-\frac{1}{4}a^3\right)$$

$$= \frac{5\pi}{6}a^3$$

In spherical coordinate: $z = \rho\cos\phi$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\rho\cos\phi\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$M_{xy} = \iiint_D z\,\rho(x,y,z)\,dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2}\frac{1}{2}\sin 2\phi d\phi \int_0^{a}\left(2\rho^3-\frac{1}{a}\rho^4\right)d\rho$$

$$2\sin\phi\cos\phi = \sin 2\phi$$

$$= 2\pi\left(-\frac{1}{4}\cos 2\phi\Big|_0^{\pi/2}\left(\frac{1}{2}\rho^4-\frac{1}{5a}\rho^5\Big|_0^{a}\right.\right.$$

$$= -\frac{1}{4}(2\pi)(-2)\left(\frac{1}{2}a^4-\frac{1}{5}a^4\right)$$

$$= \frac{3\pi}{10}a^4$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\frac{3\pi a^4}{10}}{\frac{5\pi a^3}{6}}$$

$$= \frac{9a}{25}$$

However, the center of mass of a uniform-density hemisphere solid of radius a is $\frac{3a}{8} = 0.375a$ units above the base. In this case, the variable density shifts the center of mass.

Example 6

Find the moment of inertia about the x-axis of the curve: $4x = 2y^2 - \ln y$ from $y = 2$ to $y = 4$.

Solution

$$x = \frac{1}{2}y^2 - \frac{1}{4}\ln y$$

$$\frac{dx}{dy} = y - \frac{1}{4y}$$

$$ds = \sqrt{\left(y - \frac{1}{4y}\right)^2 + 1} \qquad ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

$$= \sqrt{y^2 - \frac{1}{2} + \frac{1}{16y^2} + 1}$$

$$= \sqrt{y^2 + \frac{1}{2} + \frac{1}{16y^2}}$$

$$= \sqrt{\left(y + \frac{1}{4y}\right)^2}$$

$$= y + \frac{1}{4y}$$

$$I_x = \int_2^4 y^2\left(y + \frac{1}{4y}\right)dy$$

$$= \int_2^4 \left(y^3 + \frac{1}{4}y\right)dy$$

$$= \frac{1}{4}y^4 + \frac{1}{8}y^2 \Big|_2^4$$

$$= 64 + 2 - 4 - \frac{1}{2}$$

$$= \frac{123}{2}$$

11.5–5 The Radius of Gyration

The radius of gyration is a measure of the elastic stability, and it is defined as the resistance of a cross section of any material to bending. In pipes, this involves the walls of the pipe developing the stress necessary to resist deformation either elastically or plastically. There are several values of the radius of gyration for a cross-section, depending on which axis is being considered.

Mathematically the radius of gyration is the root mean square distance of the object's parts from either its center of mass or a given axis, depending on the relevant application. It is the perpendicular distance from point mass to the axis of rotation. One can represent a trajectory of a moving point as a body. Then radius of gyration can be used to characterize the typical distance travelled by this point.

The ***radius of gyration*** $\bar{\bar{r}}$ $\left(\bar{\bar{x}} \text{ or } k\right)$ of a revolving mass m with moment of inertia I is defined:

$$\bar{\bar{r}} = \sqrt{\frac{I}{m}}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} \qquad \text{about } y-\text{axis}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} \qquad \text{about } x-\text{axis}$$

Example 7

Find the radius of gyration about the $y-axis$ for the lamina corresponding to the region $R = \left\{ (x, y) \mid 0 \le x \le \pi,\ \ 0 \le y \le \sin x \right\}$, where the density at (x, y) is given by $\delta(x, y) = x$.

Solution

$$I_y = \int_0^{\pi} \int_0^{\sin x} x^2 (x)\, dy\, dx$$

$$= \int_0^{\pi} x^3 \left(y \Big|_0^{\sin x}\right) dx$$

$$= \int_0^{\pi} x^3 \sin x\ dx$$

$$= \left(\left(-x^3 + 6x\right) \cos x + \left(3x^2 - 6\right) \sin x\right) \Big|_0^{\pi}$$

$$= \pi^3 - 6\pi$$

		$\int \sin x\, dx$
+	x^3	$-\cos x$
−	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
−	6	$\sin x$

$$m = \int_0^{\pi} \int_0^{\sin x} x \, dy \, dx$$

$$= \int_0^{\pi} x \sin x \; dx$$

$$= \left(-x \cos x + \sin x \right|_0^{\pi}$$

$$= \pi$$

		$\int \sin x \, dx$
+	x	$-\cos x$
−	1	$-\sin x$

Thus, the radius of gyration about the $y-axis$ is:

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$$

$$= \sqrt{\frac{\pi^3 - 6\pi}{\pi}}$$

$$= \sqrt{\pi^2 - 6} \quad unit$$

Exercises *Section* 11.5 – Integrals for Mass Calculations

$(\mathbf{1-2})$ Find the location of the center of mass.

1. $m_1 = 10\ kg$ located at $x = 3\ m$; $m_2 = 3\ kg$ located at $x = -1\ m$

2. $m_1 = 8\ kg$ located at $x = 2\ m$; $m_2 = 4\ kg$ located at $x = -4\ m$; $m_2 = 1\ kg$ located at $x = 0\ m$

$(\mathbf{3-6})$ Find the ***mass*** of the following objects with given density functions.

3. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 4,\ \ 0 \le z \le 10\}$ with density $\delta(r,\ \theta,\ z) = 1 + \frac{z}{2}$

4. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 3,\ \ 0 \le z \le 2\}$ with density $\delta(r,\ \theta,\ z) = 5e^{-r^2}$

5. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 6,\ \ 0 \le z \le 6 - r\}$ with density $\delta(r,\ \theta,\ z) = 7 - z$

6. The solid cylinder $D = \left\{(r,\ \theta,\ z):\ \ 0 \le r \le 3,\ \ 0 \le z \le 9 - r^2\right\}$ with density $\delta(r,\ \theta,\ z) = 1 + \frac{z}{9}$

$(\mathbf{7-12})$ Find the ***mass*** and ***center of mass*** of the thin rods with the following density functions.

7. $\delta(x) = 1 + \sin x \quad for\ \ 0 \le x \le \pi$

8. $\delta(x) = 1 + x^3 \quad for\ \ 0 \le x \le 1$

9. $\delta(x) = 2 - \frac{x^2}{16} \quad for\ \ 0 \le x \le 4$

10. $\delta(x) = 2 + \cos x \quad for\ \ 0 \le x \le \pi$

11. $\delta(x) = \begin{cases} x^2 & if\ \ 0 \le x \le 1 \\ 2x - x^2 & if\ \ 1 \le x \le 2 \end{cases}$

12. $\delta(x) = \begin{cases} 1 & if\ \ 0 \le x \le 2 \\ 1 + x & if\ \ 2 < x \le 4 \end{cases}$

$(\mathbf{13-18})$ Find the mass and center mass centroid of the lamina bounded by the graphs of the given equation for the given density.

13. $y = \sqrt{a^2 - x^2}$, $\ y = 0$, $\ \delta(x,\ y) = k$

14. $y = \sqrt{a^2 - x^2}$, $\ y = 0$, $\ \delta(x,\ y) = k\left(ay - y^2\right)$

15. $y=\sqrt{a^2-x^2}$, $y=0$, $y=x$, $\delta(x,y)=k\sqrt{x^2+y^2}$

16. $x^2+y^2=a^2$, $x\ge 0,\ y\ge 0$, $\delta(x,y)=k\left(x^2+y^2\right)$

17. $y=\sqrt{x}$, $x=4$, $y=0$, $\delta(x,y)=kxy$

18. $y=\dfrac{1}{1+x^2}$, $x=\pm 1$, $y=0$, $\delta(x,y)=k$

(19 – 26) Find the ***mass***, ***center mass*** (centroid), the ***moment of inertia***, and the ***radius of gyration*** of the lamina bounded by the graphs of the given equation for the given density.

19. $xy=4$, $x=1$, $x=4$, $\delta(x,y)=2x^2$

20. $y=\sqrt{x}$, $x=4$, $y=0$, $\delta(x,y)=2xy$

21. $y=-\frac{1}{4}x+\frac{5}{2}$, $x=0$, $y=0$, $y=2$, $\delta(x,y)=3xy$

22. $y=-x+3$, $y=x$, $y=0$, $y=1$, $\delta(x,y)=2x+y$

23. $y=\sin\frac{\pi x}{L}$, $x=0$, $x=L$, $y=0$, $\delta(x,y)=ky$

24. $y=\cos\frac{\pi x}{L}$, $x=0$, $x=\frac{1}{2}L$, $y=0$, $\delta(x,y)=k$

25. $y=e^x$, $1\le x\le 3$, $\delta(x,y)=1$

26. $y=x^2$, $0\le x\le 2$, $\delta(x,y)=x+y$

(27 – 42) Find the ***mass*** and ***centroid*** (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry, when possible, to simplify your work.

27. The region bounded by $y=\sin x$ and $y=1-\sin x$ between $x=\frac{\pi}{4}$ *and* $x=\frac{3\pi}{4}$

28. The region bounded by $y=1-|x|$ and the x-axis.

29. The region bounded by $y=e^x$, $y=e^{-x}$, $x=0$, *and* $x=\ln 2$

30. The region bounded by $y=\ln x$, $x-axis$, *and* $x=e$

31. The region bounded by $x^2+y^2=1$ *and* $x^2+y^2=9$, *for* $y\ge 0$

32. The region bounded by $y=\sin x$ and $y=0$ between $x=0$ and $x=\pi$.

33. The region bounded by $y=x^3$ and $y=x^2$ between $x=0$ and $x=1$.

34. The half annulus $\{(r,\theta):\ 2 \le r \le 4,\ 0 \le \theta \le \pi\}$

35. The region bounded by $y = x^2$ and $y = a^2 - x^2$

36. The semicircular disk $R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ \ 0 \le \theta \le \pi\}$

37. The quarter-circular disk $R = \left\{(r,\ \theta):\ \ 0 \le r \le 2,\ \ \ 0 \le \theta \le \frac{\pi}{2}\right\}$

38. The region bounded by the cardioid $r = 1 + \cos\theta$

39. The region bounded by the cardioid $r = 3 - 3\cos\theta$

40. The region bounded by one leaf of the rose $r = \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$

41. The region bounded by the limaçon $r = 2 + \cos\theta$

(42 – 56) Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

42. $R = \{(x,\ y):\ 0 \le x \le 4,\ 0 \le y \le 2\}$; $\delta(x,\ y) = 1 + \frac{x}{2}$

43. $R = \{(x,\ y):\ -1 \le x \le 1,\ 0 \le y \le 1\}$; $\delta(x,\ y) = 2 - y$

44. $R = \{(x,\ y):\ 0 \le x \le 1,\ 0 \le y \le 5\}$; $\delta(x,\ y) = 2e^{-y/2}$

45. $R = \{(x,\ y,\ z):\ 0 \le x \le 4,\ 0 \le y \le 1,\ 0 \le z \le 1\}$; $\delta(x,\ y,\ z) = 1 + \frac{x}{2}$

46. The triangular plate in the first quadrant bounded by $x + y = 4$ with $\delta(x,\ y) = 1 + x + y$

47. The upper half $(y \ge 0)$ of the disk bounded by the circle $x^2 + y^2 = 4$ with $\delta(x,\ y) = 1 + \frac{y}{2}$

48. The upper half $(y \ge 0)$ of the disk bounded by the ellipse $x^2 + 9y^2 = 9$ with $\delta(x,\ y) = 1 + y$

49. The quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with $\delta(x,y) = 1 + x^2 + y^2$

50. The upper half of a ball $\left\{(\rho,\varphi,\theta):\ 0 \le \rho \le 16,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$ with density

$f(\rho,\ \varphi,\ \theta) = 1 + \frac{\rho}{4}$

51. The region bounded by the upper half of the sphere $\rho = 6$ and $z = 0$ with density

$f(\rho,\ \varphi,\ \theta) = 1 + \frac{\rho}{4}$

52. The cube in the first octant bounded by the planes $x = 2$, $y = 2$, $z = 2$, with $\delta(x,y,z) = 1 + x + y + z$

53. The interior of the cube in the first octant formed by the planes $x = 1$, $y = 1$, $z = 1$ with $\delta(x,\ y,\ z) = 2 + x + y + z$

54. The region bounded by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$ with $\delta(x, y, z) = 5 - z$

55. The interior of the prism formed by $x = 1$, $y = 4$, $z = x$, and the coordinate planes with $\delta(x,\ y,\ z) = 2 + y$

56. The region bounded by the cone $z = 9 - r$ and $z = 0$ with $\delta(r,\ \theta,\ z) = 1 + z$

(57 – 65) Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

57. The upper half of the ball $x^2 + y^2 + z^2 \le 16$ $(\textit{for } z \ge 0)$

58. The region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 25$

59. The tetrahedron in the first octant bounded by $z = 1 - x - y$ and the coordinate planes.

60. The solid bounded by the cone $z = 16 - r$ and the plane $z = 0$

61. The paraboloid bowl bounded by $z = x^2 + y^2$ and $z = 36$

62. The tetrahedron bounded by $z = 4 - x - 2y$ and the coordinate planes.

63. The solid bounded by the cone $z = 4 - \sqrt{x^2 + y^2}$ and the plane $z = 0$

64. The sliced solid cylinder bounded by $x^2 + y^2 = 1,\ z = 0,\ \textit{and}\ \ y + z = 1$

65. The solid bounded by the upper half $(z \ge 0)$ of the ellipsoid $4x^2 + 4y^2 + z^2 = 16$

(66 – 76) Consider the following two- and three- dimensional regions. Compute the center of mass assuming constant density. All parameters are positive real numbers.

66. A region is bounded by a paraboloid with a circular base of radius R and height h. How far from the base is the center of mass?

67. Let R be the region enclosed by an equilateral triangle with sides of length s. what is the perpendicular distance between the center of mass of R and the edges of R?

68. An isosceles triangle has two sides of length s and a base of length b. how far from the base is the center of mass of the region enclosed by the triangle?

69. A tetrahedron is bounded by the coordinate planes and the plane $x + \frac{y}{2} + \frac{z}{3} = 1$. What are the coordinates of the center of mass?

70. A solid box has sides of length a, b, and c. Where is the center of mass relative to the faces of the box?

71. A solid cone has a base with a radius of *r* and a height of *h*. How far from the base is the center of mass?

72. A solid is enclosed by a hemisphere of radius *a*. How far from the base is the center of mass?

73. A tetrahedron is bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$. What are the coordinates.

74. A solid is enclosed by the upper half of an ellipsoid with a circular base of radius *r* and a height of *a*. How far from the base is the center of mass?

75. A thin (one-dimensional) wire of constant density is bent into the shape of a semicircular of radius *r*. Find the location of its center of mass.

76. A thin plate of constant density occupies the region between the parabola $y=ax^2$ and the horizontal line $y=b$, where $a>0$ and $b>0$. Show that the center of mass is $\left(0,\ \frac{3b}{5}\right)$, independent of *a*.

77. Find the center of mass of the region in the first quadrant bounded by the circle $x^2+y^2=a^2$ and the lines $x=a$ and $y=a$, where $a>0$

78. Find the mass and the center of mass of the lamina corresponding to the first quadrant portion of the circle $x^2+y^2=4$, where density at $(x,\ y)$ is proportional to the distance between the point on inside the circle and the origin.

79. Find the centroid and the moment of inertia corresponding to the paraboloid region $y=4-x^2$ with $y\geq 0$, where density at $(x,\ y)$ is proportional to the distance between the point on the $x-axis$.

80. Find the mass, the center mass, and the moments of inertia of the triangular with vertices $(0,\ 0)$, $(0,\ 3)$, and $(2,\ 3)$, given that the density at $(x,\ y)$ is $\delta(x,\ y)=2x+y$

81. Find the mass, the center mass, and the moments of inertia of the triangular region with vertices $(0,\ 0)$, $(1,\ 1)$, and $(0,\ 5)$ with density $\delta(x,y)=x+y$.

82. Find the mass, the center mass, and the moments of inertia of the triangular region with vertices $(0,\ 0)$, $(0,\ 3)$, and $(6,\ 0)$ with density $\delta(x,y)=xy$.

83. Find the mass, the center mass, and the moments of inertia of the triangular region with vertices $(0,\ 0)$, $(2,\ 2)$, and $(2,\ 0)$ with density $\delta(x,y)=xy$.

84. Find the mass, the center mass, and the moments of inertia of the region bounded by $xy = 1$, $y = \frac{2}{x}$, $y = 1$, and $y = 2$, with density $\delta(x, y) = 4x + 4y$.

85. Find the mass, the center mass, and the moments of inertia of the tetrahedron about the $xyz - plane$ of the solid region that is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\delta(x, y, z) = x^2 y z$.

86. Find the mass, the center mass, and the moments of inertia of the disk of radius 2 centered at $(1, 2)$ with density $\delta(x, y) = x^2 + y^2 - 2x - 4y + 5$.

87. Find the mass and centroid of the solid bounded by the $xy - plane$, the cylinder $x^2 + y^2 = R^2$ and the plane $\frac{1}{R}x + \frac{1}{h}z = 1$.

88. Find the moments of inertia of the plate about three coordinates where the plate projects into $xy - plane$ that forms a parallelogram. The parallelogram is bounded by the lines:

$x + y = 0$, $x + y = 1$ and $3x - 2y = 0$, $3x - 2y = 2$.

89. Find the moments of inertia about three coordinates axes for the solid parallelepiped of constant density *k*.

(90 – 91) Find the mass and center of mass of the thin constant-density of the plate.

90.

91.

92. A thin rod of length *L* has a linear density given by $\delta(x) = 2e^{-x/3}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

93. A thin rod of length L has a linear density given by $\delta(x)=\dfrac{10}{1+x^2}$ on the interval $0\le x\le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L\to\infty$?

94. A thin plate is bounded by the graphs of $y=e^{-x}$, $y=-e^{-x}$, $x=0$, and $x=L$. Find its center of mass. How does the center of mass change as $L\to\infty$?

95. Consider the thin constant-density plate $\{(r,\ \theta):\ a\le r\le 1,\ 0\le\theta\le\pi\}$ bounded by two semicircles and the *x*-axis.

a) Find the graph the *y*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the plate?

96. Consider the thin constant-density plate $\left\{(\rho,\ \phi,\ \theta):\ 0<a\le\rho\le 1,\ 0\le\phi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\right\}$ bounded by two hemispheres and the *xy*-axis.

a) Find the graph the *z*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the solid?

97. A cylindrical soda can have a radius of 4 *cm* and a height of 12 *cm*. When the can is full of soda, the center of mass of the contents of the can is 6 *cm* above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 *cm* above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ g/cm^3$ and the density of air is $0.001\ g/cm^3$.

98. For $0\le r\le 1$, the solid bounded by the cone $z=4-4r$ and the solid bounded by the paraboloid $z=4-4r^2$ have the same base in the *xy*-plane and the same height. Which object has the greater mass if the density of both objects is $\delta(r,\ \theta,\ z)=10-2z$

99. For $0\le r\le 1$, the solid bounded by the cone $z=4-4r$ and the solid bounded by the paraboloid $z=4-4r^2$ have the same base in the *xy*-plane and the same height. Which object has the greater mass if the density of both objects is $\delta(r,\ \theta,\ z)=\dfrac{8}{\pi}e^{-z}$

100. A right circular cylinder with height 8 *cm* and radius 2 *cm* is filled with water. A heated filament running along its axis produces a variable density in the water given by $\delta(r)=1-0.05e^{-0.01r^2}$ g/cm^3 (ρ stands for density, not the radial spherical coordinate). Find the mass of the water in the cylinder. Neglect the volume of the filament.

101. A triangular region has a base that connects the vertices $(0,\ 0)$ and $(b,\ 0)$, and a third vertex at $(a,\ h)$, where $a > 0,\ b > 0$, and $h > 0$

a) Show that the centroid of the triangle is $\left(\frac{a+b}{3},\ \frac{h}{3}\right)$

b) Recall that the three medians of a triangle extend from each vertex to the midpoint of the opposite side. Knowing that the medians of a triangle intersect in a point *M* and that each median bisects the triangle, conclude that the centroid of the triangle is *M*.

102. Geographers measure the geographical center of a country (which is the centroid) and the population center of a country (which is the center of mass computed with the population density). A hypothetical country is shown below with the location and population of five towns.

Assuming no one lives outside the towns, find the geographical center of the country and the population center of the country.

103. A disk radius *r* is removed from a larger disk of radius *R* to from an earring. Assume the earing is a thin plate of uniform density.

a) Find the center of mass of the earring in terms of *r* and *R*. (Hint: Place the origin of a coordinate system either at the center of the larger disk or at *Q*; either way, the earring is symmetric about the *x*-axis.)

b) Show that the ratio $\frac{R}{r}$ such that the center of mass lies at the point *P* (on the edge of the inner disk) is the golden mean $\frac{1+\sqrt{5}}{2}$.

104. Find the moment of inertia and the radius of gyration of a solid ball of radius a, and constant density δ about a diameter of that ball.

With what linear acceleration will the ball roll (without slipping) down a plane inclined at angle α to the horizontal?

105. Find the mass and the centroid of the region, in the first octant, corresponding to $x^2 + y^2 + z^2 \le 4$, if the density is equl to xyz.

106. For the given solid bounded by $z = 8 - x^2 - y^2$ and $x^2 + y^2 = z$, with the mass density $\delta(x, y, z) = \sqrt{x^2 + y^2}$

a) The mass,
b) The centroid,
c) The moments of inertia.

107. The density, of a square plate of side ***a***, is numerically equal to the distance from the extremity of that side. Find

a) The mass,
b) The centroid,
c) The moments of inertia.

108. The density, of a cube of edge ***a***, is numerically equal to the square of the distance from one extremity of that edge. Find

a) The mass,
b) the centroid,
c) the moments of inertia,
d) And the radius of gyration about *all* axes.

109. For the region, in the first octant, bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Find

a) The volume,
b) the mass,
c) the centroid,
d) the moments of inertia,
e) And the radius of gyration about *all* axes.

110. You baked a nice cake and when the cake is almost done, someone tilts the baking pan in the oven with 60° angle (see the picture).

The pan's dimensions are given: $L = 12\ in, \quad w = 6\ in \quad d = 2\ in$.

Find the ***centroid*** (center of the pan's mass).

Hint:

Let the $x-$axis to be the horizontal axis and the pan length lay on it that axis.
Let $y-$axis to be the vertical axis and the pan length lay on it that axis.
Let $z-$axis to be toward you and parallel to the pan width (and center).

Section 11.6 – Surface Area

In this section, we will cover surfaces of functions of two variables, and instead the revolution what we learn so far. The area of these surfaces $z = f(x, y)$ can be calculated by applying double integrals.

11.6 - 1 Surface Area

Consider a surface S given by $z = f(x, y)$ defined over region R in $xy - plane$.

To determine the surface S, by projection S onto $xy - plane$, we can construct a parallelogram region R in $xy - plane$.

To find the approximation, we can partition the surface S in ΔS_{ij}, the partition projection is ΔR_{ij}.

$$S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta S_{ij}$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta R_{ij}$$

$$\vec{a} = \Delta x\ \hat{i} + f_x(x_i, y_j)\,\Delta x\ \hat{k}$$

$$\vec{b} = \Delta y\ \hat{j} + f_y(x_i, y_j)\,\Delta y\ \hat{k}$$

$$\Delta R_{ij} = \vec{a} \times \vec{b}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= -(f_x\ \Delta x\,\Delta y)\ \hat{i} - (f_y\ \Delta x\,\Delta y)\ \hat{j} + (\Delta x\,\Delta y)\ \hat{k}$$

$$= \left(-f_x\ \hat{i} - f_y\ \hat{j} + \hat{k}\right)\Delta x\,\Delta y \qquad \Delta A = \Delta x\,\Delta y$$

$$= \left(-f_x\ \hat{i} - f_y\ \hat{j} + \hat{k}\right)\Delta A$$

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{f_x^2 + f_y^2 + 1}\ \Delta A$$

11.6-2 *Definition*

If the function $f(x,y)$ is continuous, and its first partial derivative exits on the closed region in the $xy-plane$, then the surface area of $f(x,y)$ is given by:

$$Surface = \iint_R \sqrt{1+\left(f_x\right)^2+\left(f_y\right)^2}\ dA$$

Example 1

Find the surface area of the plane $x+y+z=2$ that lies above the circle $x^2+y^2=4$ in the first quadrant.

Solution

$$x+y+z=2$$

$$f(x,y) = z = 2-x-y$$

$$f_x = -1 \quad \& \quad f_y = -1$$

$$x^2+y^2=4$$

$$0 \le r \le 2$$

$$QI \quad \rightarrow \quad 0 \le \theta \le \frac{\pi}{2}$$

$$Surface = \iint_R \sqrt{1+(-1)^2+(-1)^2}\ dA$$

$$= \sqrt{3}\int_0^{\frac{\pi}{2}} d\theta \int_0^2 r\ dr$$

$$= \frac{\sqrt{3}}{4}\pi \left(r^2\right|_0^2$$

$$= \pi\sqrt{3} \quad unit^2$$

$$Surface = \iint_R \sqrt{1+\left(f_x\right)^2+\left(f_y\right)^2}\ dA$$

$$\iint dx\,dy = \iint r\ dr\,d\theta$$

Example 2

Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$.

Solution

The intersection of the paraboloid and the plane:

$$z = 9 - x^2 - y^2 = 5$$

$$x^2 + y^2 = 4$$

$$0 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

$$f(x, y) = 9 - x^2 - y^2$$

$$f_x = -2x \quad \& \quad f_y = -2y$$

$$Surface = \iint_R \sqrt{1 + 4x^2 + 4y^2}\ dA$$

$$Surface = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\ dA$$

$$= \int_0^{2\pi} d\theta \int_0^2 \sqrt{1 + 4r^2}\ r\ dr$$

$$\iint dx\,dy = \iint r\ dr\,d\theta$$

$$= 2\pi\left(\frac{1}{8}\right)\int_0^2 \left(1 + 4r^2\right)^{3/2}\ d\left(1 + 4r^2\right)$$

$$= \frac{\pi}{6}\left(1 + 4r^2\right)^{3/2}\Big|_0^2$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \quad unit^2$$

Example 3

Find the surface area of that portion of the surface $f(x, y) = 1 - x^2 + y$ that lies above the triangular region R in the $xy - plane$ with vertices $(1,\ 0)$, $(0,\ -1)$, and $(0,\ 1)$.

Solution

$$f(x, y) = 1 - x^2 + y$$

$$f_x = -2x \quad \& \quad f_y = 1$$

Line between $(1,\ 0)$ & $(0,\ -1)$:

$$\underline{y = x - 1}$$

Line between $(1,\ 0)$ & $(0,\ 1)$:

$$\underline{y = -x + 1}$$

$x - 1 \le y \le 1 - x$

$0 \le x \le 1$

$$Surface = \int_0^1 \int_{x-1}^{1-x} \sqrt{2 + 4x^2}\ dy\, dx$$

$$Surface = \iint_R \sqrt{1 + \left(f_x\right)^2 + \left(f_y\right)^2}\ dA$$

$$= \int_0^1 \sqrt{2 + 4x^2}\ \left(y\ \Big|_{x-1}^{1-x}\right) dx$$

$$= \int_0^1 \sqrt{2 + 4x^2}\ (2 - 2x)\ dx$$

$$= 2\int_0^1 \sqrt{2 + 4x^2}\ dx - \frac{1}{4}\int_0^1 \left(2 + 4x^2\right)^{1/2}\ d\left(2 + 4x^2\right)$$

$$2x = \sqrt{2}\ \tan\theta \qquad \sqrt{2 + 4x^2} = \sqrt{2}\ \sec\theta$$

$$dx = \frac{\sqrt{2}}{2}\sec^2\theta\ d\theta$$

$$= 2\int_0^1 \sec^3\theta\ d\theta - \frac{1}{6}\left(2 + 4x^2\right)^{3/2}\ \Big|_0^1 \qquad \int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\ \tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|$$

$$= \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\ \Big|_0^1 - \frac{1}{6}\left(6\sqrt{6} - 2\sqrt{2}\right)$$

$$= \left(x\sqrt{2 + 4x^2} + \ln\left|\frac{\sqrt{2 + 4x^2} + 2x}{\sqrt{2}}\right|\right)\Big|_0^1 - \sqrt{6} + \frac{1}{3}\sqrt{2}$$

$$= \sqrt{6} + \ln\left(\frac{\sqrt{6} + 2}{\sqrt{2}}\right) - \sqrt{6} + \frac{1}{3}\sqrt{2}$$

$$\underline{= \ln\left(\sqrt{6} + 2\right) - \frac{1}{2}\ln 2 + \frac{1}{3}\sqrt{2} \quad unit^2}$$

Example 4

Find the surface area of the plane $y+z=2$ inside the cylinder $x^2+z^2=1$.

Solution

$$y+z=2 \quad \Rightarrow \quad y=f(x,z)=2-z$$

$$f_x=0 \quad \& \quad f_z=-1$$

$$x^2+z^2=1$$

$$0\le r\le 1$$

$$0\le \theta\le 2\pi$$

$$Surface=\iint_R \sqrt{1+(0)^2+(-1)^2}\ dA$$

$$=\sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r\ dr$$

$$=\sqrt{2}\ (2\pi)\left(\frac{1}{2}r^2\right)\Big|_0^1$$

$$=\pi\sqrt{2}\quad unit^2$$

$$Surface=\iint_R \sqrt{1+\left(f_x\right)^2+\left(f_y\right)^2}\ dA$$

$$\iint dx\,dy=\iint r\ dr\,d\theta$$

Exercises *Section* 11.6 – Surface Area

1. Find the surface area of that portion of the surface $f(x, y) = 4x + y^2$ that lies above the triangular region R in the $xy - plane$ with vertices $(0,\ 0)$, $(0,\ 2)$, and $(2,\ 2)$.

2. Find the surface area of that portion of the surface $f(x, y) = 2x + y^2$ that lies above the triangular region R in the $xy - plane$ with vertices $(0,\ 0)$, $(0,\ 1)$, and $(1,\ 1)$.

3. Find the surface area of that portion of the surface $z = 1 - x^2 - y^2$ that lies above the triangular region R in the $xy - plane$ with vertices $(0,\ 0)$, $(1,\ 0)$, and $(1,\ 1)$.

4. Find the surface area of that portion of the surface $z = 9 - x^2 - y^2$ that lies above the triangular region R in the $xy - plane$ with vertices $(0,\ 0,\ 0)$, $(0,\ 1,\ 0)$, and $(1,\ 1,\ 0)$.

5. Find the surface area of that portion of the surface $z = \frac{1}{2}x^2 + y$ that lies above the triangular region R in the $xy - plane$ with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, and $(1,\ 1,\ 0)$.

6. Find the surface area of the portion of the paraboloid $z = 1 + x^2 + y^2$ that lies below the plane $z = 5$.

7. Find the surface area of the portion of the paraboloid $z = \sqrt{x^2 + y^2}$ that lies below the plane $z = 2$.

8. Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$.

9. Find the surface area of the portion of the surface $f(x, y) = 1 + x^2 + y^2$ that lies above the unit circle.

10. Find the surface area of the portion of the surface $f(x, y) = \sqrt{49 - x^2 - y^2}$ that lies above the circle $x^2 + y^2 = 9$.

11. Find the surface area of the portion of the surface $f(x, y) = 16 - x^2 - y^2$ that lies above the circle $x^2 + y^2 = 9$.

12. Find the surface area of the portion of the surface $f(x, y) = 9 - x^2 - y^2$ that lies above the $xy - plane$.

13. Find the surface area of the portion of the sphere surface $x^2 + y^2 + z^2 = 9$ that lies above the plane $z = 2$.

14. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 8$ that lies inside the cone $z^2 = x^2 + y^2$.

15. Find the surface area of the portion of the sphere surface $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 - ax + y^2 = 0$.

16. Find the surface area of the sphere with a radius a, whose top hemisphere has:

$$f(x, y) = \sqrt{a^2 - x^2 - y^2}\ .$$

17. Find the surface area of the cone with a base radius a, and height h is:

$$f(x, y) = h - \frac{h}{a}\sqrt{x^2 + y^2}$$

18. Find the surface area of the upper half of the cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 - 2x + y^2 = 0$

19. Find the surface area of the upper half of the cone $z^2 = x^2 + y^2$ above the interior of one loop of $r = \cos 2\theta$

20. Find the surface area of the upper hemisphere of $x^2 + y^2 + z^2 = 1$ above the interior of one loop of $r = \cos 2\theta$

21. Find the surface area of the portion of the cylinder $x^2 + z^2 = 1$ that lies within the cylinder $y^2 + z^2 = 1$.

22. Find the surface area of the portion of the surface $x = yz$ that lies inside the cylinder $y^2 + z^2 = 16$.

23. Find the surface area of the portion of the surface $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

24. Find the surface area of the portion of the surface $z = mx$ that lies inside the cylinder $x^2 + y^2 = a^2$

25. Find the surface area of the portion of the surface $f(x, y) = 9 - x^2 - y^2$ that lies above the region $R = \{ (x, y) \mid -1 \le x \le 1,\ -1 \le y \le 1 \}$.

26. Find the surface area of the portion of the surface $f(x, y) = \ln|\sec x|$ that lies above the region $R = \left\{ (x, y) \mid 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \tan x \right\}$.

27. Find the surface area of the portion of the surface $f(x, y) = 8 + x^2 - y^2$ that lies above the region $R = \left\{ (x, y) \mid x^2 + y^2 = 4 \right\}$.

28. Find the surface area of the portion of the surface $f(x, y) = \sqrt{x^2 + y^2}$ that lies above the region $R = \left\{ (x, y) \mid 0 \le \sqrt{x^2 + y^2} \le 1 \right\}$.

29. Find the surface area of the portion of the surface $f(x, y) = x^2 + y^2$ that lies above the region:

$$R = \left\{ (x, y) \mid 0 \le x^2 + y^2 \le 25 \right\}$$

30. Find the surface area of the portion of the surface $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ that lies above the region $R = \left\{ (x, y) \mid x^2 + y^2 \le b^2,\ b < a \right\}$.

31. Find the surface area of the portion of the surface $f(x, y) = 4 - x^2 - y^2$ that lies above the region $R = \left\{ (x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1 \right\}$.

32. Find the surface area of the portion of the surface $f(x, y) = x^2 + y^2$ that lies above the region $R = \left\{ (x, y) \mid 0 \le x^2 + y^2 \le 4 \right\}$.

33. Find the surface area of that portion of the surface $f(x, y) = 9 - x^2$ over the region *R*:
R: *square with vertices* $(0, 0)$, $(3, 0)$, $(0, 3)$, *and* $(3, 3)$.

34. Find the surface area of that portion of the surface $f(x, y) = y^2$ over the region *R*:
R: *square with vertices* $(0, 0)$, $(3, 0)$, $(0, 3)$, *and* $(3, 3)$.

35. Find the surface area of the portion of the surface $f(x, y) = 4 - x - 2y$ bounded by the region:

$$R\text{: } x = 0,\ y = 0, \text{ and } y = 2 - \frac{1}{2}x$$

36. Find the surface area of the portion of the surface $f(x, y) = x^2 + 2y$ between $y = x$, $y = 0$, and $x = 4$.

37. Find the surface area of the portion of the surface $f(x, y) = \sqrt{x^2 + y^2}$ between $y = x^2$, $y = 4$
.

38. Find the surface area of the portion of the surface $x+3y+z=6$ in the first octant.

39. Find the surface area of the portion of the surface $2x+y+z=8$ in the first octant.

40. Find the surface area of the portion of the surface $x+y+z=1$ in the first octant.

41. Find the surface area of the triangle that forms from the intersection of the plane $ax+by+cz=d$ and the first octant.

42. Find the surface area of the portion of the surface $3x+2y+z=6$ in the first octant.

43. Find the surface area S of the part of the sphere $x^2+y^2+z^2=36$ inside the cylinder $x^2+y^2=6y$, and above the $xy-plane$.

44. Find the surface area of the part of the surface $f(x,y)=x^2+2y$ that lies above the triangular region T in the $xy-plane$ with vertices $(0,0)$, $(1,0)$, *and* $(1,1)$.

45. Find the surface area of the portion of the paraboloid $f(x,y)=x^2+y^2$ that lies below the plane $z=9$.

46. Find the surface area of the portion of the surface $f(x,y)=\frac{2}{3}x^{3/2}+\cos x$ that lies above the region:

$$R=\{(x,y)\mid 0\le x\le 1,\quad 0\le y\le 1\}.$$

47. Find the surface area of the portion of the surface $f(x,y)=x^2-3xy-y^2$ that lies above the region:

$$R=\{(x,y)\mid 0\le x\le 4,\quad 0\le y\le x\}.$$

Section 11.7 – Change of Variables in Multiple Integrals

11.7-1 Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form:

$$x = g(u, v), \quad y = h(u, v)$$

Cartesian *uv*-plane ***Cartesian*** *xy* - plane

R is the image of G under the transformation, and G the ***preimage*** of R.

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

The ***Jacobian determinant*** is used when making a change of variables when evaluating a multiple integral of a function over a region within its domain. To accommodate for the change of coordinates the magnitude of the Jacobian determinant arises as a multiplicative factor within the integral.

11.7-2 *Definition*

The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u, v), \quad y = h(u, v)$ is:

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example 1

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian integral $\iint\limits_R f(x,y)\,dx\,dy$ as a polar integral.

Solution

Cartesian $r\theta$ –plane

Cartesian xy –plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1,\ 0 \le \theta \le 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI.

$$J(r,\ \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \qquad J(r,\ \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$= r$$

Example 2

Evaluate $\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\,dxdy$ by applying the transformation $u = \frac{2x-y}{2}, \quad v = \frac{y}{2}$ and integrating over an appropriate region in the uv-plane.

Solution

$$y = 2v$$

$$2u = 2x - y$$

$$x = \frac{2u+y}{2}$$
$$= \frac{2u+2v}{2}$$
$$= u+v$$

xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv-eqns*.**
$x = \frac{y}{2}$	$u+v = \frac{2v}{2} = v$	$u = 0$
$x = \frac{y}{2}+1$	$u+v = \frac{2v}{2}+1 = v+1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$J(u,v) = \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix} \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 2$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\, dx\, dy = \int_0^{v=2} \int_{u=0}^{u=1} u\, |J(u,v)|\, du\, dv$$

$$= \int_0^2 dv \int_0^1 (u)(2)\, du$$

$$= v \Big|_0^2 \quad u^2 \Big|_0^1$$

$$= (2)(1)$$

$$= 2$$

Example 3

Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

Solution

$$\begin{cases} u = x + y \\ v = y - 2x \end{cases}$$

$$\begin{cases} x = \dfrac{u}{3} - \dfrac{v}{3} \\ y = \dfrac{2u}{3} + \dfrac{v}{3} \end{cases}$$

xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 1$	$v = -2u$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$x = 1$	$u = 3 + v$	$y = 2 + v\big\vert_{v=0} = 2 > 1$

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{1}{3}$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2}\,v^2\,\left|J(u,v)\right|\,dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2}\,v^2\left(\frac{1}{3}\right)dv\,du$$

$$= \int_0^1 u^{1/2}\left(\frac{1}{9}v^3\right|_{-2u}^{u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}\left(u^3+8u^3\right)du$$

$$= \int_0^1 u^{7/2}\,du$$

$$= \frac{2}{9}\,u^{9/2}\Big|_0^1$$

$$= \frac{2}{9}$$

Example 4

Evaluate $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy$

Solution

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\dfrac{y}{x}} \end{cases}$$

$$\begin{cases} u^2 = xy \\ v^2 = \dfrac{y}{x} \end{cases}$$

$$x = \frac{u}{v}, \quad y = uv$$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{2u}{v}$$

xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x = y$	$\frac{u}{v} = uv$	$v = 1$
$x = \frac{1}{y}$	$\frac{u}{v} = \frac{1}{uv}$	$u = 1$
$y = 1$	$uv = 1$	
$y = 2$	$uv = 2$	$u = 2 \quad v = \frac{2}{u}$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2u\, e^u\, dv\, du$$

$$= 2\int_1^2 u\, e^u\, v \Big|_0^{2/u} du$$

$$= 2\int_1^2 u\, e^u \left(\frac{2}{u} - 1\right) du$$

$$= 2\int_1^2 (2-u)\, e^u\, du$$

$$= 2\left((2-u+1)e^u\right)\Big|_1^2$$

$$= 2\left((1)e^2 - 2e\right)$$

$$= 2e\,(e-2)$$

		e^u
$(+)$	$2-u$	e^u
$(-)$	-1	e^u
	0	

11.7-3 Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

Cartesian *uvw* - plane

Cartesian *xyz* - plane

$$\iiint_R f(x,y,z)\, dx\, dy\, dz = \iiint_R H(u,v,w)\, \left| J(u,v,w) \right|\, du\, dv\, dw$$

The ***Jacobian determinant*** is

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cube with sides parallel to the axes

Cube with sides parallel to the axes

Cartesian $r\theta z$ - plane

Cartesian *xyz* - plane

$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\underline{= r}$$

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz –space is given by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta) = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \qquad J(\rho,\phi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \rho^2 \cos^2\phi \sin\phi \cos^2\theta + \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin\phi \cos^2\phi \sin^2\theta + \rho^2 \sin^3\phi \cos^2\theta$$

$$= \rho^2 \cos^2\phi \sin\phi \left(\cos^2\theta + \sin^2\theta\right) + \rho^2 \sin^3\phi \left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2 \cos^2\phi \sin\phi + \rho^2 \sin^3\phi$$

$$= \rho^2 \sin\phi \left(\cos^2\phi + \sin^2\phi\right)$$

$$= \rho^2 \sin\phi$$

$$\iiint_D F(x,y,z)\, dx\, dy\, dz = \iiint_G H(\rho,\phi,\theta) \left|\rho^2 \sin\phi\right| d\rho\, d\phi\, d\theta$$

Cartesian $r\theta z$ - plane

Cartesian xyz - plane

Example 5

Evaluate $\displaystyle\int_0^3 \int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \left(\frac{2x-y}{2}+\frac{z}{3}\right) dx\,dy\,dz$ by applying the transformation

$u=\dfrac{2x-y}{2}$, $v=\dfrac{y}{2}$, $w=\dfrac{z}{3}$ and integrating over an appropriate region in the *uvw*-plane.

Solution

$$\begin{cases} u=\dfrac{2x-y}{2} & \rightarrow \quad x=u+\dfrac{y}{2}=u+v \\ v=\dfrac{y}{2} & \rightarrow \quad y=2v \\ w=\dfrac{z}{3} & \rightarrow \quad z=3w \end{cases}$$

xyz-eqns* for the boundary of *D	**Corresponding *uvw- eqns*. for the boundary of *G***	**Simplified *uvw- eqns*.**
$x=\dfrac{y}{2}$	$u+v=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 6$

$$\int_0^3 \int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dxdydz = \int_0^1 \int_0^2 \int_0^1 (u+w)\ \left|J(u,v,w)\right|\ du\, dv\, dw$$

$$= 6 \int_0^1 \int_0^2 \int_0^1 (u+w)\ du\, dv\, dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{u^2}{2} + wu\right|_0^1\ dv\, dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w\right)\ dvdw$$

$$= 6 \int_0^1 \left(\frac{1}{2}v + wv\right|_0^2\ dw$$

$$= 6 \int_0^1 (1+2w)\ dw$$

$$= 6\left(w + w^2\right|_0^1$$

$$= 6(1+1)$$

$= 12$

Example 6

Evaluate $\iiint_D xz\ dV$: D is bounded by the planes: $y - x = 0,\ \ y = 2 + x,\ \ z - y = 0,\ \ z - y = 2,$

$z = 0,$ and $z = 3$

Solution

$$\begin{cases} y - x = 0 \\ y - x = 2 \end{cases}$$

$let\quad u = y - x$

$\Rightarrow\ 0 \le u \le 2$

$$\begin{cases} z-y=0 \\ z-y=2 \end{cases}$$

$let \quad \underline{v=z-y}$

$\Rightarrow \quad \underline{0 \le v \le 2}$

$$\begin{cases} z=0 \\ z=3 \end{cases}$$

$let \quad \underline{w=z}$

$\underline{0 \le w \le 3}$

$$\begin{cases} \underline{z=w} \\ y-x=u \qquad \rightarrow \qquad \underline{x=-u-v+w} \\ z-y=v \quad \rightarrow \quad \underline{y=w-v} \end{cases}$$

$$J(u,\ v,\ w)=\begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \qquad J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\underline{=1}$$

$$\iiint_D xz\ dV = \int_0^3 \int_0^2 \int_0^2 (w-u-v)\ (w)\ du\, dv\, dw$$

$$= \int_0^3 \int_0^2 \int_0^2 \left(w^2 - uw - vw\right) du\, dv\, dw$$

$$= \int_0^3 \int_0^2 \left(w^2 u - \frac{1}{2} w u^2 - vwu \right|_0^2 dv\, dw$$

$$= \int_0^3 \int_0^2 \left(2w^2 - 2w - 2vw\right) dv\, dw$$

$$= \int_0^3 \left(2w^2 v - 2wv - wv^2 \right|_0^2 dw$$

$$= \int_0^3 \left(4w^2 - 4w - 4w\right) dw$$

$$= \int_0^3 \left(4w^2 - 8w\right) dw$$

$$= \frac{4}{3}w^3 - 4w^2 \Big|_0^3$$

$$= 36 - 36$$

$$= 0$$

Exercises *Section* 11.7 – Change of variables in Multiple Integrals

$(\mathbf{1-4})$ Let $S=\{0\le u\le 1,\ 0\le v\le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations.

1. $T:\quad x=v,\ y=u$

2. $T:\quad x=-v,\ y=u$

3. $T:\quad x=\dfrac{u+v}{2},\ \ y=\dfrac{u-v}{2}$

4. $T:\quad x=u,\ \ y=2v+2$

5. *a*) Solve the system $u=x-y,\quad v=2x+y$ for *x* and *y* in terms of *u* and *v*. Then find the value of

the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u=x-y,\ \ v=2x+y$ of the triangular region with vertices (0, 0), (1, 1), and (1, –2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

6. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy=1,\ xy=9$ and the lines $y=x,\ y=4x$. Use the transformation $x=\dfrac{u}{v},\ y=uv$ with $u>0,\ and\ v>0$ to rewrite:

$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dx\,dy$$

As an integral over an appropriate region *G* in the uv-plane.
Then evaluate the *uv*-integral over *G*.

7. The area πab of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ can be found by integrating the function $f(x,y)=1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x=au,\ y=bv$ and evaluate the transformed integral over the disk *G*: $u^2+v^2\le 1$ in the *uv*-plane. Find the area this way.

8. Use the transformation $x=u+\dfrac{1}{2}v,\ \ y=v$ to evaluate the integral

$$\int_0^2\int_{y/2}^{(y+4)/2} y^3\,(2x-y)\,e^{(2x-y)^2}\,dx\,dy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

9. Use the transformation $x = \frac{u}{v},\ y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dx\, dy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dx\, dy$$

10. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation:

a) $x = u\cos v,\quad y = u\sin v$

b) $x = u\sin v,\quad y = u\cos v$

11. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation:

a) $x = u\cos v,\quad y = u\sin v,\quad z = w$

b) $x = 2u - 1,\quad y = 3v - 4,\quad z = \frac{1}{2}(w - 4)$

12. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian xyz–space is $\rho^2 \sin\phi$

13. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

14. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in uvw-space)

15. Use the transformation $x = u^2 - v^2,\ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx$$

(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration ***R*** in the xy-plane defined by the limits of integration.)

16. Evaluate $\iint_R y^4\, dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $\frac{y}{x} = 1$, and $\frac{y}{x} = 3$

17. Evaluate $\iint\limits_R \left(y^2 + xy - 2x^2\right) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$, and $y = -2x - 3$

18. Evaluate $\iint\limits_D x\, dV$; R is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$ and $z - 4x = 3$

19. Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$

Evaluate the integral $\iint\limits_R 3xy\, dA$

20. Let R be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\iint\limits_R (x+y)^2 \sin^2(x-y)\, dA$

21. Evaluate $\iiint\limits_D yz\, dV$ D is bounded by the planes: $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$, $2y - z = 0$, and $2y - z = 3$

22. Evaluate $\iint\limits_R xy\, dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

23. Evaluate $\iint\limits_R x^2 y\, dA$; $R = \{(x,\ y):\ \ 0 \le x \le 2,\ \ x \le y \le x + 4\}$

24. Evaluate $\iint\limits_R x^2 \sqrt{x + 2y}\, dA$; $R = \left\{(x,\ y):\ \ 0 \le x \le 2,\ \ -\frac{x}{2} \le y \le 1 - x\right\}$

25. Evaluate $\iint\limits_R xy\, dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.

26. Evaluate $\int_0^1 \int_y^{y+2} \sqrt{x-y}\, dx\, dy$

27. Evaluate $\iint_R \sqrt{y^2 - x^2}\, dA$; where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

28. Evaluate $\iint_R \left(\frac{y-x}{y+2x+1}\right)^4 dA$; where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$

29. Evaluate $\iint_R e^{xy}\, dA$; where R is the region bounded by $xy = 1$, $xy = 4$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 3$

30. Evaluate $\iint_R xy\, dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$

31. Evaluate $\iint_R (x-y)\sqrt{x-2y}\, dA$; where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and $x - y = 1$

32. Evaluate $\iint_R e^{3x-2y}\, dx\, dy$; where R is the parallelogram spanned by the vectors: $\langle 1, 3 \rangle$ and $\langle 5, 1 \rangle$

33. Evaluate $\iint_R e^{4x-y}\, dx\, dy$; where R is the parallelogram spanned by the vectors: $\langle 4, 1 \rangle$ and $\langle 3, 3 \rangle$

34. Evaluate $\iint\limits_R xy\ dx\,dy$; where R is the parallelogram spanned by the vectors: $\langle 5,\ 1\rangle$ and $\langle 3,\ 4\rangle$

35. Evaluate $\iint\limits_R xy\left(x^2+y^2\right)\ dx\,dy$; where $R=\left\{\left(x,\ y\right)|\ \ -3\le x^2-y^2\le 3,\ \ 1\le xy\le 4\right\}$

36. Evaluate $\iint\limits_R e^{x+y}\ dx\,dy$; where $R=\left\{\left(x,\ y\right)|\ 1\le x+y\le 4,\ \ -4\le y-2x\le 1\right\}$

37. Evaluate $\iint\limits_R \left(x+y\right)^2\ dx\,dy$; where R bounded by the lines $x+y=0$, $x+y=1$, $2x-y=0$, and $2x-y=3$

38. Evaluate $\iint\limits_R xy\ dx\,dy$; where R bounded by $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$, and $x^2-y^2=4$

39. Evaluate $\iint\limits_R \left(x+y\right)\ \cos\left(\pi\left(x-y\right)\right)\ dx\,dy$; where R bounded by the lines $x+y=0$, $x+y=1$, $x-y=0$,

40. Evaluate $\iint\limits_R \sin\left(x-y\right)\ \cos\left(x+2y\right)\ dx\,dy$; where R bounded by $x-y=0$, $x-y=\pi$, $x+2y=0$, and $x+2y=\frac{\pi}{2}$.

41. Evaluate $\iint\limits_R \sin 3x\ dx\,dy$; where R bounded by $x-y=0$, $x-y=\pi$, $x+2y=0$, and $x+2y=\frac{\pi}{2}$.

42. Evaluate $\iint\limits_R \left(x^2+2xy\right)\ dA \qquad R:\ \left\{1\le y-2x\le 3\ \ \&\ \ 2\le x+y\le 5\right\}$

43. Evaluate $\iint\limits_R \frac{e^{x-y}}{x+y}\, dA$ R: $\{y = x,\quad y = x+5,\quad x+y=2,\quad and\quad x+y=4\}$

44. Evaluate $\iint\limits_R \frac{e^{y-4x}}{2x+y}\, dA$ R: $\{y-4x=2,\quad y-4x=5,\quad 2x+y=1,\quad and\quad 2x+y=3\}$

45. Evaluate $\iint\limits_R (y-4x)\, dA$ R: $\{y-4x=2,\quad y-4x=5,\quad 2x+y=1,\quad and\quad 2x+y=3\}$

46. Evaluate $\iint\limits_R (y+3x)^2\, dA$ R: $\{3x+y=1,\quad 3x+y=3,\quad x-y=1,\quad and\quad x-y=3\}$

47. Evaluate the integral $\int_0^1 \int_1^2 \int_z^{z+1} (y+1)\, dx\, dy\, dz$ by applying the transformation $u = x - z$, $v = 3y$, and $w = \frac{1}{2}z$

48. Evaluate the triple integral $\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5-4y)\, dx\, dz\, dy$ by applying the transformation $u = x - 3z$, $v = 4y$, and $w = z$.

49. Evaluate the integral $\iiint\limits_D \left(x^2 y + 3xyz\right) dV$; D is bounded by: $1 \le x \le 2$, $0 \le xy \le 2$, and $0 \le z \le 1$. By applying the transformation $u = x$, $v = xy$, and $w = 3z$.

50. Evaluate $\iiint_D xy\, dV$: D is bounded by the planes: $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 2$, $z = 0$, and $z = 3$.

51. Evaluate $\iiint_D dV$: D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$.

52. Evaluate: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

53. Evaluate $\iiint_D dV \iiint_D z\, dV$: D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy-plane.

54. Evaluate $\iiint_D xz\, dV$: D is bounded by the planes: $y = x,\ \ y = x + 2,\ \ x - z = 0,\ \ z = x + 3,$ $z = 0,$ and $z = 4$

$(\mathbf{55-56})$ Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

55. Find the area of R.

56. Evaluates $\iint_R |xy|\, dA$

57. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

58. Find the average square of the distance between points of R and the origin.

59. Find the average distance between points in the upper half of R and the x-axis.

$(\mathbf{60-63})$ Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers.

60. Find the Volume of D.

61. Evaluates $\iint_D |xyz|\, dV$

62. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

63. Find the average square of the distance between points of D and the origin.

$(\mathbf{64-67})$ Evaluate the given integral using:

- Change of variable.
- Any other method.

64. $\iint\limits_R \left(x^2+y^2\right)\,dx\,dy$; where $R=\left\{(x,\,y)\mid\ 1\le xy\le 4,\ \ 1\le \frac{y}{x}\le 4\right\}$

65. $\iint\limits_R \left(x^2+y^2\right)\,dx\,dy$; where R bounded by the lines $x+y=0,\ x+y=1,\ x-y=0$, and $x-y=2$

66. $\iint\limits_R \left(x^2-y^2\right)\,dx\,dy$; where R bounded by the lines $x+y=0,\ x+y=1,\ x-y=0$, and $x-y=2$

67. $\iint\limits_R (x+3y)\,dx\,dy$; where R bounded by the region $x+2y=6,\ x+2y=10$, and $3\le y\le 6$

$(\mathbf{68-81})$ Use the change of variables to evaluate the integral over the given shaded boundary.

68. $\iint\limits_R (x-y)\,e^{x^2-y^2}\,dA$

69. $\iint\limits_R (x-y)\,e^{x+y}\,dA$

70. $\iint\limits_R xy\,dA$

71. $\iint\limits_R (x-y)^2\,e^{x+y}\,dx\,dy$

72. $\displaystyle\iint_R (x+y)^3\, dx\, dy$

73. $\displaystyle\iint_R (x+y)\, dx\, dy$

74. $\displaystyle\iint_R \left(x^2+y^2\right) dA$

75. $\displaystyle\iint_R xy\, dA$

76. $\displaystyle\iint_R (x+y)^2\, dA$

77. $\displaystyle\iint_R e^{x+y}\, dA$

78. $\displaystyle\iint_R x\,(2x+y)\, dA$

79. $\displaystyle\iint_R 4\,(x+y)\, e^{x-y}\, dA$

80. $\displaystyle\iint_R y\,(y-x)\, dA$

81. $\displaystyle\iint_R (x+y)\, e^{x-y}\, dA$

$(\mathbf{82-85})$ Use the change of variables to find the ***area*** over the given shaded boundary.

82.

83.

84.

85.

$(\mathbf{86-87})$ Use the change of variables to find the ***volume*** over the given boundary D:

86. $D: \left\{ \begin{matrix} x+y+z=1 & x+2y=0 & y+z=2 \\ x+y+z=2 & x+2y=1 & y+z=4 \end{matrix} \right\}$

87. $D: \left\{ \begin{matrix} x+z=1 & 2y+3z=0 & y+2z=2 \\ x+z=2 & 2y+3z=1 & y+2z=4 \end{matrix} \right\}$

Summary

Volume: $V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

$$V = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

Average values of f over $\boldsymbol{R} = \dfrac{1}{area\ of\ R} \iint_R f\ dA$

Average value of $\boldsymbol{F}$ over $\boldsymbol{D} = \dfrac{1}{\text{volume of } D} \iiint_D F\ dV$

Area: $A = \iint_R r\ dr\ d\theta$

Cartesian Integrals into Polar:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\ r\ dr\,d\theta$$

$r^2 = x^2 + y^2, \quad \tan\theta = \dfrac{y}{x}$

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$

Coordinate Conversion Formulas

Cylindrical to Rectangular	*Spherical to Rectangular*	*Spherical to Cylindrical*
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r dr\,d\theta \\ &= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

$$\iiint_D f(\rho,\ \phi,\ \theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\ \phi,\ \theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\iiint_R f(x,y)\,dx\,dy = \iiint_R H(u,v,w)\ \left|J(u,v,w)\right|\,du\,dv\,dw$$

Jacobian: $$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Mass: $m = \int_C \delta\,ds$ $\quad \delta = \delta(x,y,z)$ *is the* density at (x,y,z)

First moments of the coordinate's planes:

$$M_{yz} = \int_C x\,\delta\,ds, \qquad M_{xz} = \int_C y\,\delta\,ds, \qquad M_{xy} = \int_C z\,\delta\,ds$$

Coordinates of the center of mass: $\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}$

Moments of inertia about axes and other lines:

$$I_x = \iint_R \delta\,y^2\,ds, \quad I_y = \iint_R \delta\,x^2\,ds, \qquad I = I_x + I_y$$

$$I_x = \int_C \left(y^2+z^2\right)\delta\,ds, \quad I_y = \int_C \left(x^2+z^2\right)\delta\,ds, \quad I_z = \int_C \left(x^2+y^2\right)\delta\,ds$$

$$I_L = \int_C r^2 \delta \, ds$$ $r(x, y, z) =$ distance from the point (x, y, z) to the line L

$$m = \iint_R \delta(x, y) \, dA$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x \, \delta(x, y) \, dA$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y \, \delta(x, y) \, dA$$

$$m = \iint_R \delta(r, \theta) \, r \, dr \, d\theta$$

$$\bar{x} = \frac{1}{m} \iint_R r \cos\theta \, \delta(r, \theta) \, r \, dr \, d\theta$$

and

$$\bar{y} = \frac{1}{m} \iint_R r \sin\theta \, \delta(r, \theta) \, r \, dr \, d\theta$$

$$m = \iiint_D \delta(x, y, z) \, dV$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x \, \delta(x, y, z) \, dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y \, \delta(x, y, z) \, dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z \, \delta(x, y, z) \, dV$$

The ***radius of gyration*** $\bar{\bar{r}}$: $\bar{\bar{r}} = \sqrt{\frac{I}{m}}$ $\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$ & $\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$

Exercises *Lecture* 11 – Review

(1 – 20) Evaluate the integral.

1. $\displaystyle\int_0^1 \int_0^x \left(xy + y^2\right) dy\, dx$

2. $\displaystyle\int_0^1 \int_0^y \left(xy + y^2\right) dx\, dy$

3. $\displaystyle\int_0^{\pi} \int_{-x}^{x} \cos y\ dy\, dx$

4. $\displaystyle\int_0^2 \int_0^y y^2 e^{xy}\, dx\, dy$

5. $\displaystyle\int_0^{\pi} \int_0^{\pi/2} \left(\sin x + \cos y\right)\ dy\, dx$

6. $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} y^2\ dy\, dx$

7. $\displaystyle\int_0^1 \int_0^{1-x^2} x\ \cos y\ dy\, dx$

8. $\displaystyle\int_{1/2}^2 \int_{1/x}^{\frac{5-2x}{2}} \ln x\ dy\, dx$

9. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\ dx\, dy$

10. $\displaystyle\int_0^2 \int_0^{4-y^2} y\ dx\, dy$

11. $\displaystyle\int_1^2 \int_0^{\ln x} x^3 e^y\, dy\, dx$

12. $\displaystyle\int_{-\pi}^{\pi} \int_0^2 r \sin\theta\ dr\, d\theta$

13. $\displaystyle\int_0^{\pi} \int_0^{\cos\theta} r \sin\theta\ dr\, d\theta$

14. $\displaystyle\int_0^{\pi/2} \int_2^{4\ \cos\theta} r^3\ dr\, d\theta$

15. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2 + y^2\right) dy\, dx$

16. $\displaystyle\int_0^1 \int_0^z \int_0^y \left(x + y + z\right) dx\, dy\, dz$

17. $\displaystyle\int_1^{\ln 8} \int_0^{\ln 4} \int_0^{\ln 2} e^{-x-y-2z}\ dx\, dy\, dz$

18. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x\ \cos y\ \sin 2z\ dy\, dx\, dz$

19. $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} e^{-\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$

20. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} e^{-\left(x^2+y^2+z^2\right)^{3/2}}\, dz\, dy\, dx$

$(\mathbf{21-27})$ Evaluate the double integral over the given region R.

21. $\displaystyle\iint_R \left(3x^2-y\right) dA \qquad R:\ [0,\ 2]\times[0,\ 3]$

22. $\displaystyle\iint_R x\, dA \qquad R:$ region bounded by $y=x^2$ *and* $y=x^3$

23. $\displaystyle\iint_R y\, dA \qquad R:$ region bounded by $y=x^2$ *and* $y=x^3$

24. $\displaystyle\iint_R (3-x-y)\, dA \qquad R:$ is triangle in $xy-$plane bounded by the $x-axis$ and the line $y=x$ *and* $x=1$

25. $\displaystyle\iint_R e^{-\left(x^2+y^2\right)} dA \qquad R:$ bounded between the two circles: $x^2+y^2=1$ & $x^2+y^2=4$

26. $\displaystyle\iint_R (x+y)\, dA \qquad R=\left\{(x,y)\ |\ 1\le x^2+y^2\le 4,\ x\le 0\right\}$

27. $\displaystyle\iint_R (x-y)\, dA \qquad R$: is the region bounded left by $y-axis$ and bounded by $y=\sqrt{3x}$ *and* $y=\sqrt{4-x}$

28. Find the volume of the solid bounded under the plane $f(x,y)=8x+6y$ over the rectangle $R=\left\{(x,y)\ |\ 0\le x\le 1,\ 0\le y\le 2\right\}$

29. Find the volume of the solid bounded between $f(x,y)=x+y$ and $f(x,y)=x-y$ over the region $R=\left\{(x,y)\ |\ y^2\le x\le y,\ 0\le y\le 1\right\}$

30. Find the volume of the solid bounded under the surface $f(x,y)=e^{x+y}$ over the rectangle $R=\{(x,y) \mid 2\le x\le 3,\ 1\le y\le 2\}$

31. Find the volume of the solid bounded by the plane $f(x,y)=4-x-y$ and below by the rectangle $R=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le 2\}$.

32. Find the volume of the solid of a tetrahedron $x+y+z=1$

33. Find the volume bounded above by the surface $z=1-xy$ and below by lines $y=x$ and $y=1$

34. Find the mass and the center mass of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

35. Find the mass, the center mass, and the moments of inertia of the disk of radius 2 centered at $(1,\ 2)$ with density $\delta(x,y)=x^2+y^2-2x-4y+5$.

36. Find the surface area of the portion of the surface $f(x,y)=3x^2+4y$ between $y=2x$, $y=0$, and $x=2$.

37. Find the surface area of the hemisphere has $f(x,y)=\sqrt{1-x^2-y^2}$.

38. Find the surface area of the portion of the surface $f(x,y)=x^2-3y+3$ over the region: $R=\{(x,y) \mid -x\le y\le x,\ 0\le x\le 4\}$.

39. Find the surface area of the portion of the hyperbolic paraboloid $y=x^2-z^2$ that lies in the first octant and inside the cylinder $x^2+z^2=9$.

40. Find the surface area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above $xy - plane$ and inside the cylinder $x^2 + y^2 - 4y = 0$.

41. Find the surface area of the portion of the surface $x - y - 2z = 4$ with $x \geq 0$, $y \leq 0$, and $z \leq 0$.

$(\mathbf{42} - \mathbf{43})$ Use the change of variables to evaluate the integral over the given shaded boundary.

42. $\iint\limits_R (x+y)^2 \sin(x-y)\, dA$

43. $\iint\limits_R \sqrt{(x+4y)(x-y)}\, dA$

Practice *Exam*

1. Evaluate the integral $\displaystyle\int_1^2 \int_0^{\ln x} x^3 e^y \, dy \, dx$

2. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$

3. Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r,\ \theta):\ 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$.

4. Evaluate the integral by ***changing to polar*** *coordinates* $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2 + y^2 + 1\right) dx \, dy$

5. Evaluate the integral $\displaystyle\int_1^{\ln 8} \int_0^{\ln 4} \int_0^{\ln 2} e^{-x-y-2z} \, dx \, dy \, dz$

6. Evaluate the integrals in ***cylindrical*** coordinates $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^3 \left(x^2 + y^2\right)^{3/2} dz \, dy \, dx$

7. Evaluate the ***spherical*** coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{4\cos\phi} \rho^2 \sin\phi \ \ d\rho \, d\phi \, d\theta$

8. Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$

Evaluate the integral $\displaystyle\iint_R 3xy \ dA$

9. Find the ***mass*** and ***center of mass*** of the thin rods with the following density functions.

$$\delta(x) = 2 - \frac{x^2}{16} \quad \textit{for} \quad 0 \le x \le 4$$

10. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$

Lecture Twelve

Vector Calculus

Section 12.1 – Vector Fields

Gravitational and electric forces have both direction and magnitude. They are represented by a vector, in a subset of Euclidean space, at each point in their domain, producing a ***vector field***.

Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point. A line integral can be used to find the rate at which the fluid flows along or across a curve within the domain.

12.1−1 Vector Fields

Suppose a region in the plane or in space occupied by a moving fluid, such air or water. The fluid is made up of many particles, where a particle has a velocity which can vary. Such a fluid flow is an example of a ***vector field***.

Vectors fields are associated with forces such as gravitational attraction, and to magnetic fields, electric fields, and purely mathematical fields.

A vector filed is a function that assigns a vector to each point in its domain.

A vector field on a three-dimensional domain in space is given by:

$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

The velocity field expression: $\vec{v}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$

The radial field $\vec{F} = x\hat{i} + y\hat{j}$ of position vectors of points in the plane.	A spin field of rotating unit vectors $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ in the plane.

12.1–2 *Definition*

Let f and g be defined a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\vec{F}$ assigns to each point in R a vector $\langle f(x,y),\ g(x,y)\rangle$. The vector field is written as

$$\vec{F}(x,y) = \langle f(x,y),\ g(x,y)\rangle \qquad \textit{or} \qquad \vec{F}(x,y) = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

A vector field $\vec{F} = \langle f,\ g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

12.1–3 Gradient Fields

The gradient vector of a differentiable scalar-valued function at a point gives the direction of greatest increase of the function. An important type of vector field is formed by all the gradient vectors of the function.

We define the gradient field of a differentiable function $f(x,y,z)$ to be the field gradient vectors.

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$$

Example 1

Suppose that the temperature T at each point (x, y, z) in a region of space is given by:

$$T = 100 - x^2 - y^2 - z^2$$

And that $\vec{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\vec{F}$.

Solution

The gradient field $\vec{F}$ is the field:

$$\begin{aligned}\vec{F} &= \nabla T \\ &= -2x\,\hat{i} - 2y\,\hat{j} - 2z\,\hat{k}\end{aligned}$$

At each point in space, the field $\vec{F}$ gives the direction for which the increase in temperature is greatest.

Section 12.2 – Line Integrals

12.2–1 *Definition*

If f is defined on a curve C given parametrically by $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$, $a \le t \le b$, then the line integral of f over C is

$$\int_C f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k, y_k, z_k\right) \Delta s_k$$

Provided this limit exists.

12.2–2 How to Evaluate a Line Integral

The area of a closed and bounded region R in the polar coordinate plane is:

1. Find a smooth parametrization of C, $\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}$, $a \le t \le b$
2. Evaluate the integral as

$$\int_C f(x,y,z)\, ds = \int_a^b f\left(g(t),\, h(t),\, k(t)\right) \left|\vec{v}(t)\right| dt$$

Example 1

Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1, 1, 1).

Solution

$\vec{r}(t) = t\,\hat{i} + t\,\hat{j} + t\,\hat{k}$, $0 \le t \le 1$

$\left|\vec{v}(t)\right| = \left|\hat{i} + \hat{j} + \hat{k}\right|$

$= \sqrt{3} \ne 0$ (The parameterization is *smooth*)

$$\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\left(\sqrt{3}\right)dt$$

$$= \sqrt{3}\int_0^1 \left(t - 3t^2 + t\right)dt$$

$$= \sqrt{3}\int_0^1 \left(2t - 3t^2\right)dt$$

$$= \sqrt{3}\left(t^2 - t^3\right|_0^1$$

$$= \sqrt{3}\,(1-1)$$

$$= 0$$

Example 2

Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$ using the path the origin to the point (1, 1, 1).

Solution

C_1: $\vec{r}_1(t) = t\,\hat{i} + t\,\hat{j}$ $\quad 0 \le t \le 1$

$$\left|\vec{v}_1\right| = \sqrt{1+1}$$

$$= \sqrt{2}$$

C_2: $\vec{r}(t) = \hat{i} + \hat{j} + t\,\hat{k}$ $\quad 0 \le t \le 1$

$$\left|\vec{v}_2\right| = 1$$

$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= \int_0^1 f(t,t,0)\,\sqrt{2}\,dt + \int_0^1 f(1,1,t)\,(1)\,dt$$

$$= \sqrt{2}\int_0^1 \left(t - 3t^2 + 0\right)dt + \int_0^1 (1 - 3 + t)\,dt$$

$$= \sqrt{2}\left(\tfrac{1}{2}t^2 - t^3\right)\Big|_0^1 + \left(-2t + \tfrac{1}{2}t^2\right)\Big|_0^1$$

$$= \sqrt{2}\left(\tfrac{1}{2} - 1\right) + \left(-2 + \tfrac{1}{2}\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2}$$

- The value of the line integral along a path joining two points can change if you change the path between them.

12.2–3 Line Integrals in the Plane

There is an interesting geometric interpretation for line integrals in the plane. If C is a smooth curve in the xy-plane parametrized by $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, $a \le t \le b$, we generate a cylindrical surface by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis.

The cylinder cuts through the surface, forming a curve on it. The part of the cylindrical surface that lies beneath the surface curve and above the xy-plane is like a ***winding wall*** or ***fence*** standing on the curve C and orthogonal to the plane.

$$\int_C f\ ds = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k, y_k\right)\Delta s_k$$

Where $\Delta s_k \to 0$ *as* $n \to \infty$, we see that the line integral $\int_C f\ ds$ is the area of the wall.

12.2–4 Line Integrals with Respect to *xyz* Coordinates

$$\int_C M(x,y,z)\,dx = \int_a^b M\big(g(t),\,h(t),\,k(t)\big)\;g'(t)\,dt$$

$$\int_C N(x,y,z)\,dy = \int_a^b N\big(g(t),\,h(t),\,k(t)\big)\;h'(t)\,dt$$

$$\int_C P(x,y,z)\,dz = \int_a^b P\big(g(t),\,h(t),\,k(t)\big)\;k'(t)\,dt$$

Example 3

Evaluate the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix:

$$\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + t\,\hat{k}\;;\quad 0 \le t \le 2\pi$$

Solution

$$x = \cos t,\qquad y = \sin t,\qquad z = t$$
$$dx = (-\sin t)\,dt,\quad dy = (\cos t)\,dt,\quad dz = dt$$

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} \big((-\sin t)(-\sin t) + t\cos t + 2\cos t\big)\,dt$$

$$= \int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t\right)\,dt$$

		$\int \cos t\,dt$
+	t	$\sin t$
–	1	$-\cos t$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t + t\cos t + 2\cos t\right)\,dt$$

$$= \frac{1}{2}t - \frac{1}{4}\sin 2t + (t\sin t + \cos t) + 2\sin t \;\Big|_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi)+1\right)-(1)$$

$$= \pi + 1 - 1$$

$$= \pi$$

Example 4

Evaluate the line integral $\int_C y^2\,dx + x^3\,dy$, where C is given by: $x = y^3 - 1$ from the point $(-1,\ 0)$ *to* $(7,\ 2)$

Solution

$$x = y^3 - 1$$

$$dx = 3y^2\,dy \qquad 0 \le y \le 2$$

$$\int_C y^2\,dx + x^3\,dy = \int_0^2 \left(3y^4 + \left(y^3-1\right)^3\right)dy$$

$$= \int_0^2 \left(3y^4 + y^9 - 3y^6 + 3y^3 - 1\right)dy$$

$$= \left(\frac{3}{5}y^5 + \frac{1}{10}y^{10} - \frac{3}{7}y^7 + \frac{3}{4}y^4 - y\right)\Bigg|_0^2$$

$$= \frac{96}{5} + \frac{512}{5} - \frac{384}{7} + 12 - 4$$

$$= \frac{608}{5} - \frac{328}{7}$$

$$= \frac{2{,}616}{35}$$

Exercises *Section* 12.2 – Line Integrals

1. Evaluate $\int_C 4xy\, ds$, where C is the line segment from $(-2,\ -1)$ to $(1,\ 3)$.

2. Evaluate $\int_C y\, ds$, where $C:\ \vec{r}(t) = 2t\,\hat{i} + t^3\,\hat{j}$ and $0 \le t \le 1$.

3. Evaluate $\int_C (x+y)\, ds$ where C is the straight-line segment $x = t$, $y = (1-t)$, $z = 0$ from $(0,\ 1,\ 0)$ to $(1,\ 0,\ 0)$.

4. Evaluate $\int_C (x-y+z-2)\, ds$ where C is the straight-line segment $x = t,\ y = (1-t),\ z = 1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

5. Evaluate $\int_C (xy+y+z)\, ds$ along the curve $\vec{r}(t) = 2t\,\hat{i} + t\,\hat{j} + (2-2t)\,\hat{k},\quad 0 \le t \le 1$

6. Evaluate $\int_C \left(xz - y^2\right) ds$; C is the line segment from $(0,\ 1,\ 2)$ to $(-3,\ 7,\ -1)$.

7. Evaluate the length $\int_C (ydx + zdy - xdz)$, where C is the line segment from $(0,\ 1,\ -1)$ to $(1,\ 2,\ 1)$.

8. Evaluate $\int_C \left(xy^2 + yx^2\right) ds$, where C is the upper semicircle $y = \sqrt{4-x^2}$.

9. Evaluate $\int_C (1+xy)\, ds$, where C is the unit circle in the first quadrant, described $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$.

10. Evaluate $\int_C xy\, ds$; C is the unit circle $\vec{r}(s) = \langle \cos s,\ \sin s \rangle;\quad 0 \le s \le 2\pi$.

11. Evaluate $\int_C (x+y)\, ds$; C is the circle of radius 1 centered at $(0,\ 0)$.

12. Evaluate $\int_C \left(x^2 - 2y^2\right) ds$; C is the line $\vec{r}(s) = \left\langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$; $0 \le s \le 4$.

13. Evaluate $\int_C x^2 y \, ds$; C is the line $\vec{r}(s) = \left\langle \frac{s}{\sqrt{2}}, 1 - \frac{s}{\sqrt{2}} \right\rangle$; $0 \le s \le 4$.

14. Evaluate $\int_C \left(x^2 + y^2\right) ds$; C is the circle of radius 4 centered at $(0, 0)$.

15. Evaluate $\int_C \left(x^2 + y^2\right) ds$; C is the line segment from $(0, 0)$ to $(5, 5)$.

16. Evaluate $\int_C \frac{x}{x^2 + y^2} ds$; C is the line segment from $(1, 1)$ to $(10, 10)$.

17. Evaluate $\int_C (xy)^{1/3} ds$; C is the curve $y = x^2$, $0 \le x \le 1$.

18. Evaluate $\int_C xy \, ds$; C is a portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$ in the first quadrant, oriented counterclockwise.

19. Evaluate $\int_C (2x - 3y) ds$; C is the line segment from $(-1, 0)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(1, 0)$

20. Evaluate $\int_C (x + y + z) ds$; C is the circle $\vec{r}(t) = \langle 2\cos t, 0, 2\sin t \rangle$ $0 \le t \le 2\pi$

21. Evaluate $\int_C (x - y + 2z) ds$; C is the circle $\vec{r}(t) = \langle 1, 3\cos t, 3\sin t \rangle$ $0 \le t \le 2\pi$

22. Evaluate $\int_C xyz \, ds$; C is the circle $\vec{r}(t) = \langle 1, 3\cos t, 3\sin t \rangle$ $0 \le t \le 2\pi$

23. Evaluate $\int_C xyz \, ds$; C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

24. Evaluate $\int_C \frac{xy}{z} ds$; C is the line segment from $(1, 4, 1)$ to $(3, 6, 3)$

25. Evaluate $\int_C (y-z)\,ds$; C is the helix $\vec{r}(t)=\langle 3\cos t,\ 3\sin t,\ t\rangle$ $\quad 0\le t\le 2\pi$

26. Evaluate $\int_C x\,e^{yz}\,ds$; C is $\vec{r}(t)=\langle t,\ 2t,\ -4t\rangle$ $\quad 1\le t\le 2$

27. $\int_C \frac{1}{x+1}\,ds$, where $C:\ x=t,\ y=\frac{2}{3}t^{3/2}$; $\quad 0\le t\le 3$

28. $\int_C \frac{x}{y^2+1}\,ds$, where $C:\ x=1+2t,\ y=t$; $\quad 0\le t\le 1$

29. $\int_C (y-x)\,dx+x^2y\,dy$, where $C:\ x^3=y^2$; *from* $(1,\ -1)$ *to* $(1,\ 1)$

30. $\int_C \left(x^2+y^2\right)dx-x\,dy$, where $C:\ x^2+y^2=4$; *from* $(2,\ 0)$ *to* $(0,\ 2)$

31. Find the integral of $f(x,y,z)=x+y+z$ over the straight–line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$.

32. $\int_C \frac{x\,dy-y\,dx}{x^2+y^2}$, where $C:\ x^2+y^2=a^2$; *counterclockwise*.

33. $\int_C 3x^2yz\,ds$, where $C:\ x=t,\ y=t^2,\ z=\frac{2}{3}t^3$; $\quad 0\le t\le 1$.

34. $\int_C \frac{e^{-z}}{x^2+y^2}\,ds$, where $C:\ x=2\cos t,\ y=2\sin t,\ z=t$; $\quad 0\le t\le 2\pi$.

35. $\int_C yz\,dx-xz\,dy+xy\,dz$, $\quad C:\ x=e^t,\ y=e^{3t},\ z=e^{-t}$; $\quad 0\le t\le 1$

36. $\int_C x^2\,dx+xy\,dy+z^2\,dz$, $C:\ x=\sin t,\ y=\cos t,\ z=t^2$; $\quad 0\le t\le \frac{\pi}{4}$

37. Find the integral of $f(x,y,z)=\dfrac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\vec{r}(t)=t\,\hat{i}+t\,\hat{j}+t\,\hat{k},\quad 1\le t\le\infty$

38. Evaluate $\displaystyle\int_C x\,ds$ where C is:

a) The straight-line segment $x=t$, $y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t,\ \ y=t^2$, from (0, 0) to (2, 4).

39. Evaluate $\displaystyle\int_C \sqrt{x+2y}\,ds$ where C is:

a) The straight-line segment $x=t$, $y=4t$, from (0, 0) to (1, 4).

b) $C_1\cup C_2$: C_1 is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

40. Find the line integral of $f(x,y)=\dfrac{\sqrt{y}}{x}$ along the curve $\vec{r}(t)=t^3\,\hat{i}+t^4\,\hat{j},\quad \frac{1}{2}\le t\le 1$.

41. Evaluate $\displaystyle\int_C (x+2)\,ds$, where C: $\vec{r}(t)=t\,\hat{i}+\frac{4}{3}t^{3/2}\,\hat{j}+\frac{1}{2}t^2\,\hat{k},\quad 0\le t\le 2$.

42. Evaluate $\displaystyle\int_C \big((x+y)dx+xydy+ydz\big)$, where C: $\vec{r}(t)=t\,\hat{i}+2t^2\,\hat{j}+3t^3\,\hat{k},\quad 0\le t\le 1$.

43. Evaluate $\displaystyle\int_C \big((x+y)dx+xydy+ydz\big)$, where C: $\vec{r}(t)=e^t\,\hat{i}+e^{-t}\,\hat{j}+2e^{2t}\,\hat{k},\quad 0\le t\le 1$.

44. Evaluate $\displaystyle\int_C \left(ydx+x^2dy\right)$, where C is the parabolic arc $y=4x-x^2$ from $(4,\,0)$ *to* $(1,\,3)$.

45. Evaluate $\displaystyle\int_C (ydx+xdy)$, where C consists of the arc parabola $y=4-x^2$ where C_1 from $(-2,\,0)$ *to* $(0,\,4)$, followed by the line segment C_2 from $(0,\,4)$ *to* $(2,\,0)$.

46. Evaluate $\displaystyle\int_C \left(x^2-y+3z\right)ds$, where C is the line segment from $(0,\,0,\,0)$ *to* $(1,\,2,\,1)$.

47. Evaluate $\int_C \left(xydx + (x+y)dy\right)$, where C consists of the line segment C_1 from $(1, 2)$ *to* $(3, 4)$, followed by the line segment C_2 from $(3, 4)$ *to* $(4, 0)$.

48. Evaluate $\int_C \left(xydx - yzdy + x^2dz\right)$, where C is the line segment C_1 from $(0, 0, 0)$ *to* $(1, 1, 0)$, followed by the line segment C_2 from $(1, 1, 0)$ *to* $(2, 3, 5)$.

49. Evaluate $\int_C \left(ydx + zdy + xdz\right)$, where C is the twisted cubic C_1 with parametric equations $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ where $0 \le t \le 1$, followed by the line segment C_2 from $(1, 1, 1)$ *to* $(0, 1, 0)$.

$(\mathbf{50-51})$ Evaluate where C is:

50. $\int_C \left(x + \sqrt{y}\right) ds$

51. $\int_C 2x\, ds$

52. Evaluate $\int_C \left(y\,dx + x^2\,dy\right)$ where:

a) C is the segment C_1 from $(1, -1)$ *to* $(4, 2)$.

b) C is the arc C_2 from of the parabola $x = y^2$ from $(1, -1)$ *to* $(4, 2)$.

c) C is the arc C_3 from of the parabola $x = y^2$ from $(4, 2)$ *to* $(1, -1)$.

53. Evaluate $\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is:

54. Find the line integral of $f(x,y)=\frac{x^3}{y}$ over the curve $C:\quad y=\frac{x^2}{2},\quad 0\le x\le 2$.

55. Find the line integral of $f(x,y)=x^2-y$ over the curve $C:\quad x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$.

56. Evaluate the line integral $\int_C \left(x^2-2xy+y^2\right)ds$; C is the upper half of a circle $\vec{r}(t)=\langle 5\cos t,\ 5\sin t\rangle$, $0\le t\le \pi$ (ccw).

57. Evaluate the line integral $\int_C y\,e^{-xz}\,ds$; C is the path $\vec{r}(t)=\langle t,\ 3t,\ -6t\rangle$, $0\le t\le \ln 8$.

58. Find the line integral: $\int_C (x+y)\,dx+\frac{z}{x}\,dy+xy\,dz$

Along the spiral C given by: $x=\cos t$, $y=\sin t$, $z=2t$, $0\le t\le \frac{\pi}{2}$.

59. Find the line integral: $\int_C \left(\frac{1}{xy}+\frac{1}{x+y}\right)ds$

Along the C given by: $x=2t$, $y=5t$, $1\le t\le 4$

60. Find the line integral: $\int_C \left(x\,e^y\,dx+x^2y\,dy\right)$ $\quad C:\quad x=e^t,\quad y=e^t,\quad 0\le t\le 1$

61. Integrate $f(x,y,z)=\sqrt{x^2+z^2}$ over the circle $\vec{r}(t)=(a\cos t)\hat{j}+(a\sin t)\hat{k}$, $0\le t\le 2\pi$.

62. Integrate $f(x,y,z)=\sqrt{x^2+y^2}$ over the involute curve.

$\vec{r}(t)=\langle \cos t+t\sin t,\ \sin t-t\cos t\rangle$, $0\le t\le \sqrt{3}$

$(\mathbf{63-66})$ Find the average of the function on the given curve.

63. $f(x, y) = x + 2y$ on the line segment from $(1, 1)$ to $(2, 5)$.

64. $f(x, y) = x^2 + 4y^2$ on the circle of radius 9 centered at the origin.

65. $f(x, y) = x\,e^y$ on the circle of radius 1 centered at the origin.

66. $f(x, y) = \sqrt{4 + 9y^{2/3}}$ on the curve $y = x^{3/2}$, for $0 \le x \le 5$.

$(\mathbf{67-68})$ Find the length of the curve.

67. $\vec{r}(t) = \left\langle 20\sin\frac{t}{4},\ 20\cos\frac{t}{4},\ \frac{t}{2} \right\rangle \quad 0 \le t \le 2$.

68. $\vec{r}(t) = \langle 30\sin t,\ 40\sin t,\ 50\cos t \rangle \quad 0 \le t \le 2\pi$.

69. Find the mass of a spring in the shape of the circular helix:

$$\vec{r}(t) = \frac{1}{\sqrt{2}}\left(\cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}\right) \qquad 0 \le t \le 4\pi$$

Where the density of the wire is $\rho(x, y, z) = 1 + z$.

70. Find the mass and the center mass of a thin wire of a semicircle $y = \sqrt{a^2 - x^2}$, given the mass density *k*.

71. Find the mass and the center mass of a thin wire of a circle $x^2 + y^2 = 4$ in the first quadrant, given the density $\rho(x, y) = x + y$.

72. Find the mass and the center mass of a thin wire of a circle $y = \sqrt{9 - x^2}$, given the mass density of the wire at any point is proportional to its distance from the line $y = 3$.

Section 12.3 – Conservative Vector Fields

12.3–1 Line Integrals of Vector Fields

Assume the vector field $\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ has a continuous component, and the curve C has a smooth parametrization:

$$\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}\,, \quad a \le t \le b\,.$$

$\vec{r}(t)$ defines along the path C which we call the ***forward direction***. At each point along the path C, the tangent vector $\vec{T} = \dfrac{d\vec{r}}{ds} = \dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector tangent to the path and pointing is this forward direction. The tangential component is given by the dot product.

$$\vec{F} \bullet \vec{T} = \vec{F} \bullet \frac{d\vec{r}}{ds}$$

12.3–2 *Definition*

Let $\vec{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\vec{r}(t),\ a \le t \le b$. Then the line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \bullet \vec{T}\, ds = \int_C \left(\vec{F} \bullet \frac{d\vec{r}}{ds}\right) ds$$

$$= \int_C \vec{F} \bullet d\vec{r}$$

12.3–3 Evaluating the Line of $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ along *C*:

$$\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}$$

1. Express the vector field $\vec{F}$ in terms of the parametrized curve C as $\vec{F}(\vec{r}(t))$ by substituting the components $x = g(t),\ y = h(t),\ z = k(t)$ of $\vec{r}$ into the scalar components $M(x,y,z),\ N(x,y,z),\ P(x,y,z)$ of $\vec{F}$.

2. Find the derivative (velocity) vector $\dfrac{d\vec{r}}{dt}$.

3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain:

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}\left(\vec{r}(t)\right) \bullet \frac{d\vec{r}}{dt}\, dt$$

Example 1

Evaluate $\int_C \vec{F} \bullet d\vec{r}$, where $\vec{F} = z\,\hat{i} + xy\,\hat{j} - y^2\,\hat{k}$ along the curve C given by:

$$\vec{r}(t) = t^2\,\hat{i} + t\,\hat{j} + \sqrt{t}\,\hat{k} \quad 0 \le t \le 1.$$

Solution

$$\vec{F}\left(\vec{r}(t)\right) = \sqrt{t}\,\hat{i} + t^3\,\hat{j} - t^2\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\,\hat{k}$$

$$\vec{F}\left(\vec{r}(t)\right) \bullet \frac{d\vec{r}}{dt} = \left(\sqrt{t}\,\hat{i} + t^3\,\hat{j} - t^2\,\hat{k}\right) \bullet \left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\,\hat{k}\right)$$

$$= 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}$$

$$= 2\,t^{3/2} + t^3 - \frac{1}{2}\,t^{3/2}$$

$$= \frac{3}{2}\,t^{3/2} + t^3$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \vec{F}\left(\vec{r}(t)\right) \bullet \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 \left(\frac{3}{2}t^{3/2} + t^3\right) dt$$

$$= \frac{3}{5}\,t^{5/2} + \frac{1}{4}\,t^4 \Big|_0^1$$

$$= \frac{3}{5} + \frac{1}{4}$$

$$= \frac{17}{20}$$

12.3–4 Work Done by a Force over a Curve in Space

12.3–5 *Definition*

Let C be a smooth curve parametrized by $\vec{r}(t), \quad a \le t \le b$, and $\vec{F}$ be a continuous force field over a region containing C. Then the ***work*** done in moving an object from point $A = \vec{r}(a)$ to the point $B = \vec{r}(b)$ along C is

$$Work = \int_a^b \left(\vec{F} \bullet \frac{d\vec{r}}{dt}\right) dt$$

Different ways to write the work integral for $\vec{F} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$ ***over the curve C:*** $\vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j} + k(t)\,\hat{k}$	
$\textbf{Work} = \int_C \vec{F} \bullet \vec{T}\, ds$	***The definition***
$= \int_C \vec{F} \bullet d\vec{r}$	***Vector differential form***
$= \int_a^b \vec{F} \bullet \frac{d\vec{r}}{dt}\, dt$	***Parametric vector evaluation***
$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt$	***Parametric scalar evaluation***
$= \int_C Mdx + Ndy + Pdz$	***Scalar differential form***

Example 2

Find the work done by the force field $\vec{F} = \left(y - x^2\right)\hat{i} + \left(z - y^2\right)\hat{j} + \left(x - z^2\right)\hat{k}$ along the curve $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k} \quad 0 \le t \le 1$, form (0, 0, 0) to (1, 1, 1).

Solution

$$\vec{F} = \left(y - x^2\right)\hat{i} + \left(z - y^2\right)\hat{j} + \left(x - z^2\right)\hat{k}$$

$$= \left(t^2 - t^2\right)\hat{i} + \left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}$$

$$= \left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t\,\hat{i} + z^2\,\hat{j} + t^3\,\hat{k}\right)$$

$$= \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left[\left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}\right] \bullet \left(\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}\right)$$

$$= 2t\left(t^3 - t^4\right) + 3t^2\left(t - t^6\right)$$

$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$Work = \int_0^1 \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^1 \left(2t^4 - 2t^5 + 3t^3 - 3t^8\right)dt$$

$$= \frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \Big|_0^1$$

$$= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}$$

$$= \frac{29}{60}$$

Example 3

Find the work done by the force field $\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ in moving an object along the curve C parametrized by $\vec{r}(t) = \cos(\pi t)\,\hat{i} + t^2\,\hat{j} + \sin(\pi t)\,\hat{k} \quad 0 \le t \le 1$.

Solution

$$\vec{F}(\vec{r}(t)) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = -\pi\sin(\pi t)\,\hat{i} + 2t\,\hat{j} + \pi\cos(\pi t)\,\hat{k}$$

$$\begin{aligned}\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} &= \left(\cos(\pi t)\,\hat{i} + t^2\,\hat{j} + \sin(\pi t)\,\hat{k}\right) \bullet \left(-\pi\sin(\pi t)\,\hat{i} + 2t\,\hat{j} + \pi\cos(\pi t)\,\hat{k}\right) \\ &= -\pi\,\cos(\pi t)\,\sin(\pi t) + 2t^3 + \pi\cos(\pi t)\sin(\pi t) \\ &= \underline{2t^3}\end{aligned}$$

The work done is the line integral

$$\begin{aligned}Work &= \int_0^1 2t^3\,dt \\ &= \frac{1}{2}t^4 \Big|_0^1 \\ &= \underline{\frac{1}{2}}\end{aligned}$$

12.3−6 Conservative Fields

To determine whether a vector fields is conservative or not.

If the vector field $\vec{F}(x, y) = M(x, y)\,\hat{i} + N(x, y)\,\hat{j}$ is conservative vector field in an open region *R*, and both *M* and *N* are continuous in the first-order partial derivative in *R*, then:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the vector field $\vec{F}(x, y, z) = M(x, y, z)\,\hat{i} + N(x, y, z)\,\hat{j} + P(x, y, z)\,\hat{k}$ is conservative vector field in an open region *D*. $\left(in\ \mathbb{R}^3\right)$, then:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad and \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

Or

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{0} \qquad (Curl)$$

12.3–7 Independence of Path and Conservative Vector Fields

Let $\vec{F}$ be continuous vector field in an open region *R*. The line integral $\int_C \vec{F} \bullet \vec{r}$ is independent of path if and only if $\vec{F}$ is conservative, if and only if $\vec{F} = \nabla f$ for some scalar function *f*.

$$M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k} = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$$

And if the $\vec{F}$ is a force field, then the work done by $\vec{F}$ from point $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ is given by the formula:

$$\begin{aligned} Work &= \int_C \vec{F} \bullet \vec{r} \\ &= \int_C \nabla f \bullet \vec{r} \\ &= f(x_2, y_2, z_2) - f(x_1, y_1, z_1) \end{aligned}$$

Example 4

Let the vector field is given by $\vec{F}(x,y) = 2xy^2\,\hat{i} + 2x^2y\,\hat{j}$.

a) Determine whether the vector field $\vec{F}$ is conservative.

b) Find a potential finction $f(x,y)$ that saistfy $\vec{F} = \nabla f$.

c) If $\vec{F}$ is a force field, find the work done by $\vec{F}$ in moving particle along the path from $(1,\ 0)$ *to* $(2,\ 3)$

Solution

a) $M(x,y) = 2xy^2$ & $N(x,y) = 2x^2y$

$M_y = 4xy \qquad N_x = 4xy$

Since $M_y = N_x = 4xy$, therefore the vector field $\vec{F}$ is conservative.

b) $\vec{F} = \nabla f$

$$2xy^2\,\hat{i} + 2x^2y\,\hat{j} = f_x\,\hat{i} + f_y\,\hat{j}$$

$$f_x = 2xy^2 \quad \& \quad f_y = 2x^2y$$

$$f_x = 2xy^2$$

$$f(x,y) = \int 2xy^2\,dx$$

$$= x^2y^2 + g(y)$$

$$f_y = 2x^2y + g'(y) = 2x^2y$$

$$g'(y) = 0$$

$$g(y) = C$$

$$f(x,y) = x^2y^2 + C$$

c) $Work = \int_C \nabla f \bullet \vec{r}$

$$= f(2,\ 3) - f(1,\ 0)$$

$$= (4)(9) - 0$$

$$= 36$$

12.3–8 Flow integrals and Circulation for Velocity Fields

12.3–9 *Definitions*

If $\vec{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\vec{F}$, the ***flow*** along the curve point $A = \vec{r}(a)$ to $B = \vec{r}(b)$ is

$$Flow = \int_C \vec{F} \bullet \vec{T}\,ds$$

The integral in this case is called a ***flow integral***. If the curve starts and ends at the same point, so that $A = B$, the flow is called the ***circulation*** around the curve.

***Example* 5**

A fluid's velocity field is $\vec{F} = x\,\hat{i} + z\,\hat{j} + y\,\hat{k}$. Find the flow along the helix

$$\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + t\,\hat{k}\,, \quad 0 \le t \le \frac{\pi}{2}$$

Solution

$$\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$$

$$= (\cos t)\,\hat{i} + t\,\hat{j} + (\sin t)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\,\hat{i} + (\cos t)\,\hat{j} + \hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((\cos t)\,\hat{i} + t\,\hat{j} + (\sin t)\,\hat{k}\right) \bullet \left((-\sin t)\,\hat{i} + (\cos t)\,\hat{j} + \hat{k}\right)$$

$$= -\cos t\,\sin t + t\,\cos t + \sin t$$

$$Flow = \int_0^{\pi/2} (-\cos t\,\sin t + t\,\cos t + \sin t)\,dt$$

		$\int \cos t\,dt$
+	t	$\sin t$
–	1	$-\cos t$

$$\int -\cos t\,\sin t\,dt = \int \cos t\,d(\cos t)$$

$$= \frac{1}{2}\cos^2 t$$

$$= \frac{1}{2}\cos^2 t + t\sin t + \cos t - \cos t \Big|_0^{\pi/2}$$

$$= \frac{1}{2}\cos^2 t + t\sin t \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

Example 6

Find the circulation of the field $\vec{F} = (x - y)\,\hat{i} + x\,\hat{j}$ around the circle:

$$\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}\,, \quad 0 \le t \le 2\pi$$

Solution

$$\vec{F} = (x - y)\,\hat{i} + x\hat{j}$$

$$= (\cos t - \sin t)\,\hat{i} + (\cos t)\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\,\hat{i} + (\cos t)\,\hat{j}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((\cos t - \sin t)\,\hat{i} + (\cos t)\,\hat{j}\right) \bullet \left((-\sin t)\,\hat{i} + (\cos t)\,\hat{j}\right)$$

$$= -\cos t\,\sin t + \sin^2 t + \cos^2 t$$

$$= 1 - \frac{1}{2}\sin 2t$$

$$Circulation = \int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2t\right) dt$$

$$= t + \frac{1}{4}\cos 2t \Big|_0^{2\pi}$$

$$= 2\pi + \frac{1}{4} - \frac{1}{4}$$

$$= 2\pi$$

12.3–10 Flux across a Simple Plane Curve

A curve in the *xy*-plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a ***closed curve*** or ***loop***.

Simple, not closed	Simple, not closed
Not simple, not closed	Not simple, closed

12.3–11 *Definition*

If C is a smooth simple closed curve in the domain of a continuous velocity field in $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ in the plane, and if $\vec{n}$ is the outward-pointing unit normal vector on C, the flux of $\vec{F}$ across C is

$$\textbf{Flux of } \vec{F} \text{ across } C = \int_C \vec{F} \cdot \vec{n}\, ds$$

For clockwise motion,
$\hat{k} \times \vec{T}$ points outwards

For counterclockwise motion,
$\hat{k} \times \vec{T}$ points outwards

$$\begin{aligned}\vec{n} &= \vec{T} \times \hat{k} \\ &= \left(\frac{dx}{ds}\,\hat{i} + \frac{dy}{ds}\,\hat{j}\right) \times \hat{k} \\ &= \frac{dy}{ds}\,\hat{i} - \frac{dx}{ds}\,\hat{j}\end{aligned}$$

$$\vec{F} \cdot \vec{n} = M(x,y)\,\frac{dy}{ds} - N(x,y)\,\frac{dx}{ds}$$

12.3–12 Calculating Flux Across a Smooth Closed Plane Curve

$$\left(\textit{Flux of } \vec{F} = M\,\hat{i} + N\,\hat{j} \textit{ across } C\right) = \oint_C M\,dy - N\,dx$$

The integral can be evaluated from any smooth parametrization $x = g(t),\ y = h(t),\ a \le t \le b$, that traces C counterclockwise exactly once.

Example 7

Find the flux of $\vec{F} = (x-y)\,\hat{i} + x\,\hat{j}$ across the circle $x^2 + y^2 = 1$ in the *xy*-plane.

(The vector field and curve)

Solution

The parametrization $\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}, \quad 0 \le t \le 2\pi$ traces the circle counterclockwise exactly once.

$$M = x - y \qquad\qquad N = x$$
$$= \cos t - \sin t \qquad\qquad = \cos t$$

$$dx = d(\cos t) \qquad\qquad dy = d(\sin t)$$
$$= -\sin t\ dt \qquad\qquad = \cos t\ dt$$

$$Flux = \int_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} \left(\cos^2 t - \sin t\,\cos t + \cos t\,\sin t\right) dt$$

$$= \int_0^{2\pi} \cos^2 t\ dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right) dt$$

$$= \left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \pi$$

Exercises *Section* 12.3 – Conservative Vector Fields

1. Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

2. Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

3. Find the gradient field of the function $f(x,y,z)=e^z-\ln\left(x^2+y^2\right)$

4. Find the line integral of $\int_C (x-y)\,dx$ where $C:\ x=t,\quad y=2t+1,\quad for\ 0\le t\le 3$

5. Find the line integral of $\int_C \left(x^2+y^2\right)dy$ where C is:

6. Find the line integral of $\int_C \sqrt{x+y}\,dx$ where C is:

7. Find the work done by the force field $\vec{F}=xy\,\hat{i}+y\,\hat{j}-yz\,\hat{k}$ over the curve $\vec{r}(t)=tz\hat{i}+t^2\,\hat{j}+t\,\hat{k},\quad 0\le t\le 1$.

8. Find the work done by the force field $\vec{F}=2y\,\hat{i}+3x\,\hat{j}+(x+y)\,\hat{k}$ over the curve:

$$\vec{r}(t)=(\cos t)\,\hat{i}+(\sin t)\,\hat{j}+\frac{t}{6}\,\hat{k},\quad 0\le t\le 2\pi$$

9. Find the work done by the force field $\vec{F}=z\,\hat{i}+x\,\hat{j}+y\,\hat{k}$ over the curve:

$$\vec{r}(t)=(\sin t)\hat{i}+(\cos t)\,\hat{j}+t\,\hat{k},\quad 0\le t\le 2\pi.$$

10. Find the work required to move an object with given force field $\vec{F}(x, y, z) = \langle -y,\ z,\ x \rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

11. Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle$ for $1 \le t \le 2$

12. Find the work done by a force $\vec{F} = \langle xy,\ yz,\ xz \rangle$ acting on an object moving along the curve:

$$\vec{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle \quad \textit{for} \quad 0 \le t \le 1$$

(13 – 16) Determine whether the vector field $\vec{F}$ is conservative.

13. $\vec{F}(x, y) = 2xy^2\ \hat{i} + xy^2\ \hat{j}$

14. $\vec{F}(x, y) = 2xy\ \hat{i} + 4xy\ \hat{j}$

15. $\vec{F}(x, y) = x^2y^3\ \hat{i} + x^3y^2\ \hat{j}$

16. $\vec{F}(x, y) = \left(e^{-x} - y\cos 2x\right)\ \hat{i} + \left(\sin 2x + ye^{-x}\right)\ \hat{j}$

(17 – 35) For the given vector field $\vec{F}$.

a) Determine whether the vector field $\vec{F}$ is conservative.

b) Find a potential finction $f(x, y)$ that saistfy $\vec{F} = \nabla f$.

c) If $\vec{F}$ is a force field, find the work done by $\vec{F}$ in moving particle along the given path.

d) Evaluate the $\textit{work} = \displaystyle\int_C \vec{F} \bullet \vec{r}$

17. $\vec{F}(x, y) = 2xy\ \hat{i} + \left(1 + x^2 + y^2\right)\ \hat{j}$ *from* $(1,\ 0)$ *to* $(2,\ 4)$

18. $\vec{F}(x, y) = (x + 3y)\ \hat{i} + (3x - y)\ \hat{j}$ *from* $(1,\ 0)$ *to* $(1,\ 2)$

19. $\vec{F}(x, y) = \left(2x^2 + 4y\right)\ \hat{i} + \left(4x - y^2\right)\ \hat{j}$ *from* $(1,\ 1)$ *to* $(2,\ 2)$

20. $\vec{F}(x, y) = \left(y^2\cos x\right)\ \hat{i} + (2y\sin x - 3)\ \hat{j}$ *from* $(0,\ 1)$ *to* $\left(\frac{\pi}{2},\ 3\right)$

21. $\vec{F}(x, y) = \left(x^2 + \frac{y}{x}\right)\ \hat{i} + \left(y^2 + \ln x\right)\ \hat{j}$ *from* $(1,\ 1)$ *to* $(2,\ 3)$

22. $\vec{F}(x, y) = (2y + 1)\ \hat{i} + (2x + 3)\ \hat{j}$ *from* $(0,\ 0)$ *to* $(-1,\ 1)$

23. $\vec{F}(x, y) = (x - 2y)\ \hat{i} + (y - 2x)\ \hat{j}$ *from* $(0,\ 0)$ *to* $(1,\ 1)$

24. $\vec{F}(x,y)=\left(2xy^2+2y\right)\hat{i}+\left(2x^2y+2x\right)\hat{j}$ *from* $(-1,\ 1)$ *to* $(1,\ 2)$

25. $\vec{F}(x,y)=\left(2xy^3\right)\hat{i}+\left(3x^2y^2+1\right)\hat{j}$ *from* $(1,\ 1)$ *to* $(2,\ 0)$

26. $\vec{F}(x,y)=\left(xe^{2y}\right)\hat{i}+\left(x^2e^{2y}\right)\hat{j}$ *from* $(0,\ 0)$ *to* $(-1,\ 1)$

27. $\vec{F}(x,y)=\left(e^x\sin y\right)\hat{i}+\left(e^x\cos y+y\right)\hat{j}$ *from* $(0,\ 0)$ *to* $(0,\ \pi)$

28. $\vec{F}(x,y)=2\sqrt{y}\,\hat{i}+\dfrac{x}{\sqrt{y}}\,\hat{j}$ *from* $(1,\ 1)$ *to* $(2,\ 4)$

29. $\vec{F}(x,y)=\left(x^2-2xy+1\right)\hat{i}+\left(y^2-x^2\right)\hat{j}$ *from* $(1,\ 0)$ *to* $(2,\ 1)$

30. $\vec{F}(x,y)=(\sinh x\cosh y)\,\hat{i}+(\cosh x\sinh y)\,\hat{j}$ *from* $(0,\ 0)$ *to* $(1,\ 1)$

31. $\vec{F}(x,y)=\left(\dfrac{y}{x^2+1}\right)\hat{i}+\left(y^2+\arctan x\right)\hat{j}$ *from* $(0,\ 0)$ *to* $(1,\ 1)$

32. $\vec{F}(x,y)=\dfrac{y^2}{x}\,\hat{i}+2y\ln x\,\hat{j}$ *from* $(1,\ 0)$ *to* $(2,\ 1)$

33. $\vec{F}(x,y,z)=2xyz^2\,\hat{i}+x^2z^2\,\hat{j}+2x^2yz\,\hat{k}$ *from* $(0,\ 1,\ 0)$ *to* $(1,\ 2,\ -1)$

34. $\vec{F}(x,y,\ z)=2xy\,\hat{i}+\left(x^2+z^2\right)\hat{j}+2yz\,\hat{k}$ *from* $(1,\ 1,\ 0)$ *to* $(0,\ 2,\ 3)$

35. $\vec{F}(x,y,\ z)=e^x\cos y\,\hat{i}-e^x\sin y\,\hat{j}+2\,\hat{k}$ *from* $\left(0,\ \frac{\pi}{2},\ 1\right)$ *to* $(1,\ \pi,\ 3)$

36. Find the work done by the force field $\vec{F}(x,\ y)=-\frac{1}{8}(x-y)\,\hat{i}-\frac{1}{8}(x+y)\,\hat{j}$ along a quarter-circle of radius 1 at the origin and is moving:

a) Counterclockwise in direction from $(1,\ 0)$ *to* $(0,\ 1)$.

b) Clockwise in direction from $(0,\ 1)$ *to* $(1,\ 0)$.

37. Find the work done by the force field $\vec{F}(x,\ y,\ z)=-y\,\hat{i}+x\,\hat{j}+z\,\hat{k}$ in moving a particle along a helix C described by the parametric equation $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+t\,\hat{k}$ from $(1,\ 0,\ 0)$ *to* $\left(0,\ 1,\ \frac{\pi}{2}\right)$.

38. Find the work done by a force field $\vec{F}(x,y)=-\frac{1}{2}x\,\hat{i}-\frac{1}{2}y\,\hat{j}+\frac{1}{4}\,\hat{k}$ on a particle moves along the helix given by $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+t\,\hat{k}$ *from* $(1,\ 0,\ 0)$ *to* $(-1,\ 0,\ 3\pi)$

39. Evaluate $\int_C \left(y^3\,dx + \left(x^3 + 3xy^2\right)dy\right)$. C: is a circle with radius 3 and $0 \le t \le 2\pi$.

40. Evaluate $\int_C \vec{F} \cdot \vec{T}\,ds$ for the vector field $\vec{F} = x^2\,\hat{i} - y\,\hat{j}$ along the curve $x = y^2$ from $(4,\ 2)$ to $(1,\ -1)$.

41. Find the circulation and flux of the fields $\vec{F}_1 = x\,\hat{i} + y\,\hat{j}$ *and* $\vec{F}_2 = -y\,\hat{i} + x\,\hat{j}$ around and across each of the following curves.

a) The circle $\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$, $0 \le t \le 2\pi$

b) The ellipse $\vec{r}(t) = (\cos t)\,\hat{i} + (4\sin t)\,\hat{j}$, $0 \le t \le 2\pi$

42. Find the circulation and flux of the fields $\vec{F}_1 = 2x\,\hat{i} - 3y\,\hat{j}$ *and* $\vec{F}_2 = 2x\,\hat{i} + (x - y)\,\hat{j}$ across the circle $\vec{r}(t) = (a\ \cos t)\,\hat{i} + (a\ \sin t)\,\hat{j}$, $0 \le t \le 2\pi$

43. Find a field $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ in the *xy*-plane with the property that at each point $(x,y) \ne (0,0)$, $\vec{F}$ points toward the origin and $\left|\vec{F}\right|$ is:

a) The distance from (x, y) to the origin

b) Inversely proportional to the distance from (x, y) to the origin. (The field is undefined at (0, 0).)

44. A fluid's velocity field is $\vec{F} = -4xy\,\hat{i} + 8y\,\hat{j} + 2\,\hat{k}$. Find the flow along the curve $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + \hat{k}$, $0 \le t \le 2$

45. A fluid's velocity field is $\vec{F} = x^2\,\hat{i} + yz\,\hat{j} + y^2\,\hat{k}$. Find the flow along the curve $\vec{r}(t) = 3t\,\hat{j} + 4t\,\hat{k}$, $0 \le t \le 1$

46. Find the circulation of $\vec{F} = 2x\,\hat{j} + 2z\,\hat{j} + 2y\,\hat{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t.

$C_1: \quad \vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + t\,\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$

$C_2: \quad \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\,\hat{k}, \quad 0 \le t \le 1$

$C_3: \quad \vec{r}(t) = t\,\hat{i} + (1-t)\,\hat{j}, \quad 0 \le t \le 1$

47. The field $\vec{F} = xy\,\hat{i} + y\,\hat{j} - yz\,\hat{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (***Hint***: Use $t = x$ as the parameter.)

48. Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x \rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

49. Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle$ for $1 \le t \le 2$

50. Evaluate $\displaystyle\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

(51 – 56) Evaluate the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

51. $\vec{F} = \nabla\left(x^2 y\right); \quad C: \vec{r}(t) = \left\langle 9 - t^2,\ t \right\rangle, \quad for \ \ 0 \le t \le 3$

52. $\vec{F} = \nabla(xyz); \quad C: \vec{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi} \right\rangle, \quad for \ \ 0 \le t \le \pi$

53. $\vec{F} = \langle x,\ -y \rangle$; C is the square with vertices $(\pm 1,\ \pm 1)$ with counterclockwise orientation.

54. $\vec{F} = \langle y,\ z,\ -x \rangle$; $C: \vec{r}(t) = \langle \cos t,\ \sin t,\ 4 \rangle$, *for* $0 \le t \le 2\pi$

55. $\vec{F} = \left\langle y^2,\ x \right\rangle$; where C is the arc of the parabola $x = 4 - y^2$ from $(-5,\ -3)$ to $(0,\ 2)$

56. $\vec{F} = \left\langle x^2 + y^2,\ 4x + y^2 \right\rangle$; where C is the straight-line segment from $(6,\ 3)$ to $(6,\ 0)$

(57 – 62) Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

57. $\vec{F} = \langle x,\ y \rangle$ on the parabola $\vec{r}(t) = \left\langle 4t,\ t^2 \right\rangle$ $0 \le t \le 1$

58. $\vec{F} = \langle -y,\ x \rangle$ on the semicircle $\vec{r}(t) = \langle 4\cos t,\ 4\sin t \rangle$ $0 \le t \le \pi$

59. $\vec{F} = \langle y,\ x \rangle$ on the line segment from $(1,\ 1)$ to $(5,\ 10)$

60. $\vec{F} = \langle -y,\ x \rangle$ on the parabola $y = x^2$ from $(0,\ 0)$ to $(1,\ 1)$

61. $\vec{F} = \dfrac{\langle x,\ y \rangle}{\left(x^2 + y^2\right)^{3/2}}$ on the curve $\vec{r}(t) = \left\langle t^2,\ 3t^2 \right\rangle$ $1 \le t \le 2$

62. $\vec{F} = \dfrac{\langle x,\ y \rangle}{x^2 + y^2}$ on the line $\vec{r}(t) = \langle t,\ 4t \rangle$ $1 \le t \le 10$

(63 – 73) Find the work required to move an object on the given oriented curve.

63. $\vec{F} = \langle y,\ -x \rangle$ on the path consisting of the line segment from $(1,\ 2)$ to $(0,\ 0)$ followed by the line segment from $(0,\ 0)$ to $(0,\ 4)$.

64. $\vec{F} = \langle x,\ y \rangle$ on the path consisting of the line segment from $(-1,\ 0)$ to $(0,\ 8)$ followed by the line segment from $(0,\ 8)$ to $(2,\ 8)$

65. $\vec{F} = \left\langle x^2,\ -xy \right\rangle$ on runs from $(1,\ 0)$ to $(0,\ 1)$ along the unit circle and then from $(0,\ 1)$ to $(0,\ 0)$ along the y-axis.

66. $\vec{F} = \langle y,\ x \rangle$ on the parabola $y = 2x^2$ from $(0,\ 0)$ to $(2,\ 8)$

67. $\vec{F} = \langle y,\ -x \rangle$ on the line $y = 10 - 2x$ from $(1,\ 8)$ to $(3,\ 4)$

68. $\vec{F} = \langle x,\ y,\ z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle$ $0 \le t \le 2\pi$

69. $\vec{F} = \langle -y,\ x,\ z \rangle$ on the helix $\vec{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \dfrac{t}{2\pi} \right\rangle$ $0 \le t \le 2\pi$

70. $\vec{F} = \dfrac{\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the line segment from $(1, 1, 1)$ to $(10, 10, 10)$.

71. $\vec{F} = \dfrac{\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2, 3t^2, -t^2\right\rangle$, $1 \le t \le 2$

72. $\vec{F} = \dfrac{\langle x, y\rangle}{\left(x^2+y^2\right)^{3/2}}$ over the plane curve $\vec{r}(t) = \left\langle e^t \cos t, e^t \sin t\right\rangle$ from the point $(1, 0)$ to the point $\left(e^{2\pi}, 0\right)$ by using the parametrization of the curve to evaluate the work integral.

73. $\vec{F} = \dfrac{\langle x, y, z\rangle}{x^2+y^2+z^2}$ on the line segment from $(1, 1, 1)$ to $(8, 4, 2)$.

74. Let C be the circle of radius 2 centered at the origin with counterclockwise orientation.

a) Give the unit outward vector at any point (x, y) on C.

b) Find the normal component of the vector field $\vec{F} = 2\langle y, -x\rangle$ at any point on C.

c) Find the normal component of the vector field $\vec{F} = \dfrac{\langle x, y\rangle}{x^2+y^2}$ at any point on C.

75. Find the flow of the field $\vec{F} = \nabla\left(x^2 z e^y\right)$.

a) Once around the ellipse C in which the plane $x+y+z=1$ intersects the cylinder $x^2+z^2=25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\vec{r}(r, \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + \theta\,\hat{k}$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

76. Show that $\displaystyle\int_C (ydx + xdy)$ depends only on the endpoints of C, where C is any curve in the $xy-plane$.

77. Prove that any gradient field ∇f in a region C is conservative. If C is any curve in the region from a point P to a point Q, then $\displaystyle\int_C \left(\nabla f \bullet \vec{T}\right) ds = f(Q) - f(P)$.

Section 12.4 – Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve *C* and a double integral over the plane region *D* bounded by *C*. It is the two-dimensional special case of the more general ***Stokes' theorem*** and is named after British mathematician *George Green*.

George Green

(14 July 1793 – 31 May 1841)

Green's theorem applies to any vector field, independent of any interpretation of the field, provided assumptions of the theorem are satisfied. We introduce two new ideas for Green's theorem: *divergence* and *circulation density* around an axis perpendicular to the plane.

12.4–1 Divergence

Vector fields are provided by using the example if the velocity field of a fluid, such as a liquid or gas.

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a theorem which relates the *flux* of a vector field through a closed surface to the *divergence* of the field in the volume enclosed.

Suppose that $\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ is the velocity field of fluid flowing in the plane and that the first partial derivatives of *M* and *N* are continuous at each point of a region *R*.

Fluid Flow Rates: Top: $\vec{F}(x,\ y+\Delta y)\cdot\hat{j}\ \Delta x = N(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x,\ y)\cdot(-\hat{j})\ \Delta x = -N(x,y)\Delta x$

Right: $\vec{F}(x+\Delta x, y)\cdot\hat{i}\ \Delta y = M(x+\Delta x, y)\Delta y$

Left: $\vec{F}(x, y)\cdot(-\hat{i})\Delta y = -M(x, y)\Delta y$

Top and Bottom: $\left(N(x, y+\Delta y) - N(x, y)\right)\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\left(M(x+\Delta x, y) - M(x, y)\right)\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$

Adding the last two equations gives the *net effect* of the flow rates.

Divergence

$$\textit{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y$$

The estimate of the total flux per unit area or flux density for the rectangle:

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

12.4–2 *Definition*

The divergence (flux density) of a vector field $\vec{F} = M\,\hat{i} + N\,\hat{j}$ at the point (x, y) is:

$$div\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Source: $div\,\vec{F}(x_0, y_0) > 0$

Sink: $div\,\vec{F}(x_0, y_0) < 0$

12.4–3 Spin Around an Axis: the $\hat{k}$ – Component of Curl

The Green's Theorem has to do with measuring how a floating paddle wheel, with axis perpendicular to the plane, spins at a point in fluid flowing in a plane region. Sometimes refer to *circulation density* of a vector field $\vec{F}$ at a point.

$$\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$$

The circulation rate of $\vec{F}$ around the boundary of A is the sum of flow rates along the sides in the tangential direction.

Top: $\vec{F}(x, y+\Delta y)\cdot(-\hat{i})\,\Delta x = -M(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x, y)\cdot\hat{i}\,\Delta x = M(x, y)\Delta x$

Right: $\vec{F}(x+\Delta x, y)\cdot\hat{j}\,\Delta y = N(x+\Delta x, y)\Delta y$

Left: $\vec{F}(x, y)\cdot(-\hat{j})\Delta y = -N(x, y)\Delta y$

Top and Bottom: $-\left(M(x, y+\Delta y) - M(x, y)\right)\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\left(N(x+\Delta x, y) - N(x, y)\right)\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Curl $\vec{F}\bullet\hat{k} > 0$
Counterclockwise circulation

Curl $\vec{F}\bullet\hat{k} < 0$
Clockwise circulation

12.4–4 *Definition*

The ***circulation density*** of a vector field $\vec{F} = M\,\hat{i} + N\,\hat{j}$ at the point $(x,\ y)$ is the scalar expression.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

This expression is also called the $\hat{k}$ ***– component of the curl***, denoted by $\left(curl\ \vec{F}\right) \bullet \hat{k}$.

Proof:

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = \left(\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}\right) \bullet \left(\hat{k}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Example 1

The following vector fields represent the velocity of a gas flowing in the *xy*-plane. Find the ***divergence*** and the ***circulation*** density of each vector field and interpret its physical meaning.

a) Uniform expansion or compression:
$\vec{F}(x,\ y) = cx\,\hat{i} + cy\,\hat{j}$

b) Uniform rotation: $\vec{F}(x,\ y) = -cy\,\hat{i} + cx\,\hat{j}$

c) Shearing flow: $\vec{F}(x, y) = y\,\hat{i}$

d) Whirlpool effect:

$$\vec{F}(x, y) = \frac{-y}{x^2 + y^2}\,\hat{i} + \frac{x}{x^2 + y^2}\,\hat{j}$$

Solution

a) *Uniform expansion*: $\vec{F}(x, y) = cx\,\hat{i} + cy\,\hat{j}$

$$div\vec{F} = \frac{\partial}{\partial x}(cx) + \frac{\partial}{\partial y}(cy)$$

$$= c + c$$

$$= 2c$$

If $c > 0$, the gas is undergoing uniform expansion.

If $c < 0$, the gas is undergoing uniform compression.

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx)$$

$$= 0$$

$\therefore$ The gas is not circulating at very small scales.

b) $\vec{F}(x, y) = -cy\,\hat{i} + cx\,\hat{j}$

$$div\vec{F} = \frac{\partial}{\partial x}(-cy) + \frac{\partial}{\partial y}(cx)$$

$$= 0$$

$\therefore$ The gas is neither expanding nor compressing.

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy)$$

$$= 2c$$

$\therefore$ The constant circulation density indicates rotation at every point.

If $c > 0$, the rotation is counterclockwise.

If $c < 0$, the rotation is clockwise.

c) *Shear*: $\vec{F}(x, y) = y\,\hat{i}$

$$div\vec{F} = \frac{\partial}{\partial x}(y)$$

$$\underline{= 0}$$

∴ The gas is neither expanding nor compressing.

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = -\frac{\partial}{\partial y}(y)$$

$$\underline{= -1}$$

∴ The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spin clockwise. The rate of rotation is the same at each point. The average effect of the fluid flow is to push fluid clockwise around each of the small circles.

d) *Whirlpool*: $\vec{F}(x, y) = \dfrac{-y}{x^2 + y^2}\,\hat{i} + \dfrac{x}{x^2 + y^2}\,\hat{j}$

$$div\vec{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right)$$

$$= \frac{2xy}{\left(x^2 + y^2\right)^2} - \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\underline{= 0}$$

∴ The divergence is zero at all points in the domain of the velocity field.

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right)$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} - \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\underline{= 0}$$

∴ The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined.

12.4–5 *Theorem* – Green's Theorem (Flux-Divergence or Normal Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\hat{i} + N\hat{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\vec{F}$ across C equals the double integral of div $\vec{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \vec{F} \bullet \vec{N}\, ds}_{\textit{Outward flux}} = \oint_C M dy - N dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy}_{\textit{Divergence}\ \text{integral}}$$

12.4–6 *Theorem* – Green's Theorem (Circulation-Curl or Tangential Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\hat{i} + N\hat{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\vec{F}$ around C equals the double integral of $\left(curl\ \vec{F}\right) \bullet \hat{k}$ over R.

$$\underbrace{\oint_C \vec{F} \bullet \vec{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M dx + N dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy}_{\textit{Curl}\ \text{integral}}$$

Example 3

Verify both forms of Green's Theorem for the vector field $\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$

And the region R bounded by the unit circle C: $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

Solution

$M = x - y$ $\qquad$ $N = x$

$= \cos t - \sin t$ $\qquad$ $= \cos t$

$dx = d(\cos t)$ $\qquad$ $dy = d(\sin t)$

$= -\sin t\ dt$ $\qquad$ $= \cos t\ dt$

$\frac{\partial M}{\partial x} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = -1$ $\qquad$ $\frac{\partial N}{\partial y} = 0$

1. $$\oint_C M dy - N dx = \int_0^{2\pi} (\cos t - \sin t)(\cos t\, dt) - (\cos t)(-\sin t\, dt)$$

$$= \int_0^{2\pi} \cos^2 t\ dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$$

$$= \frac{1}{2}t + \frac{1}{4}\sin 2t \Big|_0^{2\pi}$$

$$= \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R (1+0)\, dx\, dy$$

$$= \iint_R dx\, dy$$

$= area$ inside the unit circle

$$= \pi$$

2. $$\oint_C M\ dx + N\ dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t\ dt) + \cos t\ (\cos t\ dt)$$

$$= \int_0^{2\pi} \left(-\cos t\ \sin t + \sin^2 t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(-\frac{1}{2}\sin 2t + 1\right) dt$$

$$= \frac{1}{4}\cos 2t + t \Big|_0^{2\pi}$$

$$= 2\pi$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left(1-(-1)\right) dx\,dy$$

$$= 2\iint_R dx\,dy$$

$$= 2\pi$$

Example 4

Evaluate the line integral $\oint_C xy\,dy - y^2\,dx$

Where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$

Solution

With the Normal Form Equation: $M = xy \quad N = y^2$

$$\oint_C xy\,dy - y^2\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(y + 2y\right) dx\,dy$$

$$= \int_0^1 \int_0^1 3y\,dx\,dy$$

$$= \int_0^1 \left(3xy\right.\Big|_0^1\,dy$$

$$= 3\int_0^1 y\ dy$$

$$= \frac{3}{2}y^2 \Big|_0^1$$

$$= \frac{3}{2}$$

With the Tangential Form Equation: $M = -y^2 \quad N = xy$

$$\oint_C -y^2\,dx + xy\,dy = \iint_R \left(y - (-2y)\right)\,dx\,dy$$

$$= \int_0^1\int_0^1 3y\ dx\,dy$$

$$= \frac{3}{2}$$

Example 5

Calculate the outward flux of the vector field $\vec{F}(x,y) = x\,\hat{i} + y^2\,\hat{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

Solution

$M = x \quad N = y^2$

$$Flux = \oint_C \vec{F} \bullet \vec{n}\ ds \qquad = \oint_C M\,dy - N\,dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy \qquad \textit{Green's Theorem}$$

$$= \int_{-1}^1\int_{-1}^1 (1+2y)\ dx\,dy$$

$$= \int_{-1}^1 (1+2y)\,x \Big|_{-1}^1\ dy$$

$$= \int_{-1}^{1} (1+2y)\,(1-(-1))\,dy$$

$$= 2\int_{-1}^{1} (1+2y)\,dy$$

$$= 2\left(y+y^2\right)\Big|_{-1}^{1}$$

$$= 2\,(1+1-(-1+1))$$

$$= 4$$

Exercises *Section* 12.4 – Green's Theorem.

$(\mathbf{1-18})$ Use Green's theorem to find the counterclockwise circulation and outward flux for the field.

1. $\vec{F} = (x-y)\,\hat{i} + (y-x)\,\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

2. $\vec{F} = \left(x^2+4y\right)\hat{i} + \left(x+y^2\right)\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

3. $\vec{F} = (x+y)\,\hat{i} - \left(x^2+y^2\right)\hat{j}$ and curve C is the triangle bounded by $y=0,\ x=1,\ y=x$

4. $\vec{F} = x^2\,\hat{i} + y^2\,\hat{j}$ and curve C: $x^2+y^2=49$

5. $\vec{F} = \left(xy+y^2\right)\hat{i} + (x-y)\,\hat{j}$ and curve C.

6. $\vec{F} = (x+3y)\,\hat{i} + (2x-y)\,\hat{j}$ and curve C.

7. $\vec{F} = \left(x+e^x\sin y\right)\hat{i} + \left(x+e^x\cos y\right)\hat{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$.

8. *Square*: $\vec{F} = (2xy+x)\,\hat{i} + (xy-y)\,\hat{j}$ C: The square bounded by $x=0,\ x=1,\ y=0,\ y=1$.

9. *Triangle*: $\vec{F} = \left(y-6x^2\right)\hat{i} + \left(x+y^2\right)\hat{j}$.

C: *The triangle made by the lines* $y=0,\ y=x,$ *and* $x=1$

10. $\vec{F} = \langle y-x,\ y\rangle$ for the curve $\vec{r}(t) = \langle 2\cos t,\ 2\sin t\rangle,\quad 0 \le t \le 2\pi$

11. $\vec{F} = \langle x,\ y\rangle$; where R is the half-annulus $\{z(r,\ \theta):\ \ 1 \le r \le 2,\ \ 0 \le \theta \le \pi z\}$

12. $\vec{F} = \langle -y,\ x\rangle$; where R is the annulus $\{z(r,\ \theta):\ \ 1 \le r \le 3,\ \ 0 \le \theta \le 2\pi\}$

13. $\vec{F} = \langle 2x+y,\ x-4y\rangle$; where R is the quarter-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 4,\ \ 0 \le \theta \le \frac{\pi}{2}\right\}$

14. $\vec{F} = \langle x-y,\ 2y-x\rangle$; where R is the parallelogram $\{(x,\ y):\ \ 1-x \le y \le 3-x,\ \ 0 \le x \le 1\}$

15. $\vec{F} = \left\langle \ln\left(x^2 + y^2\right),\ \tan^{-1}\frac{y}{x} \right\rangle$; where R is the annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 2,\ \ 0 \le \theta \le 2\pi\right\}$

16. $\vec{F} = \nabla\left(\sqrt{x^2 + y^2}\right)$; where R is the half-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 3,\ \ 0 \le \theta \le \pi\right\}$

17. $\vec{F} = \langle y\cos x,\ -\sin x\rangle$; where R is the square $\left\{(x,\ y):\ \ 0 \le x \le \frac{\pi}{2},\ \ 0 \le y \le \frac{\pi}{2}\right\}$

18. $\vec{F} = \left\langle x + y^2,\ x^2 - y \right\rangle$; where $R = \left\{(x,\ y):\ \ 3y^2 \le x \le 36 - y^2\right\}$

19. Find the outward flux for the field $\vec{F} = \left(3xy - \frac{x}{1+y^2}\right)\hat{i} + \left(e^x + \tan^{-1} y\right)\hat{j}$ across the cardioid $r = a(1+\cos\theta),\ \ a > 0$

20. Find the work done by $\vec{F} = 2xy^3\,\hat{i} + 4x^2y^2\,\hat{j}$ in moving a particle once counterclockwise around the curve C: The boundary of the triangular region in the first quadrant enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$.

(21 – 73) Apply Green's Theorem to evaluate the integral.

21. $\oint_C \left(y^2\,dx + x^2\,dy\right)$ C: The triangle bounded by $x = 0,\ x + y = 1,\ y = 0$

22. $\oint_C (3y\,dx + 2x\,dy)$ C: The boundary of $0 \le x \le \pi,\ \ 0 \le y \le \sin x$

23. $\oint_C xy^2\,dx + x^2y\,dy$; C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 2)$ with counterclockwise orientation.

24. $\oint_C \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy$; C is the boundary of the half disk $\left\{(x,y):\ x^2 + y^2 \le 2,\ y \ge 0\right\}$ with counterclockwise orientation.

25. $\oint_C \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx$: C is the boundary of the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$ with counterclockwise orientation.

26. $\oint_C (2x-3y)\,dy-(3x+4y)\,dx$: C is the unit circle.

27. $\oint_C f\,dy-g\,dx$ where $\langle f,\,g\rangle=\langle 0,\,xy\rangle$ and C is the triangle with vertices $(0,\,0)$, $(2,\,0)$, $(0,\,4)$ with counterclockwise orientation.

28. $\oint_C f\,dy-g\,dx$ where $\langle f,\,g\rangle=\left\langle x^2,\,2y^2\right\rangle$ and C is the upper half of the unit circle and the line segment $-1\le x\le 1$ with clockwise orientation.

29. The circulation line integral of $\vec{F}=\left\langle x^2+y^2,\,4x+y^3\right\rangle$, where C is the boundary of $\{(x,\,y):\ 0\le y\le \sin x,\quad 0\le x\le \pi\}$.

30. The circulation line integral of $\vec{F}=\left\langle 2xy^2+x,\,4x^3+y\right\rangle$, where C is the boundary of $\{(x,\,y):\ 0\le y\le \sin x,\quad 0\le x\le \pi\}$

31. The flus line integral of $\vec{F}=\left\langle e^{x-y},\,e^{y-x}\right\rangle$, where C is the boundary of $\{(x,\,y):\ 0\le y\le x,\quad 0\le x\le 1\}$.

32. $\oint_C \left(3y-e^{\sin x}\right)dx+\left(7x+\sqrt{y^4+1}\right)dy$: where C is the circle $x^2+y^2=9$.

33. $\oint_C (3x-5y)\,dx+(x-6y)\,dy$: where C is the ellipse $\frac{x^2}{4}+y^2=1$.

34. $\oint_C x^2y\,dx+(x+y)\,dy$ $\quad$ C: The rectangle $0\le x\le 2$ & $0\le y\le 1$, counterclockwise.

35. $\oint_C \frac{y}{x+1}\,dx+2xy\,dy$

C: The region bounded by the curve $y=x^2$ and the line $y=x$, counterclockwise.

36. $\oint_C e^{2x+3y}\,dx+e^{xy}\,dy$ $\quad$ C: $\{(x,y)\ |\ 0\le x\le 2\ \ \&\ \ 0\le y\le 1\}$

37. $\oint_C y\cos x\,dx + y\sin x\,dy \qquad C\colon \left\{ (x, y) \mid 0 \le x \le \frac{\pi}{2} \ \ \& \ \ 1 \le y \le 2 \right\}$

38. $\oint_C x\cos y\,dx - y\sin x\,dy$

C: The square with vertices $(0,\ 0)\ \left(\frac{\pi}{2},\ 0\right)\ \left(\frac{\pi}{2},\ \frac{\pi}{2}\right)\ \left(0,\ \frac{\pi}{2}\right)$.

39. $\oint_C x^2y\,dx + xy^2\,dy \quad C\colon \left\{ (x, y) \mid 0 \le x \le 1 \ \ \& \ \ 0 \le y \le x \right\}$

40. $\oint_C -x^2y\,dx + xy^2\,dy \qquad C\colon \left\{ (x, y) \mid x^2 + y^2 \le 1 \right\}$

41. $\oint_C (y - x)\,dx + xy\,dy \qquad C$: The segment line from $(-1,\ -1)$ *to* $(1,\ 1)$

42. $\oint_C \frac{x}{y}\,dx + (2 + 3x)\,dy \qquad C\colon \left\{ (x, y) \mid 1 \le x \le 2 \ \ \& \ \ 1 \le y \le x^2 \right\}$

43. $\oint_C -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \qquad C\colon \left\{ (x, y) \mid x^2 + y^2 \le 1 \right\}$

44. $\oint_C x\ln y\,dx \qquad C\colon \left\{ (x, y) \mid 1 \le x \le 2 \ \ \& \ \ e^x \le y \le e^{x^2} \right\}$

45. $\oint_C x^2y\,dx + x\,dy \qquad C$: The triangle with vertices $(0,\ 0)\ (1,\ 0)\ (1,\ 2)$

46. $\oint_C -\frac{4x}{\sqrt{1 + y^2}}\,dx + \frac{2x^2y}{\left(1 + y^2\right)^{3/2}}\,dy$

C: The triangle with vertices $(0,\ 0)\ (1,\ 1)\ (1,\ 0)$

47. $\oint_C y^2\,dx + x^2\,dy$

C: The square with vertices $(0,\ 0)\ (1,\ 0)\ (1,\ 1)\ (0,\ 1)$ oriented counterclockwise.

48. $\oint_C 3xy\,dx + 2xy\,dy$

C: The rectangle bounded by $x=-2$, $x=4$, $y=1$, *and* $y=2$.

49. $\oint_C \ln(1+y)\,dx - \dfrac{xy}{1+y}\,dy$

C: The triangle with vertices $(0,\ 0)$, $(2,\ 0)$, & $(0,\ 4)$

50. $\oint_C \tan^{-1} y\,dx - \dfrac{xy^2}{1+y^2}\,dy$ C: The square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, *and* $(0,\ 1)$.

51. $\oint_C \cos x \sin y\,dx + \sin x \cos y\,dy$ C: The triangle with vertices $(0,\ 0)$, $(3,\ 3)$, & $(0,\ 3)$

52. $\oint_C x^2 y\,dx + \left(y + xy^2\right)dy$ C: The region bounded by $y = x^2$, & $x = y^2$

53. $\oint_C -\dfrac{4x}{\sqrt{1+y^2}}\,dx + \dfrac{2x^2 y}{\left(1+y^2\right)^{3/2}}\,dy$ C: The rectangle $0 \le x \le 2$ & $0 \le y \le 1$, counterclockwise.

54. $\oint_C xy\,dx + (x-y)\,dy$; C: *The rectangle* $\{(x,\ y)\ |\ 0 \le x \le 1,\ 1 \le y \le 3\}$

55. $\oint_C \left(x^3 + y^3\right)dx + \left(2y^3 - x^3\right)dy$; C: *Unit circle*

56. $\oint_C \cos y\,dx + \cos x\,dy$; C: $\left\{(x,y)\ |\ 0 \le x \le \frac{\pi}{4}\quad 0 \le y \le \frac{\pi}{3}\right\}$

57. $\oint_C x^2 y\,dx - xy^2\,dy$; C: *Disk* $\left\{(x,y)\ |\ x^2 + y^2 \le 4\right\}$

58. $\oint_C y \ln x\,dy$; C: $\left\{(x,y)\ |\ e^y \le x \le e^{y^3}\quad 1 \le y \le 3\right\}$

59. $\oint_C x^2 y\, dx + (y-4)\, dy$; C *is* the rectangle with vertices $(1,\ 1)$, $(4,\ 1)$, $(4,\ 5)$, *and* $(1,\ 5)$ oriented counterclockwise.

60. $\oint_C \sin x^2\, dx + (3x-y)\, dy$; C *is* the triangle with vertices $(-1,\ 2)$, $(4,\ 2)$, *and* $(4,\ 5)$ oriented counterclockwise.

61. $\oint_C \left(\sin x - \frac{1}{3}y^3\right) dx + \left(\frac{1}{3}x^3 + \cos y\right) dy$; C *is* given by $1 \le r \le 2$ & $0 \le \theta \le 2\pi$.

62. $\oint_C 2xy\, dx + (x+y)\, dy$; C *is* boundary between $y = 4 - x^2$ & $y = 0$ oriented clockwise direction.

63. $\oint_C 2 \arctan \frac{y}{x}\, dx + \ln\left(x^2 + y^2\right) dy$; C *is* defined by $x = 4 + 2\cos\theta$ & $y = 4\sin\theta$ oriented counterclockwise direction.

64. $\oint_C (\sin x \cos y)\, dx + (xy + \cos x \sin y)\, dy$; C *is* region bounded between $y = x$ & $y = \sqrt{x}$ oriented clockwise direction.

65. $\oint_C \left(-y^3 + \sin(xy) + xy\cos(xy)\right) dx + \left(x^3 + x^2\cos(xy)\right) dy$; where C is the circle $x^2 + y^2 = 1$.

66. $\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \sin\left(y^3\right)\right) dy$; where C is the region bounded between $y = x^2$ *and* $x = y^2$.

67. $\oint_C y^2\,dx + 3xy\,dy$; where R is the region where C is the region bounded between the semi-circle $x^2 + y^2 = 1$ and the semi-circle $x^2 + y^2 = 4$ where $y \geq 0$.

68. $\oint_C \left(\cos x^3 - \frac{1}{3}y^3\right) dx + \frac{1}{3}x^3\,dy$; where R is the region where C is the region bounded between the semi-circle $x^2 + y^2 = 1$ and the semi-circle $x^2 + y^2 = 4$ where $y \geq 0$.

69. $\oint_C \left(3y - e^{\cos x}\right) dx + \left(5x + \sqrt{1 + y^2}\right) dy$; where C is the region bounded the disk $x^2 + y^2 \leq 9$.

70. $\oint_C e^y\,dx + 2xe^y\,dy$; where R is the square with the sides $x = 0$, $x = 1$, $y = 0$, *and* $y = 1$.

71. $\oint_C 2e^y\,dx + 3x\,e^y\,dy$; where R is the square with the vertices $(1,\ 0)$, $(2,\ 1)$, $(1,\ 2)$, *and* $(0,\ 1)$.

72. $\oint_C y^3\,dx - x^3\,dy$; where C is the circle $x^2 + y^2 = 4$.

73. $\oint_C \left(y^3 + \sqrt{x}\right) dx + \left(x^2 + \sqrt{y}\right) dy$; where C is the curve of $y = \sin x$ and $0 \leq x \leq \pi$.

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74. Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3 + xy\right)dy + \left(2y^2 - 2x^2y\right)dx$; C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation.

75. Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3dy - 3y^3dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

76. Evaluate $\int_C y^2 dx + x^2 dy$ C is the circle $x^2 + y^2 = 4$.

77. Use both Green's Theorem to evaluate the integral: $\oint_C 3x^2 e^y\, dx + e^y\, dy$;

Where R: the region bounded between the two squares. $(1,\ 1)$, $(-1,\ 1)$, $(-1,\ -1)$, $(1,\ -1)$ and the square $(2,\ 2)$, $(-2,\ 2)$, $(-2,\ -2)$, $(2,\ -2)$.

78. Use Green's theorem to find the work done by the force field $\vec{F} = xy\,\hat{i} + (x+y)\hat{j}$ on a particle that travels once around the circle $x^2 + y^2 = 4$.

79. Use Green's theorem to find the work done by the force field $\vec{F} = \left(e^x - 3y\right)\hat{i} + \left(e^y + 6x\right)\hat{j}$, where $C:\ r = 2\cos\theta$.

80. Find the work done by the force field $\vec{F}(x, y) = \left(e^x - y^3\right)\hat{i} + \left(\cos y + x^3\right)\hat{j}$ on a particle that travels once around the *unit* circle in the *counterclockwise* direction.

81. Find the work done by the force field $\vec{F}(x, y) = (y + \sin x)\,\hat{i} + \left(e^y - x\right)\hat{j}$ on a particle that travels once around the circle with a radius 2 center at the origin in the *counterclockwise* direction.

82. Use Green's theorem to find the work done by the force field:

$\vec{F} = \left(x^{3/2} - 3y\right)\hat{i} + \left(6x + 5\sqrt{y}\right)\hat{j}$; C: the boundary of the triangle with vertices $(0,\ 0)$, $(5,\ 0)$, *and* $(0,\ 5)$.

83. Use Green's theorem to find the work done by the force field:

$\vec{F} = \left(3x^2 + y\right)\hat{i} + \left(4xy^2\right)\hat{j}$; C: the region bounded by the graph $y = \sqrt{x}$, $y = 0$, *and* $x = 9$

.

84. Use Green's theorem to find the work done by the force field

$\vec{F} = \left(y + e^{\sqrt{x}}\right)\hat{i} + \left(2x + \cos y^2\right)\hat{j}$ and curve:

C is the region enclosed by $y = x^2$ *and* $x = y^2$.

85. Use Green's theorem to find the work done by the force field $\vec{F}=\left(e^{2x}-y^3\right)\hat{i}+\left(\cos y^2+x^3\right)\hat{j}$ on a particle that travels once around the unit circle with counterclockwise direction.

86. Find the flux of the force field $\vec{F}(x,y)=\left(x^2+e^y\right)\hat{i}+(x+y)\hat{j}$ on a particle that travels along the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, *and* $(0,\ 3)$

87. Use the flux form to Green's Theorem to evaluate $\iint_R \left(2xy+4y^3\right)dA$, where R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, and $(0,\ 1)$.

88. Find the area of the elliptical region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1$.

89. Find the area of the cap cut from the paraboloid $x^2+y^2+z^2=1$ by the plane $z=\frac{\sqrt{2}}{2}$

(90 – 95) Evaluate both integrals in *Green's* theorem of the vector field. Is the vector field conservative?

90. $\vec{F}=\langle x,\ y\rangle;\quad R=\left\{(x,\ y):\quad x^2+y^2\le 2\right\}$

91. $\vec{F}=\langle y,\ x\rangle$; R is the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$

92. $\vec{F}=\langle 2y,\ -2x\rangle$; R is the region bounded by $y=\sin x$ and $y=0$ for $0\le x\le \pi$

93. $\vec{F}=\langle -3y,\ 3x\rangle$; R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, $(0,\ 2)$

94. $\vec{F}=\left\langle 2xy,\ x^2-y^2\right\rangle$; R is the region bounded by $y=x(2-x)$ and $y=0$

95. $\vec{F}=\left\langle 0,\ x^2+y^2\right\rangle;\quad R=\left\{(x,\ y):\quad x^2+y^2\le 1\right\}$

(96 – 111) Use Green's Theorem to find the area of the regions using line integral.

96. The region enclosed by the ellipse $x^2+4y^2=16$.

97. The region bounded by the hypocycloid $\vec{r}(t)=\left\langle \cos^3 t,\ \sin^3 t\right\rangle$ for $0\le t\le 2\pi$.

98. The region enclosed by a disk of radius 5.

99. A region bounded by an ellipse with semi-major and semi-minor axes of length 12 and 8, respectively.

100. The region bounded by an ellipse $9x^2+25y^2=225$

101. $\left\{ (x,\ y):\quad x^2 + y^2 \le 16z \right\}$

102. The region bounded by the parabolas $\vec{r}(t) = \left\langle t,\ 2t^2 \right\rangle$ and $\vec{r}(t) = \left\langle t,\ 12 - t^2 \right\rangle$ for $-2 \le t \le 2$.

103. The region bounded by the curve $\vec{r}(t) = \left\langle t\left(1 - t^2\right),\ 1 - t^2 \right\rangle$ for $-1 \le t \le 1$.

104. The shaded region:

105. The region of *one* loop of a four-leaf rose $r = 3\sin 2\theta$.

106. The region enclosed by the curve: $\vec{r}(t) = t^2\ \hat{i} - \left(t - \frac{1}{3}t^3\right)\hat{j}$; $\quad -\sqrt{3} \le t \le \sqrt{3}$

107. The region enclosed by the curve: $\vec{r}(t) = (t - \sin t)\ \hat{i} + (1 - \cos t)\ \hat{j}$; $\quad t \ge 0$

108. The region swept out by the line from the origin to the ellipse with $x = a\cos t$ *and* $y = b\sin t$ where $0 \le t \le t_0$

109. The region of the ellipse $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

110. The region enclosed by the asteroid: $\quad x = a\cos^3 t$, $\quad y = a\sin^3 t$, $\quad 0 \le t \le 2\pi$.

111. The region of bounded by vertices $(0,\ 0)$, $(2,\ 1)$, $(1,\ 3)$, and $(-1,\ 1)$.

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112. Prove the identity $\oint_C dx = \oint_C dy = 0$, where C is a simple closed smooth-oriented curve.

113. Prove the identity $\oint_C f(x)dx + g(y)dy = 0$, where f and g have continuous derivatives on the region enclosed by C (is a simple closed smooth-oriented curve).

114. Show that the value of $\oint_C xy^2 dx + \left(x^2 y + 2x\right)dy$ depends only on the area of the region enclosed by C.

115. In terms of the parameters a and b, how is the value of $\oint_C aydx + bxdy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

116. Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0, \ \frac{f(x)}{c} \right\rangle$ and $R = \{(x, y): \ a \le x \le b, \ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

117. Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c}, \ 0 \right\rangle$ and $R = \{(x, y): \ a \le x \le b, \ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

118. Show that $\oint_C \ln x \sin y \, dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

119. Prove that the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ where $\vec{r} = \langle x, y \rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\vec{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

120. Show that the area between two points (x_1, y_1) and (x_2, y_2) is $Area = \frac{1}{2}(x_2 y_1 - x_1 y_2)$.

Section 12.5 – Divergence and Curl

Green's Theorem out of the plane $\left(\mathbb{R}^2\right)$ and into space $\left(\mathbb{R}^3\right)$, it is done as follows:

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Stoke's Theorem relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is the same curve.

- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the Divergence Theorem relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by the surface.

12.5–1 *Definition*

The divergence of a vector field $\vec{F} = \langle f,\ g,\ h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is:

$$\begin{aligned} div\ \vec{F} &= \nabla \bullet \vec{F} \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \end{aligned}$$

If $\nabla \bullet \vec{F} = 0$, the vector field is ***source free***.

Example 1

Compute the divergence of the following vector fields.

a) $\vec{F} = \langle x,\ y,\ z \rangle \quad (radial\ field)$

b) $\vec{F} = \langle -y,\ x - z,\ y \rangle \quad (rotation\ field)$

c) $\vec{F} = \langle -y,\ x,\ z \rangle \quad (spiral\ flow)$

Solution

a) The divergence is

$$\begin{aligned} \nabla \bullet \vec{F} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

Because the divergence is positive, the flow expands outward at all points.

$$\vec{F} = \langle x,\ y,\ z \rangle \quad (\textit{radial field})$$

b) The divergence is:

$$\nabla \bullet \vec{F} = \nabla \bullet \langle -y,\ x - z,\ y \rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 0$$

The field is *source free*.

c) The divergence is:

$$\nabla \bullet \vec{F} = \nabla \bullet \langle -y,\ x,\ z \rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 1$$

Spiral flow

$\vec{F} = \langle -y,\ x,\ z \rangle$

The rotational part of the field in x and y does not contribute to the divergence.

However, the z-component of the field produces a nonzero divergence.

Example 2

Compute the divergence of the radial vector field. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{(x,\ y,\ z)}{\sqrt{x^2 + y^2 + z^2}}$

Solution

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{\left(x^2 + y^2 + z^2\right)^{1/2} - x^2\left(x^2 + y^2 + z^2\right)^{-1/2}}{x^2 + y^2 + z^2}$$

$$= \frac{y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)=\frac{\left(x^2+y^2+z^2\right)^{1/2}-y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$=\boxed{\frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\frac{\left(x^2+y^2+z^2\right)^{1/2}-z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$=\boxed{\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}}$$

$$\nabla\bullet\vec{F}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$=\boxed{\frac{2}{|\vec{r}|}}$$

12.5–2 *Theorem*

For a real number p, the divergence of the radial vector field

$$\vec{F}=\frac{\vec{r}}{|\vec{r}|^p}=\frac{(x,\,y,\,z)}{\left(x^2+y^2+z^2\right)^{p/2}}\qquad\rightarrow\qquad\nabla\bullet\vec{F}=\frac{3-p}{|\vec{r}|^p}$$

Proof:

$$\nabla\bullet\vec{F}=\frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{p/2}}+\frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{p/2}}+\frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{x^2+y^2+z^2-px^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}p}} + \frac{x^2+y^2+z^2-py^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}p}} + \frac{x^2+y^2+z^2-pz^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}p}}$$

$$= \frac{3\left(x^2+y^2+z^2\right)-p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}p}}$$

$$= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}p}}$$

$$= \frac{3-p}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}p}}$$

$$= \frac{3-p}{\left(\sqrt{x^2+y^2+z^2}\right)^{p}}$$

$$= \frac{3-p}{|\vec{r}|^{p}} \quad \checkmark$$

Example 3

To gain some intuition about the divergence, consider the two-dimensional vector field $\vec{F} = \langle f, g \rangle = \left\langle x^2, y \right\rangle$ and a circle C of radius 2 centered at the origin.

a) Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1, 1)$. Why?

b) Confirm the conjecture in part (*a*) by computing the two-dimensional divergence at Q.

c) Based on part (*b*), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence is negative?

d) By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

Solution

a) At $Q(1, 1)$ the x-component and the y-component of the field are increasing $\left(f_x > 0 \text{ and } g_y > 0\right)$, so the field is expanding at that point and the two-dimensional divergence is positive.

b) $\nabla \bullet \vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(y)$

$\underline{= 2x + 1}$

$\nabla \bullet \vec{F}\Big|_{Q(1,1)} \underline{= 3}$

∴ The divergence is 3.

c) $\nabla \bullet \vec{F} = 2x + 1 > 0 \quad \Rightarrow \quad x > -\frac{1}{2}$

$\nabla \bullet \vec{F} = 2x + 1 < 0 \quad \Rightarrow \quad x < -\frac{1}{2}$

To the left of the line $x = -\frac{1}{2}$ the field is contracting and to the right of the line the field is expanding.

d) It appears that the field is tangent to the circle at two points with $x \approx -\frac{1}{2}$.

For points on the circle with $x < -\frac{1}{2}$, the flow is into the circle.

For points on the circle with $x > -\frac{1}{2}$, the flow is out the circle.

It appears that the net outward flux across C is positive.
The points where the field changes from inward to outward may be determined exactly.

12.5–3 General Rotation Vector Field

The general rotation vector field is $\vec{F} = \vec{a} \times \vec{r}$ where the nonzero constant vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\vec{r} = \langle x, y, z \rangle$.

For all choices of $\vec{a}$, $\nabla \times \vec{F} = 2|\vec{a}|$ and $\nabla \bullet \vec{F} = 0$. The constant angular speed of the vector field is

$$\begin{aligned}\omega &= |\vec{a}| \\ &= \frac{1}{2}\left|\nabla \times \vec{F}\right|\end{aligned}$$

Paddle wheel at P with axis $\vec{n}$ measures rotation about $\vec{n}$.
Rotation is a maximum when $\nabla \times \vec{F}$ is parallel to $\vec{n}$.

Example 4

Compute the curl of the rotation field $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle 1,\ -1,\ 1 \rangle$ is the axis of rotation and $\vec{r} = \langle x,\ y,\ z \rangle$. What is the direction and the magnitude of the curl?

Solution

$$\begin{aligned}\vec{F} &= \vec{a} \times \vec{r} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} \\ &= (-z-y)\hat{i} + (x-z)\hat{j} + (y+x)\hat{k}\end{aligned}$$

$$\begin{aligned}curl\ \vec{F} &= \nabla \times \vec{F} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z-y & x-z & y+x \end{vmatrix} \\ &= 2\hat{i} - 2\hat{j} + 2\hat{k} \\ &= 2\vec{a}\end{aligned}$$

The direction of the curl is the direction of $\vec{a}$, which is the axis rotation.

The magnitude of $\nabla \times \vec{F} = |2\vec{a}| = 2\sqrt{3}$

12.5–4 Working with Divergence and Curl

Theorem

Suppose that $\vec{F}$ is a conservative vector field on an open region ***D*** of $\mathbb{R}^3$ Let $\vec{F} = \nabla\phi$, where ϕ is a potential function with continuous second partial derivatives on ***D***. Then $\nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0}$; that is, the curl of the gradient is the zero vector and $\vec{F}$ is irrotational.

$$\nabla \times \nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix}$$

$$= \left(\phi_{zy} - \phi_{yz}\right)\hat{i} + \left(\phi_{xz} - \phi_{zx}\right)\hat{j} + \left(\phi_{yx} - \phi_{xy}\right)\hat{k}$$

$$= \vec{0}$$

12.5–5 Product Rule for the Divergence

Theorem

Let *u* be a scalar-valued function that is differentiable on a region *D* and let $\vec{F}$ be a vector field that us differentiable on *D*. Then

$$\nabla \bullet \left(u\vec{F}\right) = \nabla u \bullet \vec{F} + u\left(\nabla \bullet \vec{F}\right)$$

Example 5

Let $\vec{r} = \langle x, y, z \rangle$ and let $\phi = \frac{1}{|\vec{r}|} = \left(x^2 + y^2 + z^2\right)^{-1/2}$ be a potential function.

a) Find the associated gradient field $\vec{F} = \nabla\left(\frac{1}{|\vec{r}|}\right)$

b) Compute $\nabla \bullet \vec{F}$

Solution

a) $\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2}$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{x}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= -y\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{y}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= -z\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{z}{|\vec{r}|^3}$$

$$\vec{F} = \nabla\left(\frac{1}{|\vec{r}|}\right)$$

$$= -\frac{\langle x,\ y,\ z\rangle}{|\vec{r}|^3}$$

$$= -\frac{\vec{r}}{|\vec{r}|^3}$$

This result reveals that $\vec{F}$ is an inverse square vector field and its potential function is $\phi = \frac{1}{|\vec{r}|}$

b) $\nabla \bullet \vec{F} = \nabla \bullet \left(-\frac{\vec{r}}{|\vec{r}|^3}\right)$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \nabla \frac{1}{|\vec{r}|^3}$$

$$\nabla \bullet \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \bullet \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

$$\frac{\partial}{\partial x}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$= -3x\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -\frac{3x}{|\vec{r}|^5}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$= -3y\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -\frac{3y}{|\vec{r}|^5}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$= -3z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -\frac{3z}{|\vec{r}|^5}$$

$$\nabla\frac{1}{|\vec{r}|^3} = -3\frac{x\hat{i}+y\hat{j}+z\hat{k}}{|\vec{r}|^5}$$

$$= -3\frac{\vec{r}}{|\vec{r}|^5}$$

$$\nabla\bullet\vec{F} = \nabla\bullet\left(-\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\nabla\bullet\vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r}\bullet\nabla\frac{1}{|\vec{r}|^3}$$

$$= -\nabla\bullet\vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r}\bullet\left(-3\frac{\vec{r}}{|\vec{r}|^5}\right)$$

$$= -\nabla\bullet\vec{r}\,\frac{1}{|\vec{r}|^3} + 3\frac{|\vec{r}|^2}{|\vec{r}|^5}$$

$$= -\frac{3}{|\vec{r}|^3} + \frac{3}{|\vec{r}|^3}$$

$$= 0$$

12.5–6 Properties of a Conservative a Vector Field

Let $\vec{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$.

1. There exists a potential function ϕ such that $\vec{F} = \nabla\phi$
2. $\int_C \vec{F} \bullet d\vec{r} = \phi(B) - \phi(A)$ for all points A and B in $\boldsymbol{D}$ and all piecewise-smooth oriented curves C from A to B.
3. $\int_C \vec{F} \bullet d\vec{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
4. $\nabla \times \vec{F} = 0$ at all points of D.

Exercises *Section* 12.5 – Divergence and Curl

$(\mathbf{1-8})$ Find the divergence of the following vector fields.

1. $\vec{F} = \langle 2x,\ 4y,\ -3z \rangle$

2. $\vec{F} = \langle -2y,\ 3x,\ z \rangle$

3. $\vec{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

4. $\vec{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

5. $\vec{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

6. $\vec{F} = \langle yz\cos x,\ xz\cos y,\ xy\cos z \rangle$

7. $\vec{F} = \langle 12x,\ 4y,\ -3z \rangle$

8. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{1 + x^2 + y^2}$

$(\mathbf{9-12})$ Calculate the divergence of the following radial fields. Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

9. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2 + y^2 + z^2} = \dfrac{\vec{r}}{|\vec{r}|^2}$

10. $\vec{F} = \langle x,\ y,\ z \rangle \left(x^2 + y^2 + z^2 \right) = \vec{r}\,|\vec{r}|^2$

11. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2 \right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

12. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2 \right)^{2}} = \dfrac{\vec{r}}{|\vec{r}|^4}$

$(\mathbf{13-14})$ Consider the following vector fields, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (*a*).

c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

13. $\vec{F} = \langle x,\ x + y \rangle$

$P(-1,\ 1)$ & $Q(1,\ 1)$

14. $\vec{F} = \left\langle x,\ y^2 \right\rangle$

$P(-1,\ 1)$ & $Q(-1,\ -1)$

$(\mathbf{15-18})$ Consider the following vector fields, where $\vec{r} = \langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation.

b) Compute the magnitude of the curl of the field.

15. $\vec{F} = \langle 1,\ 0,\ 0\rangle \times \vec{r}$

16. $\vec{F} = \langle 1,\ -1,\ 0\rangle \times \vec{r}$

17. $\vec{F} = \langle 1,\ -1,\ 1\rangle \times \vec{r}$

18. $\vec{F} = \langle 1,\ -2,\ -3\rangle \times \vec{r}$

$(\mathbf{19-32})$ Compute the curl of the following vector fields.

19. $\vec{F} = \left\langle x^2 - y^2,\ xy,\ z\right\rangle$

20. $\vec{F} = \left\langle 0,\ z^2 - y^2,\ -yz\right\rangle$

21. $\vec{F} = \left\langle z^2 \sin y,\ xz^2 \cos y,\ 2xz \sin y\right\rangle$

22. $\vec{F} = \vec{r} = \langle x,\ y,\ z\rangle$

23. $\vec{F} = \left\langle 3xz^3 e^{y^2},\ 2xz^3 e^{y^2},\ 3xz^2 e^{y^2}\right\rangle$

24. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

25. $\vec{F} = \left\langle x^2 - z^2,\ 1,\ 2xz\right\rangle$

26. $\vec{F}(x, y, z) = \arctan\dfrac{x}{y}\ \hat{i} + \ln\sqrt{x^2 + y^2}\ \hat{j} + \hat{k}$

27. $\vec{F}(x, y, z) = \arctan\dfrac{y}{x}\ \hat{i} + \ln\sqrt{x^2 + y^2}\ \hat{j} + \hat{k}$

28. $\vec{F}(x, y, z) = \dfrac{yz}{y - z}\ \hat{i} + \dfrac{xz}{x - z}\ \hat{j} + \dfrac{xy}{x - y}\ \hat{k}$

29. $\vec{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}\ \left(\hat{i} + \hat{j} + \hat{k}\right)$

30. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \dfrac{\vec{r}}{|\vec{r}|}$

31. $\vec{F}(x, y, z) = \sin(x - y)\ \hat{i} + \sin(y - z)\ \hat{j} + \sin(z - x)\ \hat{k}$

32. $\vec{F}(x, y, z) = \cos(x - y)\ \hat{i} + \cos(y - z)\ \hat{j} + \cos(z - x)\ \hat{k}$

$(\mathbf{33-48})$ Compute the divergence, $div\,\vec{F}$, and $curl\ \vec{F}$, for the vector field $\vec{F}$.

33. $\vec{F} = \langle xy,\ yz,\ xz\rangle$

34. $\vec{F} = \left\langle xyz^2,\ x^2yz,\ -xyz\right\rangle$

35. $\vec{F} = \langle x,\ y,\ z\rangle$

36. $\vec{F} = \langle \cos 2x,\ \sin 2y,\ \tan z\rangle$

37. $\vec{F}(x, y, z) = x^3yz^2\ \hat{i} + x^4 z^3\ \hat{k}$

38. $\vec{F}(x, y, z) = xye^z\ \hat{i} + xze^y\ \hat{j}$

39. $\vec{F}(x, y, z) = \sin yz\ \hat{i} + \sin xz\ \hat{j} + \sin xy\ \hat{k}$

40. $\vec{F}(x, y, z) = \dfrac{\sqrt{x}}{1 + z}\ \hat{i} + \dfrac{\sqrt{y}}{1 + x}\ \hat{j} + \dfrac{\sqrt{z}}{1 + y}\ \hat{k}$

41. $\vec{F}(x, y, z) = \ln(y + 2z)\ \hat{i} + \ln(2x + 3z)\ \hat{j} + \ln(3x + 4y)\ \hat{k}$

42. $\vec{F}(x, y, z) = xy^2z^2\ \hat{i} + x^2yz^2\ \hat{j} + x^2y^2z\ \hat{k}$

43. $\vec{F}(x, y, z) = x^2y^2z^2\ \hat{i} + x^2y^2z^2\ \hat{j} + x^2y^2z^2\ \hat{k}$

44. $\vec{F}(x, y, z) = \hat{i} + \sin z\ \hat{j} + y\cos z\ \hat{k}$

45. $\vec{F}(x, y, z) = e^x \sin yz\ \hat{i} + e^x \cos yz\ \hat{j} + y\, e^x \cos yz\ \hat{k}$

46. $\vec{F}(x, y, z) = \arcsin y\ \hat{i} + \sqrt{1-x^2}\ \hat{j} + y^2\ \hat{k}$

47. $\vec{F}(x, y, z) = e^{x^2+y^2}\ \hat{i} + e^{x^2+z^2}\ \hat{j} + e^{y^2+z^2}\ \hat{k}$

48. $\vec{F}(x, y, z) = z^2 \sin y\ \hat{i} + xz^2 \cos y\ \hat{j} + 2xz \sin y\ \hat{k}$

(49 – 52) Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*.

49. $\vec{F} = \langle yz,\ xz,\ xy \rangle$

50. $\vec{F} = \vec{r}|\vec{r}| = \langle x,\ y,\ z \rangle \sqrt{x^2 + y^2 + z^2}$

51. $\vec{F} = \langle \sin xy,\ \cos yz,\ \sin xz \rangle$

52. $\vec{F} = \left\langle 2xy + z^4,\ x^2,\ 4xz^3 \right\rangle$

53. Let $\vec{F} = \langle z,\ x,\ -y \rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0 \rangle$ and $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$

b) In what direction is the scalar component of curl $\vec{F}$ a maximum?

54. Let $\vec{F} = \langle z,\ 0,\ -y \rangle$

c) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0 \rangle$ and $\vec{n} = \langle 1,\ -1,\ 1 \rangle$

d) In what direction $\vec{n}$ is $\left(curl\ \vec{F}\right) \bullet \vec{n}$ a maximum?

55. Within the cube $\{(x,\ y,\ z):\ \ -1 \le x \le 1,\ \ -1 \le y \le 1,\ \ -1 \le z \le 1\}$, where does $div\vec{F}$ have the greatest magnitude when $\vec{F} = \left\langle x^2 - y^2,\ xy^2 z,\ 2xz \right\rangle$.

56. Show that the general rotation field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a}$ is a nonzero constant vector and $\vec{r} = \langle x,\ y,\ z \rangle$, has zero divergence.

57. Let $\vec{a} = \langle 0,\ 1,\ 0 \rangle$, $\vec{r} = \langle x,\ y,\ z \rangle$ and consider the rotation field $\vec{F} = \vec{a} \times \vec{r}$. Use the right-hand rule for cross product to find the direction of $\vec{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$.

58. Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\vec{F} = \langle f, g \rangle = \langle x^2, y \rangle$ switches from pointing inward to outward on the circle, or vice versa.

59. Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vec{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease.

The divergence of the heat flow vector is $\vec{F} = -k\nabla \bullet \nabla T = -k\nabla^2 T$ (the Laplacian of *T*). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x, y, z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x, y, z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

60. Consider the rotational velocity field $\vec{v} = \langle -2y, 2z, 0 \rangle$

a) If a paddle is placed in the *xy*-plane with its axis normal to this plane, what is its angular speed?

b) If a paddle is placed in the *xz*-plane with its axis normal to this plane, what is its angular speed?

c) If a paddle is placed in the *yz*-plane with its axis normal to this plane, what is its angular speed?

61. Consider the rotational velocity field $\vec{v} = \langle 0, 10z, -10y \rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

62. The potential function for the gravitational force field due to a mass *M* at the origin acting on a mass *m* is $\phi = \dfrac{GMm}{|\vec{r}|}$, where $\vec{r} = \langle x, y, z \rangle$ is the position vector of the mass m and *G* is the gravitational constant.

a) Compute the gravitational force field $\vec{F} = -\nabla\phi$

b) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

63. The potential function for the force field due to a charge q at the origin is $\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ is the position vector of the mass m and G is the gravitational constant.

c) Compute the force field $\vec{F} = -\nabla\phi$

d) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

64. The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the motion of water in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}\right) = -\nabla p + \mu\left(\nabla \bullet \nabla\right)\vec{V}$$

In this notation $\vec{V} = \langle u,\ v,\ w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation.

65. One of Maxwell's equations for electromagnetic waves is $\nabla \times \vec{B} = C\frac{\partial \vec{E}}{\partial t}$, where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and C is a constant.

a) Show that the fields $\vec{E}(z,\ t) = A\sin(kz - \omega t)\hat{i}$ $\quad \vec{B}(z,\ t) = A\sin(kz - \omega t)\hat{j}$

Satisfy the equation for constants A, k, and ω, provided $\omega = \frac{k}{C}$

b) Make a rough sketch showing the directions of $\vec{E}$ and $\vec{B}$

66. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla \bullet \frac{\langle x,\ y,\ z\rangle}{|\vec{r}|^p} = \frac{3-p}{|\vec{r}|^p}$

67. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla\left(\frac{1}{|\vec{r}|^p}\right) = \frac{-p\vec{r}}{|\vec{r}|^{p+2}}$

68. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla \bullet \nabla\left(\frac{1}{|\vec{r}|^p}\right) = \frac{p(p-1)}{|\vec{r}|^{p+2}}$

69. Prove the product rule $div\left(f\,\vec{F}\right) = f\ div\,\vec{F} + \nabla f \bullet \vec{F}$. For a vector field $\vec{F}$ and a scalar field f.

70. Prove the product rule $curl\left(f\,\vec{F}\right) = f\ curl\left(\vec{F}\right) + \nabla f \times \vec{F}$. For a vector field $\vec{F}$ and a scalar field f.

71. Prove the product rule $div\left(\vec{E} \times \vec{F}\right) = curl\left(\vec{E}\right) \bullet \vec{F} - \vec{E} \bullet curl\left(\vec{F}\right)$. For a vector field $\vec{F}$ and a scalar field f.

72. Show that $curl\left(f\left(\left|\vec{r}\right|\right)\vec{r}\right)=\vec{0}$. For any function $f\left(u\right)$ where $\vec{r}=\left\langle x,y,z\right\rangle$.

73. Prove that $\nabla f\left(r\right)=\dfrac{f'\left(r\right)}{r}\vec{r}$, where $r=\sqrt{x^2+y^2+z^2}$ and $f'\left(r\right)$ does exist.

Section 12.6 – Surface Integrals

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} \quad a \le t \le b$

And

Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0$

12.6 - 1 Parameterization of Surfaces

Suppose:

We call the range of $\vec{r}$ the ***surface*** S defined or traced by $\vec{r}$.

u and v: variable parameters

R: parameter domain

Example 1

Find a parameterization of the cone $z=\sqrt{x^2+y^2}, \quad 0\le z\le 1$

Solution

$x=r\cos\theta, \quad y=r\sin\theta$

$z=\sqrt{x^2+y^2}=r$

Assume $u=r \quad and \quad v=\theta$

$\vec{r}(r,\ \theta)=(r\cos\theta)\ \hat{i}\ +(r\sin\theta)\ \hat{j}\ +r\ \hat{k}$

$0\le r\le 1, \quad 0\le\theta\le 2\pi$

Example 2

Find a parameterization of the cone $x^2+y^2+z^2=a^2$

Solution

A typical point $(x,\ y,\ z)$ on the sphere has:

$x=a\ \sin\phi\ \cos\theta, \quad y=a\ \sin\phi\ \sin\theta, \quad z=a\ \cos\phi$

$0\le\phi\le 2\pi, \quad 0\le\theta\le 2\pi$

Taking $u=\phi \quad and \quad v=\theta$

$\vec{r}(\phi,\ \theta)=(a\sin\phi\cos\theta)\ \hat{i}\ +(a\sin\phi\sin\theta)\ \hat{j}\ +(a\cos\phi)\ \hat{k}$

The parameterization is one-to-one on the interior of the domain *R*, though not on its boundary "*poles*" where $\phi=0 \quad or \quad \theta=\pi$

Example 3

Find a parameterization of the cone $x^2+(y-3)^2=9, \quad 0\le z\le 5$

Solution

$$x=r\cos\theta, \quad y=r\sin\theta, \quad z=z$$

$$x^2+y^2-6y+9=9$$

$$x^2+y^2-6y=0$$

$$r^2-6r\sin\theta=0$$

$$r(r-6\sin\theta)=0$$

$$r=6\sin\theta, \quad 0\le\theta\le\pi$$

A typical point on the cylinder has:

$$\begin{cases} x=r\cos\theta \ =6\sin\theta\cos\theta \ = \ 3\sin 2\theta \\ y=r\sin\theta \ =6\sin^2\theta \\ z=z \end{cases}$$

Taking $u=\theta \quad and \quad v=z$

$$\vec{r}(\theta,\ z)=(3\sin 2\theta)\ \hat{i}\ +\left(6\sin^2\theta\right)\hat{j}\ +z\ \hat{k} \quad 0\le\theta\le\pi, \quad 0\le z\le 5$$

12.6 – 2 Surface Area

Calculating the area of a curved surface *S* based on the parameterization.

$$\vec{r}(u,\ v)=f(u,\ v)\ \hat{i}\ +g(u,\ v)\ \hat{j}\ +h(u,\ v)\ \hat{k}\ , \quad a\le u\le b\ , \quad c\le v\le d$$

The definition of smoothness involves the partial derivatives of $\vec{r}$ with respect to *u* and *v*:

$$\begin{cases} \vec{r}_u \ =\dfrac{\partial\vec{r}}{\partial u} \\ \quad =\dfrac{\partial f}{\partial u}\ \hat{i}\ +\dfrac{\partial g}{\partial u}\ \hat{i}\ +\dfrac{\partial h}{\partial u}\ \hat{k} \\ \vec{r}_v \ =\dfrac{\partial\vec{r}}{\partial v} \\ \quad =\dfrac{\partial f}{\partial v}\ \hat{i}\ +\dfrac{\partial g}{\partial v}\ \hat{j}\ +\dfrac{\partial h}{\partial v}\ \hat{k} \end{cases}$$

12.6 – 3 *Definition*

A ***parameterized*** surface $\vec{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{i} + h(u,v)\,\hat{k}$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the interior of the parameter domain.

12.6 – 4 *Definition*

The area of the smooth surface:

$$\vec{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}, \quad a \le u \le b, \quad c \le v \le d$$

is
$$Area = \iint_R \left|\vec{r}_u \times \vec{r}_v\right| dA$$
$$= \int_c^d \int_a^b \left|\vec{r}_u \times \vec{r}_v\right| du\, dv$$

Surface area Differential for a Parameterized Surface

$$d\sigma = \left|\vec{r}_u \times \vec{r}_v\right| \, du\, dv \qquad \iint_S d\sigma$$

Surface area differential *Differntial formula for surface area*

Example 4

Find the surface area of the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$

$\vec{r}(r,\ \theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ 1\rangle$

$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle -r\cos\theta,\ \ -r\sin\theta,\ r\rangle$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$

$$= r\sqrt{2} \ \neq 0$$

$$Surface = \int_0^{2\pi}\int_0^1 \left|\vec{r}_r \times \vec{r}_\theta\right| \, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \ \int_0^1 \sqrt{2}\, r\ dr$$

$$= \frac{\sqrt{2}}{2}(2\pi)\ \left(r^2\right)\Big|_0^1$$

$$= \pi\sqrt{2} \quad unit^2$$

Example 5

Find the surface area of a sphere of radius a.

Solution

$$\vec{r}(\phi,\ \theta) = (a\ \sin\phi\ \cos\theta)\ \hat{i}\ + (a\ \sin\phi\ \sin\theta)\ \hat{j}\ + (a\cos\phi)\ \hat{k}$$

$$\vec{r}_{\phi} = \langle a\cos\phi\cos\theta,\ \ a\cos\phi\sin\theta,\ \ -a\sin\phi\rangle$$

$$\vec{r}_{\theta} = \langle -a\sin\phi\sin\theta,\ \ a\sin\phi\cos\theta,\ \ 0\rangle$$

$$0 \le \phi \le \pi\ ,\quad 0 \le \theta \le 2\pi$$

$$\vec{r}_{\phi} \times \vec{r}_{\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} + \left(a^2\sin^2\phi\sin^2\theta\right)\hat{j} + \left(a^2\cos\phi\sin\phi\cos^2\theta + a^2\cos\phi\sin\phi\sin^2\theta\right)\hat{k}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} + \left(a^2\sin^2\phi\sin\theta\right)\hat{j} + \left(a^2\cos\phi\sin\phi\right)\hat{k}$$

$$\left|\vec{r}_{\phi} \times \vec{r}_{\theta}\right| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^4\phi\left(\cos^2\theta + \sin^2\theta\right) + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right)}$$

$$= a^2\sin\phi$$

$$Surface = \int_0^{2\pi} d\theta \int_0^{\pi} a^2\sin\phi\ d\phi$$

$$= 2\,a^2\pi\ \left(-\cos\phi\ \right|_0^{\pi}$$

$$= 4\pi a^2\ \ unit^2$$

Example 6

Let S be the "football" surface formed by rotating the curve $x = \cos z, \quad y = 0, \quad -\frac{\pi}{2} \le z \le \frac{\pi}{2}$ around the z-axis. Find the parameterization for S and compute its surface area.

Solution

Let $(x,\ y,\ z)$ be an arbitrary point on the circle.

The parameters: z and θ.

We have:

$$\begin{cases} x = r\cos\theta \ = \cos z\ \cos\theta \\ y = r\sin\theta \ = \cos z\ \sin\theta \\ \quad z \end{cases}$$

$$\vec{r}(z,\ \theta) = (\cos z\ \cos\theta)\ \hat{i} + (\cos z\ \sin\theta)\ \hat{j} + z\ \hat{k}$$

$$-\frac{\pi}{2} \le z \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi$$

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin z\ \cos\theta & -\sin z\ \sin\theta & 1 \\ -\sin z\ \sin\theta & \cos z\ \cos\theta & 0 \end{vmatrix}$$

$$= (-\cos u\ \cos v)\ \hat{i} - (\cos z\ \sin\theta)\ \hat{j} - \left(\sin z\ \cos z\ \cos^2\theta + \cos z\ \sin z\ \sin^2\theta\right)\ \hat{k}$$

$$\left|\vec{r}_z \times \vec{r}_\theta\right| = \sqrt{\cos^2 z\ \cos^2\theta + \cos^2 z\ \sin^2\theta + \left(\sin z\ \cos z\ \left(\cos^2\theta + \sin^2\theta\right)\right)^2}$$

$$= \sqrt{\cos^2 z\ \left(\cos^2 z + \sin^2 z\right) + \sin^2 z\ \cos^2 z}$$

$$= \sqrt{\cos^2 z\ \left(1 + \sin^2 z\right)}$$

$$= \cos z\ \sqrt{1 + \sin^2 z}$$

$$Surface = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos z\ \sqrt{1 + \sin^2 z}\ dz$$

$$u = \sin z \ \Rightarrow \ du = \cos z\, dz$$

$$\begin{cases} z = \frac{\pi}{2} & \to u = 1 \\ z = -\frac{\pi}{2} & \to u = -1 \end{cases}$$

$$= 2\pi \int_{-1}^{1} \sqrt{1+u^2}\ du$$

$$u = \tan\alpha \qquad \sqrt{1+u^2} = \sec\alpha$$

$$du = \sec^2\alpha\, d\alpha$$

$$= 2\pi \int_{-1}^{1} \sec^3\alpha\ d\alpha$$

$$v = \sec\alpha \qquad v = \int \sec^2\alpha\, d\alpha$$

$$dv = \sec\alpha\ \tan\alpha\ dv \qquad = \tan\alpha$$

$$\int \sec^3\alpha\, d\alpha = \sec\alpha\ \tan\alpha - \int \tan\alpha\ (\sec\alpha\ \tan\alpha\ d\alpha)$$

$$= \sec\alpha\ \tan\alpha - \int \tan^2\alpha\ \sec\alpha\ d\alpha$$

$$= \sec\alpha\ \tan\alpha - \int \left(\sec^2\alpha - 1\right) \sec\alpha\ d\alpha$$

$$= \sec\alpha\ \tan\alpha - \int \sec^3\alpha\ d\alpha + \int \sec\alpha\ d\alpha$$

$$2\int \sec^3\alpha\ d\alpha = \sec\alpha\ \tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|$$

$$= \pi\ \left(\sec\alpha\ \tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|\right.\Big|_{-1}^{1}$$

$$= \pi\ \left(u\sqrt{1+u^2} + \ln\left|u + \sqrt{1+u^2}\right|\right.\Big|_{-1}^{1}$$

$$= \pi\ \left(\sqrt{2} + \ln\left(1+\sqrt{2}\right) + \sqrt{2} - \ln\left(-1+\sqrt{2}\right)\right)$$

$$\ln\left(\left(-1+\sqrt{2}\right) \bullet \frac{1+\sqrt{2}}{1+\sqrt{2}}\right) = \ln\left(\frac{1}{1+\sqrt{2}}\right)$$

$$= -\ln\left(1+\sqrt{2}\right)$$

$$= \pi\ \left(2\sqrt{2} + \ln\left(1+\sqrt{2}\right) + \ln\left(1+\sqrt{2}\right)\right)$$

$$= 2\pi\ \left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \quad unit^2$$

12.6 – 5 Implicit Surfaces

Surfaces are often presented as level sets of a function $F(x, y, z) = c$ for some constant c. Such a level surface does not come with an explicit parameterization and is called am *implicit defined surface*.

The surface is defined by the equation $F(x, y, z) = c$ and $\vec{p}$ is a unit vector normal to the plane region R.

$$\begin{aligned}\nabla F \bullet \vec{p} &= \nabla F \bullet \hat{k} \\ &= F_z \neq 0\end{aligned}$$

Define the parameters u and v by $u = x$ and $v = y$. Then $z = h(u, v)$ and

$$\vec{r}(u, v) = u\,\hat{i} + v\,\hat{j} + h(u,v)\,\hat{k}$$

Calculating the partial derivatives of $\vec{r}$,

$$\vec{r}_u = \hat{i} + \frac{\partial h}{\partial u}\hat{k} \quad \textit{and} \quad \vec{r}_v = \hat{j} + \frac{\partial h}{\partial v}\hat{k}$$

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \quad \textit{and} \quad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}$$

$$\vec{r}_u = \hat{i} - \frac{F_x}{F_z}\hat{k} \quad \textit{and} \quad \vec{r}_v = \hat{j} - \frac{F_y}{F_z}\hat{k}$$

$$\begin{aligned}\vec{r}_u \times \vec{r}_v &= \frac{F_x}{F_z}\hat{i} + \frac{F_y}{F_z}\hat{i} + \hat{k} \\ &= \frac{1}{F_z}\left(F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}\right) \\ &= \frac{\nabla F}{F_z} \\ &= \frac{\nabla F}{\nabla F \bullet \hat{k}} \\ &= \frac{\nabla F}{\nabla F \bullet \vec{p}}\end{aligned}$$

Therefore, the surface area differential is given by:

$$d\sigma = \left|\vec{r}_u \times \vec{r}_v\right| \, du \, dv$$

$$= \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|} \, dx \, dy \qquad u = x \;\; and \;\; v = y$$

12.6 – 6 Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|} \, dA$$

Where $\vec{p} = \hat{i},\ \hat{j},\ or\ \hat{k}$ is normal to R and $\nabla F \bullet \vec{p} \neq 0$

Example 7

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$

Solution

Let $F(x, y, z) = x^2 + y^2 - z = 0$ ***and*** R the disk $x^2 + y^2 \leq 4$

$\nabla F = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$|\nabla F| = \sqrt{4x^2 + 4y^2 + 1}$

$|\nabla F \bullet \vec{p}| = |\nabla F \bullet \hat{k}|$

$= |-1|$

$= 1$

In the region R, $dA = dxdy$. Therefore,

$$Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|} \, dA$$

$$= \iint_{x^2+y^2 \leq 4} \sqrt{4\left(x^2 + y^2\right) + 1} \, dx \, dy$$

$$= \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2+1}\, r\, dr$$

$$= 2\pi \left(\frac{1}{8}\right) \int_0^2 \sqrt{4r^2+1}\; d\left(4r^2+1\right)$$

$$= (2\pi)\frac{1}{12}\left(4r^2+1\right)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{6}\left(17^{3/2}-1\right)$$

$$= \frac{\pi}{6}\left(17\sqrt{17}-1\right) \text{ unit}^2$$

12.6 – 7 Formula for the Surface Area of a Graph $z = f(x,y)$

For a graph $z = f(x,y)$ over the region R in the xy-plane, the surface area formula is:

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$$

Example 8

Evaluate $\iint_S y\, dS$ where S is the surface $z = x + y^2 \quad 0 \le x \le 1, \quad 0 \le y \le 2$.

Solution

$$z = x + y^2$$

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = 2y$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{2+4y^2}$$

$$\iint_C y\, dS = \int_0^1 dx \int_0^2 y\sqrt{4y^2+2}\; dy$$

$$= \frac{1}{8}\int_0^2 \left(4y^2+2\right)^{1/2} d\left(4y^2+2\right)$$

$$= \frac{1}{12}\left(4y^2+2\right)^{3/2}\Big|_0^2$$

$$= \frac{1}{12}\left((18)^{3/2} - 2\sqrt{2}\right)$$

$$= \frac{1}{12}\left(54\sqrt{2} - 2\sqrt{2}\right)$$

$$= \frac{13}{3}\sqrt{2}$$

Surface	***Equation***	***Explicit Description***	
		Normal Vector $\pm\left\langle -z_x,\ -z_y,\ 1\right\rangle$	***Magnitude*** $\left\lvert\left\langle -z_x,\ -z_y,\ 1\right\rangle\right\rvert$
Cylinder	$x^2+y^2=a^2$ $0 \le z \le h$	$\langle x,\ y,\ 0\rangle$	a
Cone	$z^2=x^2+y^2$ $0 \le z \le h$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ -1\right\rangle$	$\sqrt{2}$
Sphere	$x^2+y^2+z^2=a^2$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$	$\frac{a}{z}$
Paraboloid	$z=x^2+y^2$ $0 \le z \le h$	$\langle 2x,\ 2y,\ -1\rangle$	$\sqrt{1+4\left(x^2+y^2\right)}$

Surface	***Equation***	***Parametric Description***	
		Normal Vector $\vec{r}_u \times \vec{r}_v$	***Magnitude*** $\left\lvert \vec{r}_u \times \vec{r}_v \right\rvert$
Cylinder	$\vec{r} = \langle a\cos u,\ a\sin u,\ z\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le h$	$\langle a\cos u,\ a\sin u,\ 0\rangle$	a
Cone	$\vec{r} = \langle v\cos u,\ v\sin u,\ v\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le h$	$\langle v\cos u,\ v\sin u,\ -v\rangle$	$\sqrt{2}\,v$
Sphere	$\vec{r} = \langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$ $0 \le u \le \pi,\quad 0 \le v \le 2\pi$	$\left\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\sin^2 u\cos u\right\rangle$	$a^2\sin u$
Paraboloid	$\vec{r} = \left\langle v\cos u,\ v\sin u,\ v^2\right\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le \sqrt{h}$	$\left\langle 2v^2\cos u,\ 2v^2\sin u,\ -v\right\rangle$	$v\sqrt{1+4v^2}$

Exercises *Section* 12.6 – Surface Integrals

$(\mathbf{1}-\mathbf{9})$ Find a parametrization of the surface:

1. The paraboloid $z = x^2 + y^2,\ \ z \le 4$

2. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

3. The sphere $x^2 + y^2 + z^2 = 8$ cuts by the plane $z = -2$

4. The plane $2x - 4y + 3z = 16$

5. The cap of the sphere $x^2 + y^2 + z^2 = 16$ for $2\sqrt{2} \le z \le 4$

6. The frustum of the cone $z^2 = x^2 + y^2$ for $2 \le z \le 8$

7. The cone $z^2 = 4\left(x^2 + y^2\right)$ for $0 \le z \le 4$

8. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

9. The cylinder $y^2 + z^2 = 36$ for $0 \le x \le 9$

$(\mathbf{10}-\mathbf{19})$ Use parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of:

10. A plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

11. A cone $z = \dfrac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \dfrac{4}{3}$

12. A cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

13. Cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

14. The half cylinder $\{(r,\ \theta,\ z):\quad r = 4,\quad 0 \le \theta \le \pi.\quad 0 \le z \le 7\}$

15. The plane $z = 3 - x - 3y$ in the first octant

16. The plane $z = 10 - x - y$ above the square $|x| \le 2,\quad |y| \le 2$

17. The hemisphere $x^2 + y^2 + z^2 = 100,\quad z \ge 0$

18. A cone with base radius *r* and height *h*, where *r* and *h* are positive constants.

19. The cap of the sphere $x^2 + y^2 + z^2 = 4,\quad 1 \le z \le 2$

$(\mathbf{20-45})$ Use a surface integral to find the area of:

20. Cut from the bottom of the paraboloid $x^2+y^2-z=0$ by the plane $z=2$.

21. Portion $x^2-2z=0$ that lies above the triangle bounded by the lines $x=\sqrt{3},\ y=0$, and $y=x$ in the *xy*-plane.

22. Cap cut from the sphere $x^2+y^2+z^2=2$ by the cone $z=\sqrt{x^2+y^2}$.

23. Ellipse cut from the plane $z=cx$ (c a constant) by the cylinder $x^2+y^2=1$.

24. From the nose of the paraboloid $x=1-y^2-z^2$ by *yz*-plane.

25. First octant cut from the cylinder $y=\frac{2}{3}z^{3/2}$ by the planes $x=1$ and $y=\frac{16}{3}$

26. Helicoid $\vec{r}(r,\ \theta)=(r\cos\theta)\hat{\boldsymbol{i}}+(r\sin\theta)\hat{\boldsymbol{j}}+\theta\hat{\boldsymbol{k}},\quad 0\le\theta\le 2\pi,\quad 0\le r\le 1$

27. Surface $f(x,\ y)=\sqrt{2}\,xy$ above the origin $\{(r,\ \theta):\ 0\le r\le 2,\ 0\le\theta\le 2\pi\}$

28. Hemisphere $x^2+y^2+z^2=9$, for $z\ge 0$ (excluding the base)

29. Frustum of the cone $z^2=x^2+y^2$, for $2\le z\le 4$ (excluding the bases)

30. Area of the plane $z=6-x-y$ above the square $|x|\le 1,\ |y|\le 1$

31. The cone $z^2=4\left(x^2+y^2\right),\quad 0\le z\le 4$

32. The paraboloid $z=2\left(x^2+y^2\right),\quad 0\le z\le 8$

33. The trough $z=x^2,\quad -2\le x\le 2,\quad 0\le y\le 4$

34. The part of the hyperbolic paraboloid $z=x^2-y^2$ above the sector $R=\left\{(r,\ \theta):\quad 0\le r\le 4,\quad -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}\right\}$

35. $f(x,\ y,\ z)=xy$, where *S* is the plane $z=2-x-y$ in the first octant.

36. $f(x,\ y,\ z)=x^2+y^2$, where *S* is the paraboloid $z=x^2+y^2,\quad 0\le z\le 4$

37. $f(x,\ y,\ z)=25-x^2-y^2$, where *S* is the hemisphere centered at the origin with radius 5, for $z\ge 0$

38. $f(x,\ y,\ z)=e^x$, where *S* is the plane $z=8-x-2y$ in the first octant.

39. $f(x,\ y,\ z)=e^z$, where *S* is the plane $z=8-x-2y$ in the first octant.

40. Find the surface area that extends upward from the parabola $y=x^2\quad 0\le x\le 2$ in the $xy-plane$ to the plane $z=3x$.

41. The portion of cylinder $x^2+y^2=9$ in the first octant $0\le z\le 3$.

42. The revolution of $y = x^2$, $0 \le x \le 3$, about the $x - axis$.

43. The revolution of $y = \sin x$, $0 \le x \le \pi$, about the $x - axis$.

44. The semi-hemisphere $x^2 + y^2 + z^2 = 100$, $z \ge 0$.

45. The semi-hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$.

$(\mathbf{46} - \mathbf{74})$ Evaluate the surface integrals.

46. $\iint_S (x+z)\, dS$ where S is the portion of the semi-circle $z = \sqrt{1-y^2}$ that lies between the first octant by the plane $x = 3$ and $yz - plane$.

47. $\iint_S x^2 yz\, dS$ where S is the portion of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $0 \le x \le 3$ *and* $0 \le y \le 2$.

48. $\iint_S \left(x^2 + y^2\right) z\, dS$ where S is the semi-hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$.

49. $\iint_S \left(xz + 2x^2 - 3xy\right) dS$ where S is the portion of the plane $2x - 3y + z = 6$ that lies over the unit square $0 \le x \le 1$ *and* $0 \le y \le 1$.

50. $\iint_S x^2 z\, dS$ where S is the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies between plane $z = 1$ *and* $z = 4$.

51. $\iint_S (x+z)\, dS$ where S is the portion of the semi-circle $z = \sqrt{1-y^2}$ that lies between the first octant by the plane $x = 3$ and $yz - plane$.

52. $\iint_S x^2 yz\, dS$ where S is the portion of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $0 \le x \le 3$ *and* $0 \le y \le 2$.

53. $\iint_S (x-y-z)\, dS$ where S is the part of the plane $x+y=1$ in the first octant between $z=0$ *and* $z=1$.

54. $\iint_S (x+y+z)\, dS$ where S is the surface of the cube defined by: $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

55. $\iint_S z^2\, dS$ where S is the cone $z=\sqrt{x^2+y^2}$ part that lies between the planes $z=1$ *and* $z=2$.

56. $\iint_S \sqrt{x^2+y^2+z^2}\, dS$ where S is the portion of the cone $z=\sqrt{x^2+y^2}$ and below the plane $z=1$.

57. $\iint_S x^2\, dS$ where S is the sphere $x^2+y^2+z^2=1$.

58. Evaluate $\iint_C z\, dS$ where C is the sphere who sides C_1 is given by the cylinder $x^2+y^2=1$, whose bottom C_2 is given by the disk $x^2+y^2 \le 1$ in the plane $z=0$, and whose top C_3 is given by the part of the plane $z=1+x$ that lies above C_2.

59. $\iint_S xz\, dS$ where S is the plane $x+y+z=1$ lies in the first octant.

60. $\iint_S y^2 z^2\, dS$ where S is the cone $z=\sqrt{x^2+y^2}$ part that lies between the planes $z=1$ *and* $z=2$.

61. $\iint_S z^2\, dS$ where S is the cone $z=\sqrt{x^2+y^2}$ part that lies between the planes $z=1$ *and* $z=2$.

62. $\iint_S x^2 y\, dS$ where S is the part of the cylinder $x^2 + z^2 = 1$ that lies between the planes $y = 1$, $y = 2$, and above the $xy - plane$.

63. $\iint_S (1 + yz)\, dS$; S is the plane $x + y + z = 2$ in the first octant.

64. $\iint_S \langle 0, y, z \rangle \cdot \vec{n}\, dS$; S is the curve surface of the cylinder $y^2 + z^2 = a^2$, $|x| \le 8$ with outward normal vectors.

65. $\iint_S (x - y + z)\, dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$.

66. $\iint_S \nabla \ln|\vec{r}| \bullet \vec{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\vec{r} = \langle x,\ y,\ z \rangle$. Assume normal vectors point either outward or in the positive z-direction.

67. $\iint_S |\vec{r}|\, dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, and where $\vec{r} = \langle x,\ y,\ z \rangle$. Assume normal vectors point either outward or in the positive z-direction.

68. $\iint_S xyz\, dS$, where S is the part of the plane $z = 6 - y$ that lies on the cylinder $x^2 + y^2 = 4$. Assume normal vectors point either outward or in the positive z-direction.

69. $\iint_S \frac{\langle x,\ 0,\ z \rangle}{\sqrt{x^2 + z^2}} \bullet \vec{n}\, dS$, where S is the cylinder $x^2 + y^2 = a^2$, $|y| \le 2$. Assume normal vectors point either outward or in the positive z-direction.

70. $\iint_S (x+z)\, dS$ where S is the portion of the semi-circle $z=\sqrt{1-y^2}$ that lies between the first octant by the plane $x=3$ and $yz-plane$.

71. $\iint_S yz\, dS$ where S is the plane $x+y+z=1$ that lies in the first octant.

72. $\iint_S e^{-z}\, dS$ where S is the portion of the semi-hemisphere $x^2+y^2+z^2=9$ and $z\geq 0$.

73. $\iint_S \frac{1}{\sqrt{1+4x^2+4y^2}}\, dS$ where S is the surface with parametrization $\vec{r}(u,\ v)=\langle u\cos v,\ u\sin v,\ u^2\rangle$ and $0\leq u\leq \sin v$ & $0\leq v\leq \pi$.

74. $\iint_S xyz\, dS$ where S is the surface with parametrization $\vec{r}(u,\ v)=\langle u\cos v,\ u\sin v,\ 3u\rangle$; $1\leq u\leq 2$ & $0\leq v\leq \frac{\pi}{2}$.

(75 – 78) Evaluate the surface integral $\iint_S f(x,\ y,\ z)dS$

75. $f(x,\ y,\ z)=x^2+y^2$, where S is the hemisphere $x^2+y^2+z^2=36,\quad z\geq 0$

76. $f(x,\ y,\ z)=y$, where S is the cylinder $x^2+y^2=9,\quad 0\leq z\leq 3$

77. $f(x,\ y,\ z)=x$, where S is the cylinder $x^2+z^2=1,\quad 0\leq y\leq 3$

78. $f(\rho,\ \varphi,\ \theta)=\cos\varphi$, where S is the part of the unit shpere in the first octant

(79 – 86) Find the flux of the vector fields across the given surface with the specified orientation.

79. $\vec{F}=\frac{\vec{r}}{|\vec{r}|}$ across the sphere of radius a centered at the origin, where $\vec{r}=\langle x,\ y,\ z\rangle$. Assume the normal vectors to the surface point outward.

80. $\vec{F}=\langle x,\ y,\ z\rangle$ across the curved surface of the cylinder $x^2+y^2=1$ for $|z|\leq 8$

81. $\vec{F} = \langle 0,\ 0,\ -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward.

82. $\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward.

83. $\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$; normal vectors point upward.

84. $\vec{F} = \left\langle e^{-y},\ 2z,\ xy \right\rangle$ across the curved sides of the surface $S = \{(x,\ y,\ z):\quad z = \cos y,\quad -\pi \le y \le \pi,\quad 0 \le x \le 4\}$; normal vectors point upward.

85. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ across the sphere of radius, a, centered at the origin, where $\vec{r} = \langle x,\ y,\ z \rangle$; normal vectors point outward.

86. $\vec{F} = \langle -y,\ x,\ 1 \rangle$ across the cylinder $y = x^2$ for $0 \le x \le 1,\quad 0 \le z \le 4$; normal vectors point in the general direction of the positive y-axis.

87. Consider the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where a, b, and c are positive real numbers.

a) Show that the surface is described by the parametric equations
$\vec{r}(u,v) = \langle a\cos u \sin v,\ b\sin u \sin v,\ c\cos v \rangle$ for $0 \le u \le 2\pi,\ 0 \le v \le \pi$

b) Write an integral for the surface area of the ellipsoid.

88. The cone $z^2 = x^2 + y^2,\quad z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a) Find the surface area of the sphere below C, for $z \ge 0$
b) Find the surface area of the sphere above C.
c) Find the surface area of the cone below C, for $z \ge 0$

89. Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$ for $z \ge 0$.

a) Find the surface area of the cylinder inside the sphere.
b) Find the surface area of the sphere inside the cylinder.

90. Find the upward flux of the field $\vec{F} = \langle x,\ y,\ z \rangle$ across the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ in the first octant. Show that the flux equals c times the area if the base of the origin.

91. Consider the field $\vec{F} = \langle x, y, z \rangle$ and the cone $z^2 = \dfrac{x^2 + y^2}{a^2}$, for $0 \le z \le 1$

a) Show that when $a = 1$, the outward flux across the cone is zero.
b) Find the outward flux (away from the *z*-axis); for any $a > 0$.

92. A sphere of radius *a* is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane. Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness *h*.

93. Consider the radial field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where $\vec{r} = \langle x, y, z \rangle$ and *p* is a real number. Let *S* be the sphere of radius *a*, centered at the origin. Show that the outward flux of $\vec{F}$ across the sphere is $\dfrac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

(94 – 96) The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surfaces *S* for the given temperature distributions. Assume $k = 1$.

94. $T(x, y, z) = 100e^{-x-y}$; *S* consists of the faces of the cube $|x| \le 1, \quad |y| \le 1, \quad |z| \le 1$

95. $T(x, y, z) = 100e^{-x^2-y^2-z^2}$; *S* is the sphere $x^2 + y^2 + z^2 = a^2$

96. $T(x, y, z) = -\ln\left(x^2 + y^2 + z^2\right)$; *S* is the sphere $x^2 + y^2 + z^2 = a^2$

97. Given: $\vec{r}(u, v) = \langle (R + r\cos u)\cos v, \ (R + r\cos u)\sin v, \ r\sin u \rangle$

a) Show that a torus with radii $R > r$ may be described parametrically by $\vec{r}(u, v)$ for $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

b) Show that the surface area of the torus is $4\pi^2 Rr$.

Section 12.7 – Stokes' Theorem

12.7-1 Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

If C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a region R and $\vec{F} = \langle M, N \rangle$ is a differentiable vector field on R. Green's Theorem says that

$$\underbrace{\oint \vec{F} \bullet d\vec{r}}_{circulation} = \underbrace{\iint_R \left(M_x - N_y \right) dA}_{curl\ or\ rotation}$$

If the fingers of your right-hand curl in the positive direction around C, then your right thumb points in the direction of the vectors normal to S.

12.7-2 *Theorem*

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\vec{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \left(\nabla \times \vec{F} \right) \bullet \vec{n} \ dS$$

Where $\vec{n}$ is the unit vector normal to S determined by the orientation of S.

Example 1

Confirm that Stokes' Theorem holds for the vector field $\vec{F} = \langle z - y,\ x,\ -x \rangle$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Solution

The orientation of C says that the vectors normal to S point in the outward direction. The vector field is a rotation field $\vec{a} \times \vec{r}$, where $\vec{a} = \langle 0,\ 1,\ 1 \rangle$ and $\vec{r} = \langle x,\ y,\ z \rangle$, so the axis of rotation points in the direction of the vector $\langle 0,\ 1,\ 1 \rangle$.

Compute first the circulation integral in Stokes' Theorem. The curve C with the given orientation is parametrized as $\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0 \rangle$, for $0 \leq t \leq 2\pi$

$$\frac{d}{dt}\vec{r}(t) = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$$

$$\vec{F} = \langle -2\sin t,\ 2\cos t,\ -2\cos t \rangle$$

$$\vec{F} \bullet \vec{r}'(t) = 4\sin^2 t + 4\cos^2 t$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \vec{F} \bullet \vec{r}'(t)\ dt$$

$$= \int_0^{2\pi} 4\left(\sin^2 t + \cos^2 t\right)\ dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= 4\int_0^{2\pi} dt$$

$$= 8\pi$$

$$\nabla \times \vec{F} = \nabla \times \langle z - y,\ x,\ -x \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(x)\right)\hat{i} + \left(\frac{\partial}{\partial z}(z - y) - \frac{\partial}{\partial x}(-x)\right)\hat{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z - y)\right)\hat{k}$$

$$= 0\hat{i} + 2\hat{j} + 2\hat{k}$$

$$= \langle 0,\ 2,\ 2 \rangle$$

The region of integration is the base of the hemisphere in the *xy*-plane, which is

$$R = \left\{ (x, y) : x^2 + y^2 \le 4 \right\} = \left\{ (r, \theta) :\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi \right\}$$

$$z = \pm\sqrt{4 - \left(x^2 + y^2\right)}$$

The normal vector from the table: $\vec{n} = \left\langle -z_x,\ -z_y,\ 1 \right\rangle$

$$x^2 + y^2 + z^2 = 4$$

$$2x + 2zz_x = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2y + 2zz_x = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 0,\ 2,\ 2 \rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2 \right) dA \qquad x^2 + y^2 + z^2 = 4 \ \rightarrow\ z = \sqrt{4 - x^2 - y^2}$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4 - r^2}} + 2 \right) r\ dr\ d\theta$$

$$= \int_0^2 \int_0^{2\pi} \left(\frac{2r^2}{\sqrt{4 - r^2}} \sin\theta + 2r \right) d\theta\ dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4 - r^2}} \cos\theta + 2r\theta \right)_0^{2\pi} dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4 - r^2}} + 4\pi r + \frac{2r^2}{\sqrt{4 - r^2}} \right) dr$$

$$= 4\pi \int_0^2 r\ dr$$

$$= 2\pi r^2 \Big|_0^2$$

$$= 8\pi$$

Example 2

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = \langle z,\ -z,\ x^2 - y^2 \rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown

Solution

$$\nabla \times \vec{F} = \nabla \times \langle z,\ -z,\ x^2 - y^2 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(x^2 - y^2\right) - \frac{\partial}{\partial z}(-z) \right)\hat{i} + \left(\frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}\left(x^2 - y^2\right) \right)\hat{j} + \left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(z) \right)\hat{k}$$

$$= \langle 1 - 2y,\ 1 - 2x,\ 0 \rangle$$

$$z = 8 - 4x - 2y \ \Rightarrow\ 4x + 2y + z = 8$$

$$\vec{n} = \langle 4,\ 2,\ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \left(\langle 1 - 2y,\ 1 - 2x,\ 0 \rangle \bullet \langle 4,\ 2,\ 1 \rangle \right) dA$$

$$= \int_0^2 \int_0^{4-2x} (6 - 4x - 8y)\ dydx$$

$$= \int_0^2 \left(6y - 4xy - 4y^2\right)\Big|_0^{4-2x} dx$$

$$= \int_0^2 \left(24 - 12x - 16x + 8x^2 - 4\left(16 - 16x + 4x^2\right)\right) dx$$

$$= \int_0^2 \left(-8x^2 + 36x - 40\right) dx$$

$$= -\frac{8}{3}x^3 + 18x^2 - 40x \Big|_0^2$$

$$= -\frac{64}{3} + 72 - 80$$

$$= -\frac{88}{3}$$

Example 3

Evaluate the line integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$ where $\vec{F} = \left\langle -xz,\ yz,\ xy\, e^z \right\rangle$ and S is the cap of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 3$.

Assume $\vec{n}$ points in the upward direction on S.

Solution

$$z = 5 - x^2 - y^2 = 3 \quad \Rightarrow \quad x^2 + y^2 = 2$$

$$\vec{r}(t) = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t,\ 3 \right\rangle \qquad \vec{r}(t) = \left\langle r\cos t,\ r\sin t,\ z \right\rangle$$

$$\frac{d}{dt}\vec{r}(t) = \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t,\ 0 \right\rangle$$

$$\left\langle -xz, yz, xy\,e^{z} \right\rangle \bullet \left\langle -\sqrt{2}\ \sin t,\ \sqrt{2}\ \cos t, 0 \right\rangle = xz\sqrt{2}\ \sin t + yz\sqrt{2}\ \cos t$$

$$= 3\sqrt{2}\cos t\sqrt{2}\sin t + 3\sqrt{2}\sin t\sqrt{2}\cos t$$

$$= 12\sin t\cos t$$

$$\underline{= 6\sin 2t}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \int_0^{2\pi} 6\ \sin 2t\ dt$$

$$= -3\cos 2t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

12.7-3 Interpreting the Curl

Stokes' Theorem leads to another interpretation of the curl at a point in a vector field. We need the idea of the average circulation. If C is the boundary of an oriented surface C, we define the average circulation of $\vec{F}$ over S as

$$\frac{1}{area(S)} \oint_C \vec{F} \bullet d\vec{r} = \frac{1}{area(S)} \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$$

Where Stokes' Theorem is used to convert the circulation integral to a surface integral.

Example 4

Consider the vector field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = \left(a_1,\ a_2,\ a_3\right)$ is a nonzero vector and $\vec{r} = \left(x,\ y,\ z\right)$

$$\vec{F} = \vec{a} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \left(a_2 z - a_3 y\right)\hat{i} + \left(a_3 x - a_1 z\right)\hat{j} + \left(a_1 y - a_2 x\right)\hat{k}$$

$$curl\ \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left(2a_1\right)\hat{i} + \left(2a_2\right)\hat{j} + \left(2a_3\right)\hat{k}$$

$$= 2\vec{a}$$

Let S to be a small circular disk centered at a point P, whose normal vector $\vec{n}$ makes an angle θ with the axis $\vec{a}$.

Let C be the boundary of S with a counterclockwise orientation.

The average circulation of this vector field on S is:

$$\frac{1}{area(S)} \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \frac{1}{area(S)} \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ \cdot area(S)$$

$$= \left(2\vec{a}\right) \bullet \vec{n}$$

$$= 2\left|\vec{a}\right| \cos\theta$$

Example 5

Consider the velocity field $\vec{v} = \langle 0,\ 1-x^2,\ 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$, which represents a horizontal flow in the *y*-direction.

Horizontal channel flow $\vec{v} = \langle 0,\ 1-x^2,\ 0 \rangle$

Paddle wheel with vertical axis spins clockwise for $x > 0$ and counterclockwise for $x < 0$

a) Suppose you place a paddle wheel at the point $P\left(\frac{1}{2},\ 0,\ 0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2},\ 0,\ 0\right)$?

b) Compute and graph the curl of $\vec{v}$ and provide an interpretation.

Solution

a) If the axis of the wheel is aligned with the *x*-axis at *P*, the flow strikes the upper and lower halves of the wheel symmetrically and the wheel does not spin. If the axis of the wheel is aligned with the *z*-axis at *P*, the flow in the *y*-direction is greater for $x < \frac{1}{2}$ than it is for $x > \frac{1}{2}$ Therefore, a wheel located at $\left(\frac{1}{2}, 0, 0\right)$ spins in the clockwise direction, looking from above. Using the similar argument, we conclude that a vertically oriented paddle wheel placed at $Q\left(-\frac{1}{2}, 0, 0\right)$ spins in the counterclockwise direction (when viewing from above).

b) $$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 1-x^2 & 0 \end{vmatrix}$$

$$= -2x\,\hat{k}$$

The curl points in the *z*-direction, which is the direction of the paddle wheel axis that gives the maximum angular speed of the wheel. Consider the *z*-component of the curl, which is $(\nabla \times \vec{v}) \bullet \hat{k} = -2x$

At $x = 0$, this component is zero, meaning the wheel does not spin at any point along the y-axis when its axis of the wheel is aligned with the z-axis. For $x>0$, we see that $(\nabla\times\vec{v})\bullet\hat{k}\ <0$, which corresponds to clockwise rotation of the vector field.

For $x<0$, we see that $(\nabla\times\vec{v})\bullet\hat{k}\ >0$, which corresponds to counterclockwise rotation.

12.7-4 Proof of Stokes' Theorem

Consider the case in which the surface S is the graph of the function $z=s(x,\ y)$, defined on a region in the xy-plane. Let C be the curve that bounds S with a counterclockwise orientation, let R be the projection of S in the xy-plane, and let C_1 the projection of C in the xy-plane.

C_1 is the projection of C in the xy-plane

Let $\vec{F}=\langle f,\ g,\ h\rangle$ the line integral in Stokes' Theorem is:

$$\oint_C \vec{F}\bullet d\vec{r}=\oint_C f\ dx+g\ dy+h\ dz$$

$$=\oint_C f\ dx+g\ dy+h\left(z_x dx+z_y dy\right)$$

$$= \oint_C \underbrace{\left(f + hz_x\right)}_{M(x,y)} dx + \underbrace{\left(g + hz_y\right)}_{N(x,y)} dy$$

Where $M(x,y) = f + hz_x \quad N(x,y) = g + hz_y$

Applying Green's Theorem: $\displaystyle\oint_{C'} M\,dx + N\,dy = \iint_R \left(N_x - M_y\right) dA$

$M(x, y) = f + h\,z_x$ $\qquad\qquad$ $\frac{df}{dy} = f_x x_y + f_y y_y + f_z z_y$

$$M_y = f_y + f_z z_y + hz_{xy} + z_x\left(h_y + h_z z_y\right)$$

$N(x, y) = g + h\,z_y$

$$N_x = g_x + g_z z_x + hz_{yx} + z_y\left(h_x + h_z z_x\right)$$

$$N_x - M_y = g_x + g_z z_x + hz_{yx} + h_x z_y + h_z z_x z_y - f_y - f_z z_y - hz_{xy} - h_y z_x - h_z z_y z_x$$

$$= g_x - f_y + z_x\left(g_z - h_y\right) + z_y\left(h_x - f_z\right)$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \left(g_x - f_y + z_x\left(g_z - h_y\right) + z_y\left(h_x - f_z\right)\right) dA$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS = \iint_R \left(\left(h_y - g_z\right)\left(-z_x\right) + \left(f_z - h_x\right)\left(-z_y\right) + \left(g_x - f_y\right)\right) dA$$

Where the upward vector normal $\vec{n} = \left\langle -z_x,\ -z_y,\ 1\right\rangle$

12.7-5 *Notes* on Stokes' *Theorem*

1. Stokes' Theorem allows a surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$ to be evaluated using only the values of the vector field in the boundary *C*.

$$\iint_{S_1} \left(\nabla \times \vec{F}\right) \cdot \vec{n}_1\, dS = \iint_{S_2} \left(\nabla \times \vec{F}\right) \cdot \vec{n}_2\, dS \qquad \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = 0$$

Since $\vec{n}_1$ and $\vec{n}_2$ are equal in magnitude and of opposite sign; therefore:

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS = \underbrace{\iint_{S_1} \left(\nabla \times \vec{F}\right) \bullet \vec{n}_1\, dS + \iint_{S_2} \left(\nabla \times \vec{F}\right) \bullet \vec{n}_2\, dS}_{=0}$$

2. If $\vec{F}$ is conservative vector field, then $\nabla \times \vec{F} = 0$.

Theorem Curl $\vec{F} = 0$ Implies $\vec{F}$ is Conservative

Suppose that $\nabla \times \vec{F} = 0$ throughout an open simply connected region *D* of $\mathbb{R}^3$. Then $\oint_C \vec{F} \bullet d\vec{r} = 0$ on all closed simple smooth curves *C* in *D* and $\vec{F}$ is a conservative vector field on *D*.

$$\oint_C \vec{F} \bullet d\vec{r} = \underbrace{\iint_S \underbrace{\left(\nabla \times \vec{F}\right)}_{0} \bullet \vec{n}\, dS}_{=0}$$

Exercises *Section* 12.7 – Stokes' Theorem

(1 – 6) Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

1. $\vec{F} = \langle y,\ -x,\ 10\rangle$; *S* is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and *C* is the circle $x^2 + y^2 = 1$ in the *xy*-plane.

2. $\vec{F} = \langle 0,\ -x,\ y\rangle$; *S* is the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and *C* is the circle $x^2 + y^2 = 4$ in the *xy*-plane.

3. $\vec{F} = \langle x,\ y,\ z\rangle$; *S* is the paraboloid $z = 8 - x^2 - y^2$ for $0 \le z \le 8$ and *C* is the circle $x^2 + y^2 = 8$ in the *xy*-plane.

4. $\vec{F} = \langle 2z,\ -4x,\ 3y\rangle$; *S* is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and *C* is the boundary of *S*.

5. $\vec{F} = \langle y - z,\ z - x,\ x - y\rangle$; *S* is the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$ and *C* is the boundary of *S*.

6. $\vec{F} = \langle -y,\ -x - z,\ y - x\rangle$; *S* is the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$ and *C* is the boundary of *S*.

(7–14) Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation.

7. $\vec{F} = \langle 2y,\ -z,\ x\rangle$; *C* is the circle $x^2 + y^2 = 12$ in the plane $z = 0$.

8. $\vec{F} = \langle y,\ xz,\ -y\rangle$; *C* is the ellipse $x^2 + \frac{y^2}{4} = 1$ in the plane $z = 1$.

9. $\vec{F} = \left\langle x^2 - z^2,\ y,\ 2xz\right\rangle$; *C* is the boundary of the plane $z = 4 - x - y$ in the plane first octant.

10. $\vec{F} = \left\langle y^2,\ -z^2,\ x\right\rangle$; *C* is the circle $\vec{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ for $0 \le t \le 2\pi$.

11. $\vec{F} = \left\langle 2xy\sin z,\ x^2\sin z,\ x^2 y\cos z\right\rangle$; *C* is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

12. $\vec{F} = \langle xz,\ yz,\ xy \rangle$; C: is the circle $x^2 + y^2 = 4$ in the xy-plane.

13. $\vec{F} = \left\langle x^2 - y^2,\ x,\ 2yz \right\rangle$; C is the boundary of the plane $z = 6 - 2x - y$ in the first octant.

14. $\vec{F} = \left\langle x^2 - y^2,\ z^2 - x^2,\ y^2 - z^2 \right\rangle$; C is the boundary of the square $|x| \le 1,\quad |y| \le 1$ in the plane $z = 0$.

(15 – 20) Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$. Assume that $\vec{n}$ points in an upward direction.

15. $\vec{F} = \langle x,\ y,\ z \rangle$; S is the upper half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

16. $\vec{F} = \langle 2y,\ -z,\ x - y - z \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25$ *for* $3 \le x \le 5$

17. $\vec{F} = \langle x + y,\ y + z,\ x + z \rangle$; S is the tilted disk enclosed $\vec{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \right\rangle$

18. $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$; S is the paraboloid $x = 9 - y^2 - z^2$ *for* $0 \le x \le 9$ (excluding its base), and $\vec{r}(t) = \langle x,\ y,\ z \rangle$

19. $\vec{F} = \langle -z,\ x,\ y \rangle$, where S is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$ *for* $z \ge 0$. Assume that $\vec{n}$ is the *outward normal*.

20. $\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz \right\rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $y \ge 0$. Assume that $\vec{n}$ is the outward normal.

(21 – 24) Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C.

21. $\vec{F} = \langle 2x,\ -2y,\ 2z \rangle$

22. $\vec{F} = \nabla\left(x \sin y e^z\right)$

23. $\vec{F} = \left\langle 3x^2 y,\ x^3 + 2yz^2,\ 2y^2 z \right\rangle$

24. $\vec{F} = \left\langle y^2 z^3,\ 2xyz^3,\ 3xy^2 z^2 \right\rangle$

25. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\vec{F} = \langle -y,\ x,\ 0 \rangle$ as a function of ϕ. For what value of ϕ is the circulation a maximum?

26. A circle C in the plane $x+y+z=8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \vec{F}\bullet d\vec{r}$ for $\vec{F}=\langle 0,\ -z,\ 2y\rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

27. Begin with the paraboloid $z=x^2+y^2$, for $0\le z\le 4$, and slice it with the plane $y=0$. Let S be the surface that remains for $y\ge 0$ (including the planar surface in the xz-plane). Let C be the semicircle and line segment that bound the cap of S in the plane $z=4$ with counterclockwise orientation. Let $\vec{F}=\langle 2z+y,\ 2x+z,\ 2y+x\rangle$

a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

b) Evaluate $\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS$

c) Evaluate $\oint_C \vec{F}\bullet d\vec{r}$ and check for argument with part (*b*).

28. The French Physicist André-Marie Ampère discovered that an electrical current I in a wire produces a magnetic field B. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B}\bullet d\vec{r}=\mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\boldsymbol{J}$ as $I=\iint_S \boldsymbol{J}\bullet\vec{n}\ dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla\times\boldsymbol{B}=\mu\boldsymbol{J}$.

29. Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \geq 0$, where $a > 0$ is a real number. Let $\vec{F} = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ have its maximum value?

30. The goal is to evaluate $A = \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$, where $\vec{F} = \langle yz,\ -xz,\ xy \rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1\ \ (z \geq 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of *A*.
b) Evaluate *A* using a line integral.

31. Let $\vec{F} = \langle 2z,\ z,\ x + 2y \rangle$ and let *S* be the hemisphere of radius *a* with its base in the *xy*-plane and center at the origin.

a) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ by computing $\nabla \times \vec{F}$ and appealing to symmetry.

b) Evaluate the line integral using Stokes' Theorem to check part (*a*).

32. Let *S* be the disk enclosed by the curve $C:\ \vec{r}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$ fot $0 \leq t \leq 2\pi$, where $0 \geq \varphi \leq \frac{\pi}{2}$ is a fixed angle.

a) Find the a vector normal to *S*.
b) What is the areas of *S*?
c) Whant the length of *C*?
d) Use the Stokes' Theorem and a surface integral to find the ciurculation on *C* of the vector field $\vec{F} = \langle -y,\ x,\ 0 \rangle$ as a function of φ. For what value of φ is the circulation a maximum?
e) What is the circulation on C of the vector field $\vec{F} = \langle -y,\ -z,\ x \rangle$ as a function of φ? For what value of φ is the circulation a maximum?
f) Consider the vector field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = \left\langle a_1,\ a_2,\ a_3 \right\rangle$ is a constant nonzero vector and $\vec{r} = \langle x,\ y,\ z \rangle$. Show that the circulation is a maximum when $\vec{a}$ points in the direection of the normal to *S*.

33. Let *R* be a region in a plane that has a unit normal vector $\vec{n} = \langle a,\ b,\ c \rangle$ and boundary *C*. Let $\vec{F} = \langle bz,\ cx,\ ay \rangle$

a) Show that $\nabla \times \vec{F} = \vec{n}$

b) Use Stokes' Theorem to show that: $\text{Area of } R = \oint_C \vec{F} \bullet d\vec{r}$

c) Consider the curve C given by $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t \rangle$, for $0 \le t \le 2\pi$. Prove that C lies in a plane by showing that $\vec{r} \times \vec{r}'$ is constant for all t.

d) Use part (*b*) to find the area of the region enclosed by C in part (*c*). (*Hint*: Find the unit normal vector that is consistent with the orientation of C.)

34. Consider the radial vector fields $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a) Evaluate a line integral to show that the field has zero circulation on C.

b) For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

35. Consider the vector fierld $\vec{F} = \dfrac{-y}{x^2 + y^2}\,\hat{i} + \dfrac{x}{x^2 + y^2}\,\hat{j} + z\,\hat{k}$

a) Show that $\nabla \times \vec{F} = \vec{0}$

b) Show that $\oint_C \vec{F} \bullet d\vec{r}$ is not zero on circle C in the xy-plane enclosing the origin.

c) Explain why Stokes' Theorem does not apply in this case.

36. Let S be a small circular disk of radius R centered at the point P with a unit normal vector $\vec{n}$. Let C be the boundary of S.

a) Express the average circulation of the vector field $\vec{F}$ on S as a surface integral of $\nabla \times \vec{F}$

b) Argue for that small R, the average circulation approaches $\left(\nabla \times \vec{F}\right)\Big|_P \bullet \vec{n}$ (the component of $\nabla \times \vec{F}$ in the direction of $\vec{n}$ evaluated at P) with the approximation improving as $R \to 0$.

Section 12.8 – Divergence Theorem

12.8-1 Divergence Theorem

The Divergence Theorem is the 3-dimensional version of the flux form of Green's Theorem.
If R is a region in the xy-plane, C is the simple closed piecewise-smooth oriented boundary of R, and $\vec{F} = \langle M, N \rangle$ is a vector field, Green's Theorem says that:

$$\underbrace{\oint_C \vec{F} \bullet \vec{n}\, dS}_{\text{flux across } C} = \iint_R \underbrace{\left(M_x + N_y\right)}_{\text{divergence}} dA$$

12.8-2 *Theorem*

Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_D \nabla \bullet \vec{F}\, dV$$

Where $\vec{n}$ is the unit outward normal vector on S.

Example 1

Consider the radial field $\vec{F} = \langle x, y, z \rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\vec{n}$ is the outward normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

Solution

The divergence of $\vec{F}$:

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$
$$= 3$$

$$\iiint_D \nabla \bullet \vec{F}\, dV = \iiint_D (3)\, dV$$
$$= 3 \cdot volume(D)$$

$$= 3 \cdot \frac{4\pi}{3} a^3$$

$$= 4\pi a^3$$

$$\vec{r} = \langle x,\ y,\ z \rangle \qquad R = \{(\phi,\ \theta):\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

$$= \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$$

$$\vec{r}_\phi = \langle a\ \cos\phi\ \cos\theta,\ a\ \cos\phi\ \sin\theta,\ -a\sin\phi \rangle$$

$$\vec{r}_\theta = \langle -a\ \sin\phi\ \sin\theta,\ a\ \sin\phi\ \cos\theta,\ 0 \rangle$$

The required vector normal to the surface is:

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\ \cos\phi\ \cos\theta & a\ \cos\phi\ \sin\theta & -a\sin\phi \\ -a\ \sin\phi\ \sin\theta & a\ \sin\phi\ \cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2 \sin^2\phi\cos\theta\right)\hat{i} + \left(a^2 \sin^2\phi\sin\theta\right)\hat{j} + \left(a^2 \sin\phi\cos\phi\cos^2\theta + a^2 \sin\phi\cos\phi\sin^2\theta\right)\hat{k}$$

$$= \left\langle a^2\ \sin^2\phi\ \cos\theta,\quad a^2\ \sin^2\phi\ \sin\theta,\quad a^2\ \sin\phi\ \cos\phi \right\rangle$$

$$\vec{F} \bullet \left(t_\phi \times t_\theta\right) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle \bullet \left\langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi \right\rangle$$

$$= a^3\ \sin^3\phi\ \cos^2\theta\ +\ a^3\ \sin^3\phi\ \sin^2\theta\ +\ a^3\ \sin\phi\cos^2\phi$$

$$= a^3\ \sin\phi\left(\sin^2\phi + \cos^2\phi\right)$$

$$= a^3 \sin\phi$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \nabla\vec{F} \bullet \left(\vec{r}_\phi \times \vec{r}_\theta\right)\ dA$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} a^3 \sin\phi\ d\phi$$

$$= a^3\ \left(\theta\ \Big|_0^{2\pi}\ \left(-\cos\theta\ \Big|_0^{\pi}\right.\right.$$

$$= 4\pi a^3$$

∴ The two integrals of the Divergence Theorem are equal.

Example 2

Consider the rotation field:

$$\begin{aligned}\vec{F} &= \vec{a}\times\vec{r}\\ &= \langle 1,\ 0,\ 1\rangle \times \langle x,\ y,\ z\rangle\\ &= \langle -y,\ \ x-z,\ \ y\rangle\end{aligned}$$

Let S be the sphere $x^2+y^2+z^2=a^2$ for $z\ge 0$, together with its base in the xy-plane. Find the net outward flux across S.

Solution

$$\begin{aligned}\nabla\bullet\vec{F} &= \frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x-z)+\frac{\partial}{\partial z}(y)\\ &\underline{=0}\end{aligned}$$

$\therefore$ The flux across the hemisphere is zero.

However, with the Divergence Theorem, radial fields are interesting and have many physical applications.

Example 3

Find the net outward flux of the field $\vec{F}=xyz\,\langle 1,\ 1,\ 1\rangle$ across the boundaries of the cube $D=\{\,(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1\,\}$

Solution

$$\begin{aligned}\nabla\bullet\vec{F} &= \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}(xyz)\\ &\underline{= yz+xz+xy}\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla\bullet\vec{F}\ dV &= \int_0^1\int_0^1\int_0^1 (yz+xz+xy)\ dx\,dy\,dz\\ &= \int_0^1\int_0^1 \left(yzx+\frac{1}{2}x^2z+\frac{1}{2}x^2y\ \Big|_0^1\right.\ dy\,dz\\ &= \int_0^1\int_0^1 \left(yz+\frac{1}{2}z+\frac{1}{2}y\right)\ dy\,dz\end{aligned}$$

$$= \int_0^1 \left(\frac{1}{2}y^2 z + \frac{1}{2}zy + \frac{1}{4}y^2 \right|_0^1 dz$$

$$= \int_0^1 \left(z + \frac{1}{4} \right) dz$$

$$= \frac{1}{2}z^2 + \frac{1}{4}z \Big|_0^1$$

$$= \frac{3}{4}$$

12.8-3 Divergence *Theorem* for Hollow Regions

Suppose the vector field $\vec{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2.

Let S be the entire boundary of $D\left(S = S_1 \cup S_2\right)$ and let $\vec{n}_1$ and $\vec{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively.

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_{S_2} \vec{F} \cdot \vec{n}_2\, dS - \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS$$

12.8–4 Interpretation of the Divergence Using Mass Transport

Suppose that $\vec{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density.

The vector field $\vec{F} = \rho\vec{v} = \langle f,\ g,\ h\rangle$ describes the ***mass transport*** of the material, with units of :

$$\frac{mass}{vol.} \times \frac{length}{time} = \frac{mass}{area - time}$$ typical units of mass transport are $g / m^2 / s$.

This means that $\vec{F}$ gives the mass material flowing past a point (in each of the three coordinates direction) per unit of surface area per unit of time.

When $\vec{F}$ is multiplied by an area, the result in the *flux*, with units of mass/unit time.

Example 4

Consider the inverse square vector field $$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

a) Find the net outward flux of $\vec{F}$ across the surface of the region $D = \left\{(x, y, z):\ a^2 \le x^2 + y^2 + z^2 \le b^2\right\}$ that lies between concentric spheres with radii *a* and *b*.

b) Find the outward flux of $\vec{F}$ across any sphere that encloses the origin,

Solution

a) $$\nabla \bullet \vec{F} = \nabla \bullet \left(\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \nabla \frac{1}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3x\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3x}{|\vec{r}|^5}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3y\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -\frac{3y}{|\vec{r}|^5}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-3/2}$$

$$= -3z\left(x^2+y^2+z^2\right)^{-5/2}$$

$$= -\frac{3z}{|\vec{r}|^5}$$

$$\nabla\frac{1}{|\vec{r}|^3} = -3\frac{x\hat{i}+y\hat{j}+z\hat{k}}{|\vec{r}|^5}$$

$$= -3\frac{\vec{r}}{|\vec{r}|^5}$$

$$\nabla\bullet\vec{F} = \nabla\bullet\left(-\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\frac{1}{|\vec{r}|^3}\nabla\bullet\vec{r} + 3\vec{r}\bullet\frac{\vec{r}}{|\vec{r}|^5}$$

$$= -\frac{3}{|\vec{r}|^3} + \frac{3}{|\vec{r}|^3}$$

$$= 0$$

Let $S = S_2\left(\textit{with radius } b \textit{ larger}\right)\cup S_1\left(\textit{with radius } a\right)$

Because $\iiint_D \nabla\bullet\vec{F}\ dV = 0$, the divergence Theorem implies that:

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \iint_{S_2}\vec{F}\bullet\vec{n}_2\ dS - \iint_{S_1}\vec{F}\bullet\vec{n}_1\ dS = 0$$

Therefore, the net flux across S is zero.

b) $$\iint_{S_2} \vec{F} \cdot \vec{n}_2 \, dS = \iint_{S_1} \vec{F} \cdot \vec{n}_1 \, dS$$

$$\iint_{S_1} \vec{F} \cdot \vec{n}_1 \, dS = \iint_{S_1} \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} \, dS$$

$$= \iint_{S_1} \frac{\vec{r}^{\,2}}{|\vec{r}|^4} \, dS \qquad \left(|\vec{r}| = a\right)$$

$$= \iint_{S_1} \frac{1}{a^2} \, dS \qquad \textit{Surface Area} = 4\pi a^2$$

$$= \frac{4\pi a^2}{a^2}$$

$$= 4\pi$$

$$\iint_{S_2} \vec{F} \cdot \vec{n}_2 \, dS = \iint_{S_2} \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} \, dS$$

$$= \frac{4\pi b^2}{b^2}$$

$$= 4\pi$$

The flux of the inverse square field across any surface enclosing the origin is 4π.

12.8–5 *Gauss'* Law

Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge Q located at the origin is given by the inverse square law.

$$\vec{E}(x, y, z) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

Where $\vec{r}(x, y, z)$ and ε_0 is a physical constant called the ***permittivity of free square***.

This is one statement of Gauss' Law: If S is a surface that encloses a point charge Q, then the flux of the electric field across S is

$$\iint_S \boldsymbol{E} \cdot \vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

Gauss' Law: Flux of electric field across S due to point charge Q

$$\iint_S \boldsymbol{E} \cdot \vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

Gauss' Law: Flux of electric field across S due to charge distribution q.

$$\iint_S \boldsymbol{E} \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q\, dV$$

	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem of Line Integrals	$\int_C \nabla f \bullet d\mathbf{x} = f(B) - f(A)$	
Green's Theorem (Circulation Form)	$\iint_R \left(g_x - f_y\right) dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS = \oint_C \vec{F} \bullet d\vec{r}$	
Divergence Theorem	$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_D \nabla \bullet \vec{F}\, dV$	

Exercises *Section* 12.8 – Divergence Theorem

$(\mathbf{1-4})$ Evaluate both integrals of the Divergence Theorem for the following vector fields and region. Check for agreement.

1. $\vec{F} = \langle 2x,\ 3y,\ 4z \rangle \quad D = \left\{ (x, y, z):\ x^2 + y^2 + z^2 \le 4 \right\}$

2. $\vec{F} = \langle -x,\ -y,\ -z \rangle \quad D = \{ (x, y, z):\ |x| \le 1,\ |y| \le 1,\ |z| \le 1 \}$

3. $\vec{F} = \langle z - y,\ x,\ -x \rangle \quad D = \left\{ (x, y, z):\ \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \le 1 \right\}$

4. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle \quad D = \{ (x, y, z):\ |x| \le 1,\ |y| \le 2,\ |z| \le 3 \}$

5. Find the net outward flux of the field $\vec{F} = \langle 2z - y,\ x,\ -2x \rangle$ across the sphere of radius 1 centered at the origin.

6. Find the net outward flux of the field $\vec{F} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface $\mathbb{R}^3$, where *a*, *b*, and *c* are constants.

7. Find the net outward flux of the field $\vec{F} = \langle z - y,\ x - z,\ y - x \rangle$ across the boundary of the cube $\{ (x,\ y,\ z):\ |x| \le 1,\ \ |y| \le 1\ \ |z| \le 1 \}$

$(\mathbf{8-47})$ Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

8. $\vec{F} = \langle x,\ -2y,\ 3z \rangle$; *S* is the sphere $\left\{ (x, y, z):\ x^2 + y^2 + z^2 = 6 \right\}$.

9. $\vec{F} = \left\langle x^2,\ 2xz,\ y^2 \right\rangle$; *S* is surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

10. $\vec{F} = \langle x,\ 2y,\ z \rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$.

11. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; *S* is the sphere $\left\{ (x, y, z):\ x^2 + y^2 + z^2 = 25 \right\}$.

12. $\vec{F} = \left\langle y - 2x,\ x^3 - \mathrm{y},\ y^2 - z \right\rangle$; *S* is the sphere $\left\{ (x, y, z):\ x^2 + y^2 + z^2 = 4 \right\}$.

13. $\vec{F} = \langle x,\ y,\ z \rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

14. $\vec{F} = \langle x,\ y,\ z \rangle$; S is the surface of the cone $z^2 = x^2 + y^2$, for $0 \leq z \leq 4$, plus its top surface in the plane $z = 4$.

15. $\vec{F} = \langle z - x,\ x - y,\ 2y - z \rangle$; D is the region between the spheres of radius 2 and 4 centered at origin.

16. $\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z \rangle \sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at origin.

17. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\langle x,\ y,\ z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between the spheres of radius 1 and 2 centered at origin.

18. $\vec{F} = \langle z - y,\ x - z,\ 2y - x \rangle$; $D = \{(x, y, z):\ 1 \leq |x| \leq 3,\ 1 \leq |y| \leq 3,\ 1 \leq |z| \leq 3\}$ is the region between two cubes.

19. $\vec{F} = \langle y + z,\ x + z,\ x + y \rangle$; S consists of the faces of the cube

$\{(x,\ y,\ z):\ \ |x| \leq 1,\ \ |y| \leq 1\ \ |z| \leq 1\}$

20. $\vec{F} = \left\langle x^2,\ -y^2,\ z^2 \right\rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$

21. $\vec{F} = \langle x,\ 2y,\ 3z \rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \leq z \leq 8$.

22. $\vec{F} = \langle -x,\ x - y,\ x - z \rangle$ across S is the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1,\ and\ z = 1$.

23. $\vec{F} = \frac{1}{3}\left\langle x^3,\ y^3,\ z^3 \right\rangle$ across S is the sphere $\left\{(x,\ y,\ z):\ x^2 + y^2 + z^2 = 9\right\}$

24. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ across D is the region between two spheres of radius 1 and 2 centered at $(5,\ 5,\ 5)$.

25. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; S is the cylinder $\left\{(x, y, z):\ x^2 + y^2 = 4,\ 0 \leq z \leq 8\right\}$

26. $\vec{F} = \left\langle x^2 e^y \cos z,\ -4xe^y \cos z,\ 2xe^y \sin z \right\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$

27. $\vec{F} = \langle -yz,\ xz,\ 1 \rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$.

28. $\vec{F} = \langle x \sin y,\ -\cos y,\ z \sin y \rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$.

29. $\vec{F} = \left\langle x^2 + x \sin y,\ y^2 + 2\cos y,\ z^2 + z \sin y \right\rangle$ across the surface S that is the boundary of the prism bounded by the planes.

30. $\vec{F} = \langle x,\ -2y,\ 4z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2,\quad a > 0$.

31. $\vec{F} = \left\langle ye^z,\ x^2 e^z,\ xy \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2,\quad a > 0$.

32. $\vec{F} = \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$ of the sphere S with $x^2 + y^2 + z^2 = a^2,\quad a > 0$.

33. $\vec{F} = \left\langle x^3,\ 3yz^2,\ 3y^2 z + x^2 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2,\quad a > 0$.

$y = 1 - x,\ x = 0,\ y = 0,\ z = 0,\ z = 4$

34. $\vec{F} = \left\langle 2z,\ x,\ y^2 \right\rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, and the xy-plane.

35. $\vec{F} = \left\langle x,\ y^2,\ z \right\rangle$; S is the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$.

36. $\vec{F} = \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$; S is the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane.

37. $\vec{F} = \left\langle 2x^3,\ 2y^3,\ 2z^3 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 4$.

38. $\vec{F} = \langle x,\ y,\ z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$.

39. $\vec{F} = \langle z,\ y,\ x \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$.

40. $\vec{F} = \left\langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \right\rangle$; S is the solid region bounded by the cylinder $z = 1 - x^2$, the planes $y + z = 2$, $z = 0$, and $y = 0$.

41. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of an ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$.

42. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the tetrahedron $x + y + z \le 3 \quad \& \quad x,\ y,\ z \ge 0$.

43. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the cylinder $x^2 + y^2 \le 2y \quad \& \quad 0 \le z \le 4$.

44. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of a ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$.

45. $\vec{F} = \left\langle x^4,\ -x^3z^2,\ 4xy^2z \right\rangle$; across the boundary of the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2 \quad \& \quad z = 0$.

46. $\vec{F} = \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$; S is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

47. $\vec{F} = \left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z \right\rangle$; across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

48. Prove that $\nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = -\dfrac{4\vec{r}}{|\vec{r}|^6}$ and use the result to prove that $\nabla \bullet \nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = \dfrac{12}{|\vec{r}|^6}$.

49. Consider the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$. Let S be the sphere of radius a at the origin.

a) Use the surface integral to show that the outward flux of $\vec{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\dfrac{\vec{r}}{|\vec{r}|}$.

b) For what values of p does $\vec{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact the $\nabla \bullet \vec{F} = \dfrac{3-p}{|\vec{r}|^p}$ to compute the flux around S using the Divergence Theorem.

50. Consider the radial vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$.

a) Evaluate a surface integral to show that $\iint_S \vec{F} \bullet \vec{n}\, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b) Note that the first partial derivatives of the components of $\vec{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (*a*) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate $div\vec{F}$ over the region between two spheres of radius a and $0 < \varepsilon < a$. Then let $\varepsilon \to 0^+$ to obtain the flux computed in part (*a*).

51. The electric field due to a point charge Q is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \bullet \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x, y, z\rangle$ and ε_0 is a constant.

a) Show that the flux of the field across a sphere of radius a centered at the origin is

$$\iint_S \boldsymbol{E} \bullet \vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

b) Let S be the boundary of the origin between two spheres centered of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of charge within a region D. Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \boldsymbol{E} \bullet \vec{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q(x, y, z)\, dV$$

d) Assuming $\boldsymbol{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \bullet \boldsymbol{E} = \frac{q}{\varepsilon_0}$

e) Because the electric force is conservative, it has a potential function ϕ. From part (*d*) conclude that $\nabla^2\phi = \nabla \bullet \nabla\phi = \frac{q}{\varepsilon_0}$

(52 – 55) ***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region *D* is given. Find the net outward heat flux $\iint_S \vec{F} \bullet \vec{n}\, dS = -k \iint_S \nabla T \bullet \vec{n}\, dS$ across the boundary *S* of *D*. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

52. $T(x,\ y,\ z) = 100 + x + 2y + z;\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

53. $T(x,\ y,\ z) = 100 + x^2 + y^2 + z^2;\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

54. $T(x,\ y,\ z) = 100 + e^{-z};\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

55. $T(x,\ y,\ z) = 100e^{-x^2-y^2-z^2}$; *D* is the sphere of radius *a* centered at the origin.

56. Consider the surface *S* consisting of the quarter-sphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$ *and* $x \ge 0$, and the half disk in the *yz*-plane $y^2 + z^2 \le a^2$, for $z \ge 0$. The boundary of *S* in the *xy*-plane is *C*, which consists of the semicircle $x^2 + y^2 = a^2$, for $x \ge 0$, and the line segment $[-a,\ a]$ on the *y*-axis, with a counterclockwise orientation. Let $\vec{F} = \langle 2z - y,\ x - z,\ y - 2x \rangle$

a) Describe the direction in which the normal vectors point on *S*.

b) Evaluate $\oint_C \vec{F} \bullet d\vec{r}$

c) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$ and check for segment with part (*b*).

57. Let *S* be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and let *T* be the paraboloid $z = a - \frac{1}{a}\left(x^2 + y^2\right)$, for $z \ge 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

a) Verify that *S* and *T* have the same base $\left(x^2 + y^2 \le a^2\right)$ and the same high point $(0,\ 0,\ a)$.

b) Which surface has the greater area?

c) Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z \rangle$ across *S* is $2\pi a^3$.

d) Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z \rangle$ across *T* is $\frac{3\pi a^3}{2}$.

58. The gravitational force due to a point mass M is proportional to $\vec{F} = \frac{GM\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x, y, z \rangle$ and G is the gravitational constant.

a) Show that the flux force field across a sphere of radius a centered at the origin is

$$\iint_S \vec{F} \bullet \vec{n}\, dS = 4\pi GM$$

b) Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of mass within a region D containing the origin. Let $\rho = (x, y, z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \vec{F} \bullet \vec{n}\, dS = 4\pi G \iiint_D \rho(x, y, z)\, dV$$

d) Assuming $\vec{F}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \bullet \vec{F} = 4\pi G\rho$

e) Because the gravitational force is conservative, it has a potential function ϕ. From part (*d*) conclude that $\nabla^2\phi = 4\pi G\rho$.

59. Let $\vec{F}$ be a radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x, y, z \rangle$. With $p = 3$, $\vec{F}$ is an inverse square field.

a) Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$

b) Explain the observation in part (*a*) by finding the flux of $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ across the boundaries of a spherical box $\{(\rho, \varphi, \theta): \quad a \le \rho \le b, \quad \varphi_1 \le \varphi \le \varphi_2, \quad \theta_1 \le \theta \le \theta_2\}$ for various values of p.

60. Consider the potential function $\phi(x, y, z) = G(\rho)$, where G is any twice differentiable function and $\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, G depends only on the distance from the origin.

a) Show that the gradient vector field associated with ϕ is $\vec{F} = \nabla\phi = G'(\rho)\frac{\vec{r}}{\rho}$, where $\vec{r} = \langle x, y, z \rangle$ and $\rho = |\vec{r}|$.

b) Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S. show that the flux of $\vec{F}$ across S is $\iint_S \vec{F} \bullet \vec{n}\, dS = 4\pi a^2 G'(a)$.

c) Show that $\nabla \bullet \vec{F} = \nabla \bullet \nabla \phi = \frac{2}{\rho} G'(\rho) + G''(\rho)$

d) Use part (*c*) to show that the flux across S (as given in part (*b*)) is also obtained by the volume integral $\iiint_D \nabla \bullet \vec{F}\, dV$. (*Hint*: use spherical coordinates and integrate by parts.)

61. Prove Green's Identity for scalar-valued functions u and v defined on a region D:

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right) dV = \iint_S \left(u\nabla v - v\nabla u\right) \bullet \vec{n}\, dS$$

62. Prove the identity: $$\iiint_D \nabla \times \vec{F}\, dV = \iint_S \left(\vec{n} \times \vec{F}\right) dS$$

63. Prove the identity: $$\iint_S \left(\vec{n} \times \nabla \varphi\right) dS = \oint_C \varphi\, d\vec{r}$$

Summary (Review)

$$\int_C f(x,y,z)\, ds = \int_a^b f\big(g(t),\, h(t),\, k(t)\big)\, \left|\vec{v}(t)\right|\, dt$$

$$W = \int_C \vec{F} \bullet \vec{T}\, ds = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$$

$\vec{F}$ is conservative:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad and \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

$$\vec{F} = \nabla f \qquad M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k} = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$$

$$\text{Flux of } \vec{F} \text{ across C} = \int_C \vec{F} \bullet \vec{n}\, ds$$

$$\vec{F} \bullet \vec{n} = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

$$\left(Flux\ of\ \ \vec{F} = M\,\hat{i} + N\,\hat{j}\ \ across\ C\right) = \oint_C Mdy - Ndx$$

Divergence $\quad div\vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

$$\underbrace{\oint_C \vec{F} \bullet \vec{n}\, ds}_{Outward\ flux} = \oint_C Mdy - Ndx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy}_{Divergence\ \text{integral}}$$

$$\underbrace{\oint_C \vec{F} \bullet \vec{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C Mdx + Ndy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy}_{Curl\ \text{integral}}$$

Green's Theorem: $Area = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_R r^2 d\theta$

$$Surface\ Area = \iint_R \left|\vec{r}_u \times \vec{r}_v\right| dA = \int_c^d \int_a^b \left|\vec{r}_u \times \vec{r}_v\right| du\, dv$$

$$Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \boldsymbol{p}|} dA \qquad \hat{p} = \hat{i},\ \hat{j},\ or\ \ \hat{k}$$

$$= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dxdy$$

Exercises *Lecture* 12 – Review

$(1-18)$

Practice *Exam*

1. Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\vec{r}(t) = 2t\,\hat{i} + t\,\hat{j} + (2 - 2t)\hat{k}, \quad 0 \le t \le 1$

2. Find the ***work*** required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $r(t) = \left\langle t^2,\ 3t^2,\ -t^2\right\rangle$ for $1 \le t \le 2$.

3. Use ***Green's*** theorem to find the counterclockwise ***circulation*** and outward ***flux*** for the field $\vec{F} = (x + y)\,\hat{i} - \left(x^2 + y^2\right)\hat{j}$ and curve C is the triangle bounded by $y = 0,\ x = 1,\ y = x$

4. Use either form of ***Green's*** Theorem to evaluate the line integral
$\oint_C \left(x^3 + xy\right)dy + \left(2y^2 - 2x^2 y\right)dx$;
C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation.

5. Calculate the divergence of the radial fields. $\vec{F} = \langle x,\ y,\ z\rangle \left(x^2 + y^2 + z^2\right) = 5\left|\vec{r}\right|^2$

Express the result in terms of the position vector $\vec{r}$ and its length $\left|\vec{r}\right|$.

6. Find the area of the surface of the frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 4$ (excluding the bases). (Use a surface integral)

7. Evaluate the line integral in ***Stokes'*** Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$. Assume that $\vec{n}$ points in an upward direction, $\vec{F} = \langle 2y,\ -z,\ x - y - z\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25 \quad for\ 3 \le x \le 5$

8. Use the ***Divergence*** Theorem to compute the outward flux of the vector field $\vec{F} = \langle -x,\ x - y,\ x - z\rangle$ across S is the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1,\ and\ z = 1$.

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iiint_D \nabla \bullet \vec{F}\ dV$$